Standing in Line to Withdraw

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Abstract

This paper studies the dynamics in fundamental driven bank runs. Consumers make withdrawal decisions based on the public history of the number of withdrawals already made and the private signals on the fundamentals. The paper proves that given a demand deposit contract, there is a perfect Bayesian equilibrium in which consumers withdraw if the beliefs are above the threshold, and wait otherwise. This perfect Bayesian equilibrium also exists in a broad class of banking mechanisms that allow the bank to give out payment contingent on the history of withdrawals. The optimal contract tolerates bank run in some economies.

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1 Introduction

Bank runs are a widely observed phenomenon in the banking history. As modeled by Diamond and Dybvig (1983), the bank's major function is to provide liquidity to the consumers. The ex ante identical consumers have private information on consumption shocks ex post. The consumers who are hit by the shocks are impatient and need to consume immediately, while the others can wait. Thus, a risk averse consumer would like to smooth consumptions ex ante. The economy has a production technology which matures only in the long term. Due to the rigidity of the production, a consumer can not smooth consumption in autarky. The bank aggregates the consumption shocks, and provides a demand deposit

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contract that observes sequential service constraint. The contract serves the consumers like an insurance against being impatient and achieves the first-best allocation.

There are two sources for bank instability. One is panic driven and the other is fundamental driven. A panic driven run is unrelated to fundamentals such as the bank's choice of portfolio. When all consumers decide to withdraw deposits from the bank, the bank will have to deplete all its resource to meet the payment requirement and close early. An individual consumer fears that if everyone else withdraws the deposits, he will be unpaid if he does not do so. When all of the consumers decide to withdraw, the panic driven run is self-fulfilled. The panic driven run is well explored in Diamond and Dybvig (1983).

The fundamental driven run relates to the information on, for example, the performance of bank's portfolio. When bad news spreads around, consumers worry that the bank is unable to pay back the promised amount in the long term since its investment is unsuccessful. Therefore, the consumers prefer to withdraw as soon as possible before the bank declares bankruptcy, and bank run occurs.

However, regardless of the driving force, the existing literature on bank runs focuses on static models. Specifically, consumers decide whether to withdraw simultaneously without the observation of the decisions of the others. Whether a bank run happens is determined by a publicly acknowledged coordination device in the economy. Peck and Shell (2003) models the panic run as a sunspot phenomenon. A purely extrinsic variable tells consumers whether bank run is on-going or not. Peck and Shell also prove that an optimal contract tolerates bank runs in some economies if the probability of bank run is small. Goldstein and Pauzner (2005) construct a model in which fundamentals determine whether bank run occurs or not. Consumers receive noisy signals of production and make decisions whether to withdraw or not. A consumer runs on the bank if the signal is below the threshold, and does not run above. When the fundamentals are in the middle range, a proportion of consumers withdraw the deposits since the noisy signals they obtain fall below the threshold. In this model, a private signal on fundaments is the coordination device although the signal is imperfect.

As observed in reality, bank run is not a simultaneous move game. Consumers stand in line sequentially with observations of their positions in the line and the number of consumers ahead of them. This paper reconsiders the banking problem by modelling formally the dynamics, or the withdrawal history. Production is random. Consumers receive private signals on the productivity and also their consumption types in sequence in period 1. By watching whether other consumers get in the line, an individual consumer infers the information on productivity, and updates the prior accordingly.

A dynamic model is crucial to study the interesting features during the bank run. Starr and Yilmaz (2006) study the Turkish bank runs on the special finance houses (SFH) in 2001 empirically. SFH have the functions of a commercial bank, but the deposits are not insured. The authors show that the "increased withdrawals by moderate-size accountholders tended to boost withdrawals by small counterparts, suggesting that the latter viewed the former as informative with respect to the SFH's financial condition". Without a model which allows for consumers' decisions to be dependent on the public withdrawal history, this phenomenon can not be explained theoretically.

The dynamics, or the queueing process in bank runs, resembles herding in investment decisions. There exists a large amount of literature on herd behavior. The literature on information cascade and herding discusses information externality and the consumers' or investors' rationality of discarding private signals to follow up decisions of others. Banerjee (1992) provides a simple model of herd behavior to illustrate how investors choose the project. According to Banerjee, investors receive imperfect private signals on which is the best project, and they make decisions only when the signals are received. Because the signals are of the same quality, once two of the same investment decisions are observed, the rest of the investors will invest in the popular project. If an investor's private signal indicates a different project, it is offset by the two signals inferred from the previous investors. Compare to the simple model, herding in bank runs has three special features.

First, when a consumer gets in the line, it is not clear whether the consumer needs to consume right away or he gets a low signal. If there is a perfect Bayesian equilibrium, the description of the cutoff beliefs is more complicated. Second difference lies in the payment externality. For an investment problem, the number of investors who choose a project does not affect the quality of the project, and the return on it is ex ante determined. While in the banking setup, whether a consumer decides to withdraw affects the bank's payment to the next consumer. The more consumers withdraw the deposits, the less chance and less amount other consumers will get paid. The second difference is not obvious if only a simple demand deposit contract is allowed But if a more complicated banking mechanism such as partial suspension is allowed, a consumer's decision updates the bank's belief of the productivity and, therefore, the bank's payments to the following consumers are re-calculated. Thirdly, there is no assigned time slots to the consumers in bank runs. Consumers can get in the line any time they want. Although the signals are sent in an exogenously determined sequence, consumers' timing of decision making is endogenously determined.

The second and third deviations from the simple herd models are studied by Avery and Zemsky (1998) and Chari and Kehoe (2003), respectively. Avery and Zemsky (1998) study the payment externality in a financial markets where the asset prices are adjusted according to the number of sells and buys. They show that without the uncertainty in the economy, the price adjustment prevents herd behavior. However, if there is shocks on the economy, herd behavior still exists. This results is in accordance with mine. Herd behavior exists in the economy even though the bank can pay consumers conditionals on the withdrawal history because the information the bank inferred from consumers' withdrawal decision is not perfect. In Chari and Kehoe's (2003) paper, an investor can make decisions whether to invest in a project at any point of time. There exists a perfect Bayesian equilibrium in which an investor, either informed or not yet, updates prior by the observation of other investors' decisions, and either invests or waits depending on whether his belief is above or below a cutoff level. The endogenous timing feature of the bank runs in this paper is very similar to Chari and Kehoe's. However, this paper is the first one to piece together all three special features in herding in a banking setup.

I start with a simple demand deposit contract without the suspension of convertibility to illustrate the queueing process. A simple demand deposit contract with sequential service is widely used in the banking industry. Calomiris and Kahn (1991) show that demand deposit contract is efficient if a bank's moral hazard problem potentially exists. Since bank run is costly, consumers are motivated to monitor the bank and the moral hazard problem will be reduced. Thus, it is worthwhile as the first attempt to explain the queueing process given a contract in a narrow class of banking mechanism such as a simple demand deposit contract. A few consumers have the privilege to receive signals on productivity and consumption types, while others make decision by watching how the informed consumers behave. There exists a perfect Bayesian Nash equilibrium in which there are cutoff beliefs for the informed and uninformed consumers below which they withdraw deposits from the bank, above which they do not. An optimal demand deposit contract is calculated for an example. It is proved that in some economies, an optimal contract tolerates bank run.

The simple demand deposit contract is not optimal in this setup. Sequential service is crucial to generate bank run. Since the bank also sees the queue, it updates the belief as an uninformed consumer thus it can forecast consumers' decisions, and offer a contract depending on the history of withdrawals. The second part of the paper is to present a two-consumer two-stage version of the model in which the bank pays the consumers contingent on the withdrawal history. The sequential service is observed because consumers need to consume at the time they are informed to be impatient. In some economies, an optimal contract in this class of banking mechanism tolerates bank runs. In an extreme case where the signals are perfect, the optimal contract is run-admitting regardless of other parameters.

In this paper, I do not consider panic driven bank runs. Bank run occurs due to the weakness in fundamentals. A bank is out of funds because the productivity is low. Bank run is not undesirable as it interrupts a production crisis in progress. There is positive probability that the bank run happens due to the wrong signals or due to the fact that an unusually large number of people are informed as impatient at the first few stages. A bank is still socially desirable because it smooths consumption across consumption type and the queueing in front of the bank may reveal more information on productivity than an individual consumer can get in autarky.

There are several papers attempting to address the dynamics in banking crisis. Chari and Jagannathan (1988) analyze an economy with random productivity. Some consumers are informed of the productivity status and others are not. The uninformed consumers infer information on productivity by observing the number of withdrawals in the economy. There is a rational expectation equilibrium in the model which allows for bank run. However, although there is time sequence in the model, the equilibrium concept is static. The queueing process is not clear in Chari and Jagannathan.

Chen's (1999) paper explains contagious bank runs using information externality. Banks' assets are correlated. Some customers of some banks have private information on the asset returns. When those banks receive more than the usual number of customers, customers of other banks are worried that their investments in the banks are probably in bad shape too. Thus, they withdraw from the banks and trigger the crisis in banking sector. However, Chen's paper does not model the queueing process for an individual bank explicitly, and observing a queue in front of some banks is somewhat inconsistent with the assumption that consumers are isolated from each other.

Green and Lin (2003 (*a, b*)) models the dynamics in a banking game by assuming that consumers receive consumption signals sequentially and withdraw deposit sequentially from the bank. There exists a banking mechanism that completely eliminates panic runs. Compare to their model, this paper has the following differences.

Consumers in Green and Lin's withdraw deposits sequentially. They know the probabilities of getting in certain positions in the line, but they do not observe the line or the decisions of other consumers. Also, there is a direct revelation rule in Green and Lin's. Consumers are forced to report their consumption types to the bank. Given such a mechanism, consumers prefer to tell the truth about their liquidity needs, and bank runs are completely eliminated. In my model, consumers do not report to the bank the signals they receive. Given a simple demand deposit contract, a bank can not make the payments conditional on the withdrawal history. Bank run exists because the contract is not able to vary with the changes in the economy. Even if the bank pays consumers conditional on the history, because bank's information on productivity inferred from the withdrawal history is not the same as the consumers' information, bank run still occurs. In another word, the mechanisms considered in my model do not eliminate asymmetric information between the bank and the consumers. A direct revelation rule in Green and Lin's might not work here, because at least the impatient consumes have the incentive to always report high productivity in order to get more liquidity. There may exist a more efficient mechanism. However, it is not persued in this paper.

While this work is in progress, a paper by Yorulmazer (2003) was recently discovered. The model set-ups are similar to each other. However, there are several important differences between this paper and Yorulmazer's.

Yorumazer (2003) consider a deposit contract with countably infinite number of consumers. He illustrates that in some economies, there is a perfect Bayesian equilibrium in which bank run occurs if two consumers in a row withdraw deposits. A contract that allows for such an equilibrium is better than a run proof contract. However, the proof of the existence of the perfect Bayesian equilibrium is incorrect. It is not possible to prove the existence of the equilibrium given any contract for the banking mechanism considered since there are infinte number of consumers in the economy. Therefore, the conclusion of the paper is not credible and it is not clear whether an optimal contract tolerates bank run as a result of herd behavior. In this paper, the existence of a perfect Bayesian equilibrium is proved rigorously, and an optimal contract is found.

This paper is organized as follows: A model with simple demand deposit contract will be discussed in section 2. A perfect Bayesian Nash equilibrium will be shown to exist in such an economy. In section 3, a more complicated banking mechanism will be introduced and an optimal contract will be calculated. The last part is the conclusion.

2 A Model with Simple Demand Deposit Contract

2.1 Set Up

There are three periods, $t = 0, 1, 2$ (period 0, 1 and 2, respectively), and a mass of 1 consumers in the economy. Each consumer is endowed with 1 unit of consumption goods in period 0. Period 1 is divided into $N + 1$ stages. N is a finite integer. Consumers are identical at $t = 0$, but they face consumption shocks at some stage at $t = 1$. If a consumer is hit by the consumption shock, he is called impatient and has to consume immediately. If a consumer is not hit by the consumption shock, his consumption type is patient. In each of the first N stages of $t = 1$, only one consumer is informed of his consumption type. At stage $N+1$, all the consumers who have not received consumption information are informed. Consumers have equal opportunity to be informed at each stage. Since N is tiny compared to the infinite number of consumers, the probability of getting a consumption shock in the first N stages is zero. Each consumer has probability α ($0 < \alpha < 1$) to be impatient and probability $1 - \alpha$ to become patient at $t = 1$. By the law of large numbers, a proportion of α of the consumers are impatient. An impatient consumer's utility is described by $v(c^1)$, where c^1 is the consumption received at $t = 1$. Patient consumers derive utility from the consumption in the last period. If a patient consumer receives consumption at $t = 1$, he can reinvest it into a storage technology and consume it at $t = 2$. Thus, a patient consumer's utility is described by $u(c^1+c^2)$, where c^2 is the consumption received at $t=2$. $v(\cdot)$ and $u(\cdot)$ are strictly increasing, strictly concave and twice differentiable. The coefficients of relative risk aversion of the utility functions, $-xv''(x)/v'(x)$ and $-xu''(x)/u'(x)$, are greater than 1 for $x \geq 1$. $v(x) \geq u(x)$ and $v'(x) \geq u'(x)$ for $x \geq 0$. The utility function is normalized to 0 at $x=0$, i.e., $v(0)=u(0)=0$.

In addition to the storage, there is a risky and rigid production technology available to the consumers. Investment in the production can only be made in the first period. One unit of endowment invested at $t=0$ yields 1 unit of consumption goods at $t=1$, and R units at $t=2$. R is a random variable which is unknown at $t=0$. It is realized at $t=2$. But the distribution of R is known at $t=0$. The production and storage technologies are illustrated as

$$
\begin{array}{lcccl}
 & t=0 & t=1 & t=2 & \\
\text{Production:} & \left\{\begin{array}{c} -1 \\ -1 \end{array}\right. & \begin{array}{c} \rightarrow 1 \\ \Rightarrow \end{array} & \Rightarrow R = & \left\{\begin{array}{ll} \overline{R} > 1, & \text{with probability } p; \\ \underline{R} = 1, & \text{with probability } 1-p. \end{array}\right. \\
\text{Storage:} & \left\{\begin{array}{c} -1 \\ \end{array}\right. & \begin{array}{c} \Rightarrow 1 \\ -1 \end{array} & \begin{array}{c} \\ \Rightarrow 1 \end{array} &
\end{array}
$$

Although the return on the production is random, it strictly dominates the storage as the return is at least one unit.

The banking market is competitive. The representative bank accepts deposits from the consumers and offers a simple demand deposit contract that describes the amount of consumption goods paid to the consumers who withdraw in period 1 (c^1) and 2 (c^2) respectively. c^1 is independent of the productivity state since the productivity is realized at $t=2$. c^2 is state contingent. The bank pays c^1 to the consumers at $t=1$ until it is out of funds. The bank distributes the remaining resource in addition to the interest equally among the consumers who wait until the last period. Therefore

$$
c^2 = \left\{\begin{array}{ll} \frac{1-\beta c^1}{1-\beta} R & \text{if } \beta c^1 \leq 1; \\ 0 & \text{if } \beta c^1 > 1. \end{array}\right.
$$

where β ($0 \leq \beta \leq 1$) denotes the fraction of consumers who withdraw the deposits in period 1. In the situation that the bank can not meet payment requirement at $t=1$, the bank fails. Since the production strictly dominates the storage, bank invests all deposits in production and does not need to change the portfolio at any stage at $t=1$.

In each of the first N stages, the consumer who receives the signal on the consumption type also receives a signal on productivity. The information on consumption is precise. The signal of the production status is accurate with probability q, $q > 0.5$. That is,

$$\Pr(R = \overline{R} | S_n = H) = \Pr(R = \underline{R} | S_n = L) = q.$$

S_n denotes the signal of productivity obtained by consumer n. If the signal is high $(S_n = H)$, the probability of getting $R = \overline{R}$ at $t = 2$ is q, while the probability of getting $R = \underline{R}$ when a low signal $(S_n = L)$ is received is also q. Receiving a signal, a consumer updates his belief of the productivity by the Bayes rule. The common initial prior is p_0.

At the last stage, all consumers who have not received signals are informed of their consumption types, but no information on the productivity is released. Consumers do not communicate with each other about the signals they receive. However, a consumer's withdrawal decision is observed by all others. By watching the number of withdrawals, the uninformed consumers can infer the signals the informed obtain and Bayesian update the belief of productivity.

There are four types of consumers at each of the first N stages. The first type are those who have already withdrawn their deposits from the bank. Those are inactive consumers. They can not deposit money back to the bank later. The second type is the newly informed consumer who gets the signals at current stage. The third type are those who are previously informed but did not withdraw from the bank. The rest are uninformed consumers.

Once the consumer withdraws the deposit, he can not reverse his decision. However, if a consumer chooses to wait, he can withdraw the deposit at a later stage. The final deadline for consumers to withdraw at $t = 1$ is at the stage of $N + 1$. I impose an assumption that at the stage $N + 1$, consumers line up in front of the bank in a random order. The consumers are not allowed to make decisions after observing other consumers' decisions at stage $N + 1$. So a previously informed or uninformed consumer can not infer any information at the last stage.

A finite number of stages is necessary because it imposes a deadline to the consumers' decisions, and the expected utility can be calculated by backward induction. The specification of an infinite number of consumers tremendously simplifies the calculation. Consider a model allowing for a finite number of consumers and each of them will be informed of the consumption types and productivity signals at a stage. The law of large numbers does not apply any more. A consumer's expected utility depends on the updates of productivity as well as the updates of the number of impatient consumers. By using a continuum of consumers, the number of impatient consumers can be calculated by the law of large numbers at any point of time. Consumers do not need to update the number of impatient consumers while updating the productivity status. However, the assumption of a continuum of consumers is not essential to the results. The results can be extended to the situation where there are a finite number of consumers each of whom will be informed of consumption types and productivity status one by one. To simplify the presentation, I first consider a finite number of people to be informed one by one and a continuum to be informed at last all together. In section 3, I present a simplest case of two-stage and two-consumer with a more complicated banking contract. We will see that the results still hold.

The difference in the utility functions of an impatient and a patient consumer is not essential either; however, it helps construct the numerical example of an optimal contract in the predeposit game.

The sequence of timing of the banking game is as follows.

Bank announces the contract;	$t = 0$		pre-deposit game
Consumers make investment decision.			
Signals on consumption type and production state are revealed to the first N consumers one by one.	$t = 1$	post deposit game	
Consumers make withdrawal decisions.			
(repeat for N stages)			
At stage $N + 1$, consumption types are revealed to those who are not informed.			
Consumers make withdrawal decisions.			
Bank allocates the remaining resource to the rest of the consumers.	$t = 2$		

The post deposit game starts after consumers deposit at the bank. Only one consumer who has not received any signals before receives signals at a stage, and makes withdrawal decision that relies on the signals received and withdrawal history he observed. At stage $N + 1$, all consumers who have not been informed are informed of their consumption types. If the bank is not out of resources at $t = 1$, it distributes the remaining funds available at $t = 2$ equally among the consumers who did not withdraw at $t = 1$. Knowing the strategies of consumers in the post deposit game, the bank offers an optimal contract that describes the payments to the consumers who withdraw at $t = 1$ and $t = 2$, respectively. Consumers determine whether to deposit at the bank or stay in autarky. Starting at $t = 0$, the entire game is called the predeposit game. My paper starts with the analysis in the post deposit game. I first prove there exists a perfect Bayesian equilibrium given the contract in the post deposit game. Then the optimal contract the bank offers in the predeposit game given the equilibrium strategies in the post deposit game will be calculated.

2.2 Post Deposit Game

I start with the post deposit game, which takes the banking contract as given. In Diamond and Dybvig (1983), a demand deposit banking contract allows for a run equilibrium in the post deposit game if $c^1 > 1$. Run equilibrium exists because if all consumers withdraw deposits from the bank, the bank will be out of funds. If anyone chooses to stay, his expected payoff will be zero. Meanwhile, there exists a no-run equilibrium where people are truth-telling and the consumption is smoothed across the consumption types. Which equilibrium occurs is traditionally determined by a sunspot variable. For example, if a zero-one sunspot variable ω is realized to be 1, the run equilibrium occurs, while if it is 0, no-run equilibrium takes place. Thus, the probability of bank run is due to the distribution function of ω.

To simplify the model, the panic driven run is not considered, because it is a separable problem from the fundamental driven run studied here. If the contract allows for both run and no-run equilibria at a stage, the probability of getting a run equilibrium is assumed to be zero. Bank run occurs in the model solely due to the imperfect information on the productivity released in period 1. After stage N, if consumers believe the productivity is high enough that the expected utility at $t = 2$ is higher than that of withdrawing immediately given that all other patient consumers do not withdraw deposits, consumers do not withdraw unless they are told to be impatient at stage $N + 1$.

I follow the notations used in Chari and Kehoe (2003) as much as possible in this section. Let X_n denote the total number of withdrawals at stage n. The history of withdrawals, $h_n = (X_1, X_2, ..., X_{n-1})$, publicly records the number of withdrawals at each stage up to the beginning of stage n. The history of consumer i who receives a signal at stage r is $h_{i,n} = (h_n, s_r, r)$, and the history of an uninformed consumer is $h_{i,n} = (h_n, \varnothing, \varnothing)$.

The strategies $x_{i,n}(h_{i,n})$ and beliefs $p_{i,n}(h_{i,n})$ of consumers are sequences of functions that map the history of an individual consumer into zero-one withdrawal decisions and into probabilities of the productivity being high. Let $x_{i,n}(h_{i,n}) = 1$ represent the decision to withdraw, while $x_{i,n}(h_{i,n}) = 0$ represent the decision to wait. At stage n, the belief of an uninformed or of a previously informed consumer is based on his observation of the number of withdrawals in each stage up to n, and a newly informed consumer's belief is that of the uninformed at stage $n - 1$ updated by the signal he receives. Assume that consumers with the same history adopt the same strategy at any stage. To simplify the notation, let x_n^U and p_n^U denote the strategy and belief of a consumer who is uninformed at stage n; and let x_n^{I,S_r} and p_n^{I,S_r} denote the strategy and belief of a consumer who is informed at stage r of a productivity signal S_r. If $r = n$, the consumer is newly informed. Otherwise, he is previously informed. If the calculations apply to all of the three types of active consumers, superscripts will be omitted and the notations of x_n and p_n will be used.

The consumer informed at stage n Bayesian updates his belief when a signal is obtained by

$$p_n^{I,S_n} = \begin{cases} P_H(p_{n-1}^U) = \dfrac{p_{n-1}^U q}{p_{n-1}^U q + (1-p_{n-1}^U)(1-q)}, & \text{if } S_n = H; \\ \\ P_L(p_{n-1}^U) = \dfrac{p_{n-1}^U\,(1-q)}{p_{n-1}^U(1-q) + (1-p_{n-1}^U)q}, & \text{if } S_n = L. \end{cases}$$

P_H and P_L denote the rule of Bayesian updates when a high or a low signal is received, respectively. $p \leq P_H\,(p) \leq 1$ and $0 \leq P_L\,(p) \leq p$ for $p \in [0,1]$ and $q > 0.5$. $P_H\,(p)$ and $P_L\,(p)$ are strictly increasing in p. $P_S\,(0) = 0$ and $P_S\,(1) = 1$, $S = H$, L. The simplification of the signal accuracy has the following advantage: Because signals are of the same quality, receiving or inferring n_1 high signals and n_2 low signals is equivalent to receiving or inferring $n_1 - n_2$ high signals.

The uninformed and previously informed consumers update the probability that the productivity is high by observing the decision made by the newly informed consumer. Suppose the uninformed and the previously informed consumers believe that the newly informed consumer stays if and only if a high signal is received and he is patient, and that the newly informed consumer withdraws the deposit if he is impatient or a low signal is received. The uninformed consumers then update the belief by

$$p_n^U = \begin{cases} P_H(p_{n-1}^U) = \dfrac{p_{n-1}^U q}{p_{n-1}^U q + (1-p_{n-1}^U)\,(1-q)}, & \text{if the newly informed waits;} \\ \\ P_{\widetilde{L}}(p_{n-1}^U) = \dfrac{p_{n-1}^U\,(1-q+\alpha q)}{\alpha + (1-\alpha)\left[\left(1-p_{n-1}^U\right)q + p_{n-1}^U(1-q)\right]}, & \text{if the newly informed withdraws.} \end{cases}$$

$0 < P_L\,(p) < P_{\widetilde{L}}(p) < p$ for $p \in (0,1)$. $P_L\,(p) = P_{\widetilde{L}}(p) = 0$ for $p = 0$, and $P_L\,(1) = P_{\widetilde{L}}(1) = 1$ for $p = 1$. The previously informed consumers update the prior by the same rule.

The rule states that when the newly informed waits, the uninformed and the previously informed consumers are sure that he gets a high signal. Their beliefs are updated by a high signal given the prior. However, when the newly informed consumer is observed to withdraw deposit, the newly informed could get a low signal; he could also get to know he is impatient and have to consume right away. Hence, the update of the uninformed and the previously informed is not as fast as when a high signal is obtained. Let $P_{\widetilde{L}}$ denote the Bayesian update when people are not sure whether a low signal is actually received or the consumer has to consume immediately. The belief update rule has a good property, i.e., $P_H^{n_1}\left(P_{\widetilde{L}}^{n_2}(p)\right) = P_{\widetilde{L}}^{n_2}(P_H^{n_1}\,(p))$. The power on $P_{\widetilde{L}}$ (or P_H) denotes the number of updates by $P_{\widetilde{L}}$ (or P_H), given the prior.

Suppose, again, that the uninformed and previously informed consumers believe that the newly informed does not make decisions according to the signal on productivity. In this case, their prior belief will not be changed because the newly informed's decision carries no information of the productivity. Therefore, $p_n^U = p_{n-1}^U$, and $p_n^{I,S_r} = p_{n-1}^{I,S_r}$ for $r < n$.

Let us draw a time line for $t = 1$ to illustrate the sequence of belief updates.

Starts with p_0.

At stage 1:

Consumer 1 is informed $\left(p_1^{I,S_1} = P_{S_1}(p_0)\right)$ and decides to withdraw or not $\left(x_1^{I,S_1} = 0 \text{ or } 1\right)$

Uninformed consumers update beliefs $\left(p_1^U\right)$ and decide to withdraw or not $\left(x_1^U = 0 \text{ or } 1\right)$.

At stage 2:

Consumer 2 is informed $\left(p_2^{I,S_2} = P_{S_2}\left(p_1^U\right)\right)$ and decides to withdraw or not $\left(x_2^{I,S_2} = 0 \text{ or } 1\right)$

Uninformed consumers update beliefs $\left(p_2^U\right)$ and decide to withdraw or not $\left(x_2^U = 0 \text{ or } 1\right)$.

At stage 3:

Repeat the same process.

...

...

At stage $N+1$:

All uninformed consumers are informed of their consumption types. $x_{N+1}^I = 0$ or 1.

The end.

The amount paid to a consumer if he withdraws the deposit at stage n at $t = 1$ is

$$\begin{cases} c_1, & \text{if } \beta c^1 < 1; \\ 0, & \text{otherwise.} \end{cases}$$

where β is the measure of consumers who have withdrawn already. By the assumption of symmetric strategies, either all uninformed consumers withdraw or none of them withdraw. If none of the uninformed consumers withdraw, a consumer gets c_1 definitely if he withdraws. If all of the consumers withdraw, the payment of a consumer depends on his position in the line. If there are fewer than a measure of $\frac{1}{c^1}$ consumers in front of him, he will be paid, otherwise, he is left unpaid.

The expected utility of a consumer who does not withdraw at stage n is more complicated. The expected utility obviously depends on the current belief. Furthermore, it depends on how the active consumers behave at future stages. I start with stage N to illustrate this.

Let $v_1 = v\left(c^1\right)$, $u_1 = u\left(c^1\right)$, $\overline{u}_2 = u\left(\frac{1-\alpha c^1}{1-\alpha}\overline{R}\right)$, and $\underline{u}_2 = u\left(\frac{1-\alpha c^1}{1-\alpha}\underline{R}\right)$. Suppose that the uninformed consumers wait at the end of stage $N-1$. At the end of stage N, the uninformed consumers have the updated prior p_N^U. Consumers will not get more information of the productivity at stage $N+1$. Therefore, if $p_N^U \overline{u}_2 + \left(1-p_N^U\right) \underline{u}_2 \geq u_1$, consumers will stay unless they are told to be impatient at stage $N+1$. If $p_N^U \overline{u}_2 + \left(1-p_N^U\right) \underline{u}_2 < u_1$, regardless of the decisions of other consumers, an individual consumer will withdraw his deposits before he receives information on his consumption type. Assuming symmetric strategies, i.e. people with the same history adopt the same strategy, each consumer has a chance of $\frac{1}{c^1}$ to get in line in time if $c^1 \geq 1$. He will definitely get a payment of amount of c^1 if $c^1 < 1$. So the expected utility at the end of stage N, denoted by $w_N\left(p_N^U\right)$, is

$$w_N\left(p_N^U\right) = \begin{cases} \alpha v_1 + (1-\alpha)\left(p_N^U \overline{u}_2 + \left(1-p_N^U\right)\underline{u}_2\right), & \text{if } p_N^U \overline{u}_2 + \left(1-p_N^U\right)\underline{u}_2 > u_1; \\ \min\left\{\frac{1}{c^1}, 1\right\}\left(\alpha v_1 + (1-\alpha) u_1\right), & \text{otherwise;} \end{cases}$$

For a consumer informed at stage N with prior p_{N-1}^U, the belief is updated to $P_H\left(p_{N-1}^U\right)$ or $P_L\left(p_{N-1}^U\right)$ according to the signal he receives. Because no more information will be available after stage N, the informed consumer makes decision based on his belief and on other people's reactions to his decision. Assume firstly, that the uninformed and previously informed consumers share the same belief; secondly, $p_{N-1}^U \overline{u}_2 + \left(1-p_{N-1}^U\right)\underline{u}_2 > u_1$, which means if the uninformed's belief is not updated at stage N, they are willing to wait with belief p_{N-1}^U. If the previously informed consumers have the same belief as the uninformed, they are also willing to wait at stage N if beliefs are not updated; assume thirdly, with prior p_{N-1}^U, a newly informed consumer is willing to wait if a high signal is received. Lastly, if a consumer is willing to wait at stage $N-1$, he is also willing to wait at stage N if a high signal is recieved given all others wait. Let us verify that the following strategies and beliefs of the active consumers at stage N are part of the equilibrium.

If $c^1 \geq 1$, $\overline{u}_2 \geq u_1 \geq \underline{u}_2$, define $\underline{p}$ as

$$u_1 = \underline{p}\overline{u}_2 + \left(1-\underline{p}\right)\underline{u}_2. \tag{1}$$

$\underline{p}$ is between 0 and 1. If $\overline{u}_2 \geq \underline{u}_2 \geq u_1$, $\underline{p} < 0$; if $u_1 > \overline{u}_2 \geq \underline{u}_2$, $\underline{p} > 1$.

For a newly informed consumer at stage N.

$$x_N^{I,S_N} = \begin{cases} 1, & \text{if impatient or } p_N^{I,S_N} \leq \underline{p};. \\ 0, & \text{otherwise.} \end{cases}$$

For $n = N$, an uninformed consumer's strategy is

$$x_N^U = \begin{cases} 1, & \text{if } w_N\left(p_N^U\right) \leq \alpha v_1 + (1-\alpha) u_1. \\ 0, & \text{otherwise.} \end{cases}$$

The belief of an uninformed consumer is updated by

$$p_N^U = \begin{cases} P_{\widetilde{L}}(p_{N-1}^U), & \text{if } X_n \geq 1 \text{ and } P_L(p_{N-1}^U) \leq \underline{p}; \\[2em] P_H(p_{N-1}^U), & \text{if } X_n = 0 \text{ and } P_L(p_{N-1}^U) \leq \underline{p}; \\[2em] p_{N-1}^U, & \text{otherwise.} \end{cases}$$

Let us first look at how the newly informed consumer behaves if he is patient. Suppose the prior is very high. Even a low signal is received, the newly informed's posterior belief is still above the critical level of $\underline{p}$. Though a low signal is received, the newly informed will not withdraw. Knowing that the newly informed will not withdraw whichever the signal is, the uninformed and the previously informed consumers' beliefs are unchanged. By the assumption that $p_{N-1}^U \overline{u}_2 + \left(1 - p_{N-1}^U\right) \underline{u}_2 > u_1$, the uninformed consumers prefer to wait. The previously informed also prefer to wait if they also have the belief of p_{N-1}^U. Given all others wait, the newly informed patient consumer's best response is to wait because if he waits, his expected payoff is higher than u_1.

In the second case, the newly informed consumer's prior is moderately high. If a low signal is received, the posterior belief drops below the critical level of $\underline{p}$. Regardless of the uninformed's strategy, when a low signal is received, it is in the best interest of the newly informed to withdraw as no more information will be obtained by waiting. If a high signal is received, the newly informed is willing to wait if all others wait. When the newly informed waits, the uninformed and the previously informed consumers' belief will be updated to the same level as the newly informed consumer, and all people will agree to wait as $P_H\left(p_{N-1}^U\right) \overline{u}_2 + \left(1 - P_H\left(p_{N-1}^U\right)\right) \underline{u}_2 > u_1$. Waiting is the best response, given everyone else waits. Given the uninformed consumers' response to a newly informed consumer's decision, a newly informed consumer has the incentive to wait if a high signal is received.

In summary, an uninformed consumer's belief is updated in the following way. An uninformed consumer imagines that he is the one who is newly informed, and he deduces the newly informed consumer's strategies to the signals. On the one hand, If the newly informed adopts a "pooling" strategy, i.e., the same strategy to different signals, then an uninformed consumer's belief will not be changed by the newly informed consumer's decision. On the other hand, if the newly informed adopts a "separating" strategy, i.e., the different strategies to different signals, then an uninformed consumer updates belief by watching how the newly informed behaves. A previously informed consumer updates his belief in the same way.

Given the strategies at stage N, the expected utility to an uninformed consumer at stage $N - 1$ is

$$w_{N-1}\left(p^U_{N-1}\right) =$$

$$\begin{cases} \alpha v_1 + (1-\alpha)\left(p^U_{N-1}\overline{u}_2 + \left(1-p^U_{N-1}\right)\underline{u}_2\right), & \text{if } P_L\left(p^U_{N-1}\right) > \underline{p}; \\ \\ \pi\left(p^U_{N-1}\right) w_N\left(P_H\left(p^U_{N-1}\right)\right) + & \text{if } P_L\left(p^U_{N-1}\right) \le \underline{p} \text{ and} \\ (1-\pi\left(p^U_{N-1}\right)) w_N\left(P_{\widetilde{L}}\left(p^U_{N-1}\right)\right), & \pi\left(p^U_{N-1}\right) w_N\left(P_H\left(p^U_{N-1}\right)\right) + (1-\pi\left(p^U_{N-1}\right))\cdot \\ & \cdot\, w_N\left(P_{\widetilde{L}}\left(p^U_{N-1}\right)\right) > \alpha v_1 + (1-\alpha) u_1; \\ \min\left\{\frac{1}{c^1}, 1\right\}\left(\alpha v_1 + (1-\alpha) u_1\right), & \text{otherwise.} \end{cases}$$

where

$$\pi\left(p^U_{N-1}\right) = (1-\alpha)\left[\left(1-p^U_{N-1}\right)(1-q) + p^U_{N-1} q\right].$$

is the probability that given the prior of p^U_{N-1} at the end of stage $N-1$, the consumer informed at stage N gets a high signal and is also patient.

What happens at an arbitrary stage before N? Is there an equilibrium in which a critical level of belief for a newly informed consumer exists at each stage below which the newly informed withdraws the deposits, and above which he does not? If that is the case, an uninformed can track the decisions of the newly informed and update his prior accordingly, and the expected utility of an uninformed consumer at any stage given the belief can be written down. Furthermore, according to the calculated expected utility based on the belief at the current stage, the uninformed consumers decide whether they should wait for another stage or whether they should withdraw the deposits immediately before they infer more information from other consumers. In other word, is there a cutoff belief for the uninformed consumers at each stage below which the uninformed withdraw the deposits and above which they stay? Let us conjecture that the following strategies and belief update rules constitute a perfect Bayesian equilibrium.

Define the expected utility of an uninformed consumer with belief p_n^U at stage n as

For $n \geq N$:

$$w_N^U\left(p_N^U\right) = \begin{cases} \alpha v_1 + (1-\alpha)\left(p_N^U \overline{u}_2 + \left(1-p_N^U\right)\underline{u}_2\right), & \text{if } p_N^U > \underline{p}; \\ \min\left\{\frac{1}{c^1}, 1\right\}\left(\alpha v_1 + (1-\alpha) u_1\right), & \text{otherwise.} \end{cases}$$

For $1 \leq n < N$:

$$w_n^U\left(p_n^U\right) = \begin{cases} \alpha v_1 + (1-\alpha)\left(p_n^U \overline{u}_2 + \left(1-p_n^U\right)\underline{u}_2\right), & \text{if } P_L\left(p_n^U\right) > \underline{p}; \\ \\ \pi\left(p_n^U\right) w_{n+1}^U\left(P_H\left(p_n^U\right)\right) + & \text{if } P_L\left(p_n^U\right) \leq \underline{p} \text{ and } \pi\left(p_n^U\right) w_{n+1}^U\left(P_H\left(p_n^U\right)\right) + \\ (1-\pi\left(p_n^U\right)) w_{n+1}^U\left(P_{\widetilde{L}}\left(p_n^U\right)\right), & +(1-\pi\left(p_n^U\right)) w_{n+1}^U\left(P_{\widetilde{L}}\left(p_n^U\right)\right) > \alpha v_1 + (1-\alpha) u_1; \\ \\ \min\left\{\frac{1}{c^1}, 1\right\}\left(\alpha v_1 + (1-\alpha) u_1\right), & \text{otherwise.} \end{cases} \tag{2}$$

where

$$\pi\left(p_n^U\right) = (1-\alpha)\left[\left(1-p_n^U\right)(1-q) + p_n^U q\right]. \tag{3}$$

Define a previously informed consumer's expected utility as

For $n \geq N$:

$$w_N^{I,S_r}\left(p_N^{I,S_r}\right) = \begin{cases} p_N^{I,S_r}\overline{u}_2 + \left(1-p_N^{I,S_r}\right)\underline{u}_2, & \text{if } p_N^{I,S_r} > \underline{p} \text{ and } w_N^U\left(p_N^U\right) > \alpha v_1 + (1-\alpha) u_1; \\ \\ u_1, & \text{if } p_N^{I,S_r} \leq \underline{p} \text{ and } w_N^U\left(p_N^U\right) > \alpha v_1 + (1-\alpha) u_1; \\ \\ \min\left\{\frac{1}{c^1}, 1\right\} u_1, & \text{otherwise.} \end{cases}$$

For $1 \leq n < N$:

$$w_n^I\left(p_n^{I,S_r}\right) = \begin{cases} p_n^{I,Sr}\overline{u}_2 + \left(1-p_n^{I,Sr}\right)\underline{u}_2, & \text{if } P_L\left(p_n^U\right) > \underline{p}; \\ \\ \max\{\pi\left(p_n^{I,S_r}\right) w_{n+1}^{I,S_r}\left(P_H\left(p_n^{I,S_r}\right)\right) + & \text{if } P_L\left(p_n^U\right) \leq \underline{p} \text{ and} \\ (1-\pi\left(p_n^{I,S_r}\right)) w_{n+1}^{I,S_r}\left(P_{\widetilde{L}}\left(p_n^{I,S_r}\right)\right), u_1\} & w_n^U\left(p_n^U\right) > \alpha v_1 + (1-\alpha) u_1; \\ \\ \min\left\{\frac{1}{c^1}, 1\right\} u_1, & \text{otherwise.} \end{cases} \tag{4}$$

The newly informed consumer becomes a previously informed consumer at next stage. At the current stage, he can choose to withdraw and get u_1 or become a previously informed consumer next stage. So his expected utility is

For $n = N$:

$$w_N^{I,S_N}\left(p_N^{I,S_N}\right) = \begin{cases} p_N^{I,S_N}\overline{u}_2 + \left(1 - p_N^{I,S_N}\right)\underline{u}_2, & \text{if } p_N^{I,S_N} > \underline{p} \text{ and } w_N^U\left(p_N^U\right) > \alpha v_1 + (1-\alpha)\,u_1; \\ \\ u_1, & \text{otherwise.} \end{cases}$$

For $1 \leq n < N$:

$$w_n^I\left(p_n^{I,S_n}\right) = \begin{cases} p_n^{I,S_n}\overline{u}_2 + \left(1 - p_n^{I,S_n}\right)\underline{u}_2, & \text{if } P_L\left(p_n^U\right) > \underline{p}; \\ \\ \max\{\pi\left(p_n^{I,S_n}\right) w_{n+1}^{I,S_n}\left(P_H\left(p_n^{I,S_n}\right)\right) + & \text{otherwise.} \\ (1 - \pi\left(p_n^{I,S_n}\right))w_{n+1}^{I,S_n}\left(P_{\widetilde{L}}\left(p_n^{I,S_n}\right)\right), u_1\} & \end{cases} \tag{5}$$

For $1 \leq n \leq N$, a newly informed consumer's strategy is

$$x_n^{I,S_n} = \begin{cases} 1, & \text{if impatient or } p_n^{I,S_n} < \underline{p}. \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

For $1 \leq n \leq N$, an uninformed consumer's strategy is

$$x_n^U = \begin{cases} 1, & \text{if } w_n\left(p_n^U\right) \leq \alpha v_1 + (1-\alpha)\,u_1. \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

For $1 \leq n \leq N$, a previously informed consumer's strategy is $(r < n)$

$$x_n^{I,S_r} = \begin{cases} 1, & \text{if } w_n\left(p_n^{I,S_r}\right) \leq u_1. \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

For $n = N + 1$, an active consumer's strategy is

$$x_{N+1} = \begin{cases} 1, & \text{if impatient.} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

The belief of a newly informed consumer at stage $1 \leq n \leq N$ is updated by

$$p_n^{I,S_n} = \begin{cases} P_L(p_{n-1}^U), & \text{if } S_n = L; \\ P_H(p_{n-1}^U), & \text{if } S_n = H. \end{cases} \tag{10}$$

The belief of an uninformed consumer at stage $1 \leq n \leq N$ is updated by

$$p_n^U = \begin{cases} P_{\widetilde{L}}(p_{n-1}^U), & \text{if } X_n \geq 1 \text{ and } P_L(p_{n-1}^U) \leq \underline{p}; \\ \\ P_H(p_{n-1}^U), & \text{if } X_n = 0 \text{ and } P_L(p_{n-1}^U) \leq \underline{p}; \\ \\ p_{n-1}^U, & \text{otherwise.} \end{cases} \tag{11}$$

The belief of a previously informed consumer at stage $1 \leq n \leq N$ is updated by $(r < n)$

$$p_n^{I,S_r} = \begin{cases} P_{\widetilde{L}}(p_{n-1}^{I,S_r}), & \text{if } X_n \geq 1 \text{ and } P_L(p_{n-1}^{I,S_r}) \leq \underline{p}; \\ \\ P_H(p_{n-1}^{I,S_r}), & \text{if } X_n = 0 \text{ and } P_L(p_{n-1}^{I,S_r}) \leq \underline{p}; \\ \\ p_{n-1}^{I,S_r}, & \text{otherwise.} \end{cases} \tag{12}$$

Denote $P_L(\bar{p}) = \underline{p}$. $\bar{p}$ is the cutoff probability above which the uninformed or previously consumers' belief will not be updated by the withdrawal decision of the newly informed consumer any more. $\bar{p} > \underline{p}$ for $\underline{p} \in (0, 1)$.

The above strategies and beliefs imply the following: A newly informed consumer follows the productivity signal if $P_L(p_{n-1}^U)$ drops below the threshold level. If there are many consumers staying in the economy, the prior of the uninformed consumers will be raised to such a high level that even though a low signal is received, the newly informed's belief will not be below the threshold. From then, the consumers will be staying regardless of the signals of productivity. In the opposite case, if there are many people lining up in the queue, the expected utility of the uninformed consumers drops below $\alpha v_1 + (1 - \alpha) u_1$, and they are unwilling to wait for more information. All consumers will be lining up in front of the bank. In a situation that the uninformed observe some withdrawals and some no withdrawals, and the belief has been updated neither high enough nor low enough all the way to the end of stage N, then the uninformed make withdrawal decisions based on p_N^U. At stage $N + 1$, they either withdraw or stay depending on the expected utility. Before the equilibrium is proved formally, I first introduce the definitions of herd of withdrawals and no withdrawals.

Definition 1 *herd of non-withdrawals begins when (1) the newly informed consumers do not withdraw deposits unless he is impatient even though a low signal on productivity is received, and (2) all other consumers wait until their consumption type is revealed to be impatient.*

Definition 2 *Herd of withdrawals begins when all consumers withdraw deposits.*

Herd of withdrawals is a simultaneous move. When it begins, all consumers get in the queue at the stage it happens. They do not know whether they are lucky enough to reach the bank early. They get to know whether they will be paid when they arrive at the bank and observe their actual positions in the queue. If herd of non-withdrawals begins at stage $n < N+1$, the newly informed consumers withdraw deposits one by one if impatient, and at stage $N+1$ all impatient consumers withdraw deposits simultaneously. If neither herd of non-withdrawals nor herd of withdrawals happens before stage $N+1$, then at stage $N+1$, consumers make decisions according to the accumulated information on productivity as well as their consumption types, and the definition of herd of non-withdrawals is satisfied if patient consumers do not withdraw at stage $N+1$.

When the strategies at the last stage is illustrated, four conditions are assumed. To prove the equilibrium, we need to show that conditions 2-4 assumed are actually satisfied in the equilibrium. The first condition is also satisfied and will be proved as a part of proposition 1.

Lemma 1 *$w_n(p_n)$ is increasing in p_n on $[0,1]$. For $\underline{p} < 0$, $w_n(p_n) = \alpha v_1 + (1-\alpha)[p_n \overline{u}_2 + (1-p_n)\underline{u}_2] > \alpha v_1 + (1-\alpha) u_1$ for $p_n \in [0,1]$. For $\underline{p} > 1$, $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha) u_1]$ on $p_n \in [0,1]$. For $\underline{p} \in [0,1]$, $w_N(p_N) = \alpha v_1 + (1-\alpha)[p_N \overline{u}_2 + (1-p_N)\underline{u}_2]$ on $(\underline{p}, 1]$ and $w_N(p_N) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha) u_1]$ on $p_N \in [0, \underline{p}]$. At stage $0 \le n \le N-1$, there exist $\tilde{p}_n$ and $\hat{p}_n$ $(0 < \tilde{p}_n \le \hat{p}_n \le \bar{p})$ such that $w_n(p_n) = \alpha v_1 + (1-\alpha)[p_n \overline{u}_2 + (1-p_n)\underline{u}_2] > \alpha v_1 + (1-\alpha) u_1$ for $p_n \in (\hat{p}_n, 1]$; $w_n(p_n)$ is increasing on $(\tilde{p}_n, \hat{p}_n]$ with $\alpha v_1 + (1-\alpha) u_1 < w_n(p_n) < \alpha v_1 + (1-\alpha)[p_n \overline{u}_2 + (1-p_n)\underline{u}_2]$; $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha) u_1]$ for $p_n \in [0, \tilde{p}_n]$. $\hat{p}_n$ and $\tilde{p}_n$ are decreasing in n.*

Proof. see appendix. ■

Lemma 2 *If $w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha) u_1$, $w_n^U\left(P_H\left(p_{n-1}^U\right)\right) > w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha) u_1$.*

Proof. see appendix. ■

Lemma 1 and 2 have the following implication: given c^1, assume u_1 is a consumer's cutoff expected payoff conditional on his being patient, if a consumer is willing to wait at stage $n-1$, he is also willing to wait at stage n assuming a good signal is received.

Examples of $w_n^U\left(p_n^U\right)$:

Figure 1 shows two examples of $w_n^U\left(p_n^U\right)$ for $N = 2$.

The utility function takes the form $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $v(c) = \frac{A\left[(c+b)^{1-\gamma} - b^{1-\gamma}\right]}{1-\gamma}$, $A = 1$, $b = 0.001$[2], $\gamma = 1.01$.

---

[2] b can not equal zero because $u(c=0, b=0) = -\infty$. Any contract which has a positive probability of bank failure yield $-\infty$ ex ante. A maximization problem is not defined.

The risky production is as follows: $\overline{R} = 1.5$, $\underline{R} = 1$, $p = 0.999$.

The accuracy of the signal: $q = 0.9999$.

The proportion of impatient consumers: $\alpha = 0.01$.

In figure 1-1, $c^1 = 1$. $\overline{u}_2 = 7.5572$, $\underline{u}_2 = u_1 = v_1 = 7.1529$.

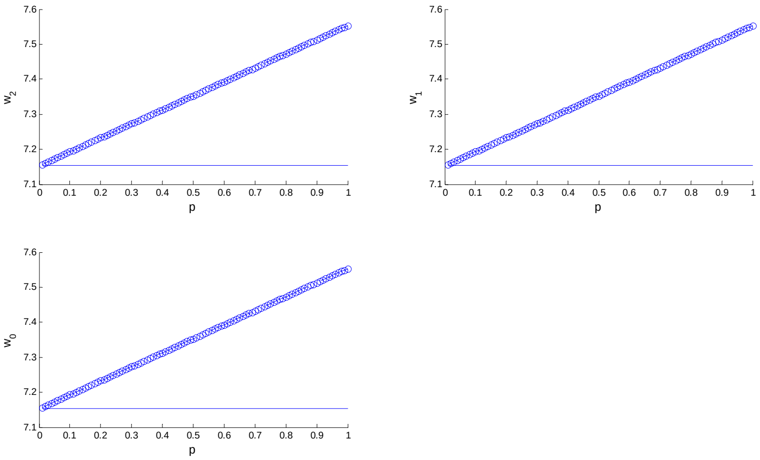

Figure 1-1: An example of $w_2(p)$, $w_1(p)$ and $w_0(p)$.
— u_1; _ _ $\alpha v_1 + (1-\alpha)\left[p_n \overline{u}_2 + (1-p_n)\underline{u}_2\right]$; ooo w_n.

In this example, $\widetilde{p}_0 = \widetilde{p}_1 = \widetilde{p}_2 = \hat{p}_0 = \hat{p}_1 = \hat{p}_2 = 0$.

Figure 1-2 continues the previous example with $c^1 = 1.04$. $\overline{u}_2 = 7.5568$, $\underline{u}_2 = 7.1525$, $u_1 = v_1 = 7.1921$.

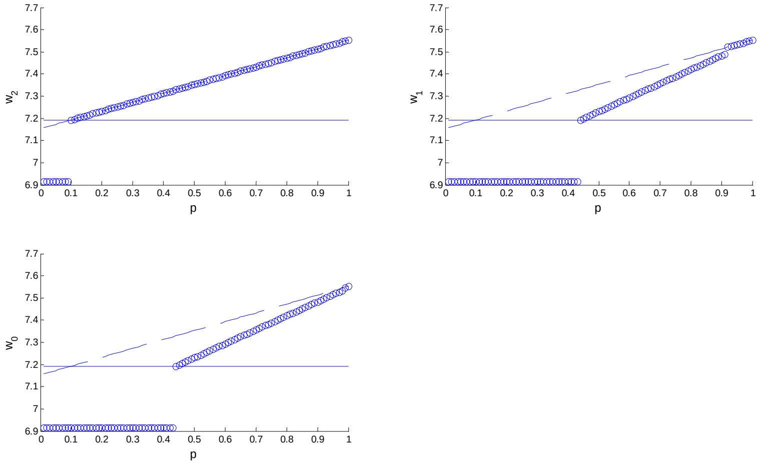

Figure 1-2: An example of $w_2(p)$, $w_1(p)$ and $w_0(p)$.
— u_1; _ _ $\alpha v_1 + (1-\alpha)\left[p_n \overline{u}_2 + (1-p_n)\underline{u}_2\right]$; ooo w_n.

In this example, $\widetilde{p}_0 = 0.43, \hat{p}_0 = 0.99$; $\widetilde{p}_1 = 0.43, \hat{p}_1 = 0.91$; $\widetilde{p}_2 = \hat{p}_2 = 0.09$.

Lemma 3 *If* $w_n^U(p) > \alpha v_1 + (1-\alpha) u_1$, $w_n^{I,S_r}(p) > u_1$.

Proof. see appendix. $\blacksquare$

The intuition behind lemma 3 is the following. Conditional on being impatient, a consumer prefers to withdraw immediately at any stage as the probability of getting $\frac{1}{c^1} v_1$ is positive. If an uninformed consumer is willing to wait, it must be true that conditional on being patient, the expected utility of waiting is strictly higher than that of withdrawing immediately. Therefore, given a prior of p_n^U, if an uninformed consumer is willing to wait at stage n, he is willing to wait as an informed consumer at the next stage if a high signal is received.

Lemma 4 *If* $w_{n-1}^U(p) > \alpha v_1 + (1-\alpha) u_1$ *and* $p_n^U = P_H(p)$, $w_n^{I,S_n}(P_H(p)) > u_1$.

Proof. see appendix. $\blacksquare$

Proposition 1 *In the post deposit game, the beliefs and strategies in* $(2) - (12)$ *constitute a perfect Bayesian equilibrium.*

Proof. The proof process is divided into several steps to facilitate reading.

Step 1: Check the beliefs.

By construction, the beliefs are updated by Bayes' rule whenever possible.

Step 2: Check the strategies of the uninformed consumers if no one deviates.

By construction, the expected payoff of the uninformed consumers at stage n is $w_n^U(p_n^U)$. If it is lower than or equal to $\alpha v_1 + (1-\alpha) u_1$, they should withdraw. Otherwise they should not.

Step 3: Check the strategies of the newly informed consumers if no one deviates.

For a newly informed consumer at stage n, $w_{n-1}^U(p_{n-1}^U) > \alpha v_1 + (1-\alpha) u_1$ at the stage before. If herd of non-withdrawals has begun already, that is, $p_n^U = p_{n-1}^U$ and $P_L(p_{n-1}^U) > \underline{p}$, the newly informed consumer's decision does not change the uninformed consumers' belief. If not, the uninformed consumers' belief will be updated by P_H or $P_{\widetilde{L}}$. The uninformed consumers' probability of getting a productivity signal is zero.

Now let us discuss cases by the signal the newly informed gets at the stage n and the prior at stage n or p_{n-1}^U.

(1) If a high signal is received, and $P_L(p_{n-1}^U) > \underline{p}$. The newly informed consumer's posterior belief is $P_H(p_{n-1}^U) > \underline{p}$. His decision will not change the uninformed or the previously informed consumers' decision as the herd of non-withdrawals has already begun, and because of which, he will not get more

information from the newly informed consumers' decision at later stages. According to his updated belief, the expected utility is $P_H\left(p^U_{n-1}\right)\overline{u}_2 + \left(1 - P_H\left(p^U_{n-1}\right)\right)\underline{u}_2 > u_1$. He should stay.

(2) If a high signal is received, and $P_L\left(p^U_{n-1}\right) \leq \underline{p}$. The newly informed consumer's belief is $P_H\left(p^U_{n-1}\right)$. If he waits, the uninformed consumers' belief will also be $p^U_n = P_H\left(p^U_{n-1}\right)$. By lemma 4, if the newly informed consumer stays, his expected utility at stage n is $w^{I,S_n}_n\left(P_H\left(p^U_{n-1}\right)\right) > u_1$. By lemma 2, the uninformed consumers will be staying too, as the expected utility is updated to $w_n\left(P_H\left(p^U_{n-1}\right)\right) > \alpha v_1 + (1-\alpha)u_1$. If the newly informed consumer withdraws, he gets c_1 immediately with the utility of u_1, but u_1 is lower than the expected utility if he stays.

(3) If the consumer gets a low signal, and $P_L\left(p^U_{n-1}\right) > \underline{p}$. His decision will not update the uninformed consumers' prior, and the uninformed consumer's expected utility will not be changed since $w_n\left(p^U_n\right) = w_{n-1}\left(p^U_{n-1}\right) = p^U_{n-1}\overline{u}_2 + \left(1 - p^U_{n-1}\right)\underline{u}_2$ for $p^U_n = p^U_{n-1}$. He will not be able to infer any more information from other consumers, and according to his belief, he should wait.

(4) If the consumer gets a low signal, and $P_L\left(p^U_{n-1}\right) \leq \underline{p}$. According to the strategies, he should withdraw and get u_1. Suppose not, he waits. Then the uninformed consumers' prior is updated to $p^U_n = P_H\left(p^U_{n-1}\right)$. From then on, $p^{I,S_n=L}_m = P^2_L\left(p^U_m\right)$ for $m \geq n$. If in the end, bank run happens at some stage m, his expected payoff is $\min\left\{\frac{1}{c^1}, 1\right\}u_1$, which is lower than or equal to u_1. If herd of non-withdrawals begins at a later stage $m < N$, the prior of the uninformed consumer at stage m satisfies $P_L\left(p^U_m\right) > \underline{p}$, but $P_L\left(p^U_{m-1}\right) \leq \underline{p}$. Otherwise, the herd of non-withdrawals could have begun earlier. Since $P_L(\cdot)$ is an increasing function, we have $p^U_{m-1} < p^U_m$ and $p^U_m = P_H\left(p^U_{m-1}\right)$. By $P_L\left(p^U_{m-1}\right) \leq \underline{p}$, we have $P^2_L\left(p^U_m\right) \leq \underline{p}$, which means $p^{I,S_n}_m \leq \underline{p}$. After the herd of non-withdrawals begins, there will be no chance of updating priors. Thus, at the stage that the herd of non-withdrawalss begins, the previously informed consumer's expected utility is still lower than u_1, and the expected utility will stay there from then on. This previously informed consumer will withdraw sooner or later. In the last case when herd of withdrawals or no withdrawal does not occur before stage N, it must be true that the uninformed's belief satisfies $P_L\left(p^U_N\right) \leq \underline{p}$. Therefore, an informed consumer weakly prefer to withdraw immediately if a low productivity signal is received.

Step 4: Check the previously informed consumers' strategies if no one deviates.

The active previously informed consumers are those who chose to wait. If they choose to wait before the herd of non-withdrawals begins, then the uninformed consumers' prior is updated in the same way as the informed consumer. That is, by $P_H\left(p^U_n\right)$. The previously informed consumers hold the same belief as the uninformed consumers, and can accurately predict when the bank fails. Additionaly, the uninformed can accurately predict when the previously informed withdraw. By lemma 4, the previously informed always wait if the uninformed wait given the same belief. If the uninformed wait because herd of non-withdrawals began before they got the signal, then they will be staying from that point on since no more updates on the belief are available.

Step 5: Check the strategies if there is a deviation.

Because the consumption types are private information, the deviations are undetectable unless more than 1 withdrawal is observed in a stage. In that case, the uninformed consumers' beliefs will be updated as if only one withdrawal is observed. By equilibrium strategies, no uninformed consumer should withdraw before $w_n\left(p_n^U\right)$ drops below $\alpha v_1 + (1-\alpha)\, u_1$.

Another plausible detectable deviation is as follows: the newly informed consumer should withdraw regardless of the signal. If he waits, the uninformed detect the deviation. If this is the case, it must be true that at stage n, $P_H\left(p_{n-1}^U\right) \leq \underline{p}$ or $p_{n-1}^U \leq P_L\left(\underline{p}\right) < \underline{p}$. However, since $\tilde{p}_{n-1} \geq \underline{p}$, the uninformed consumers must have all withdrawn from the bank already. $\blacksquare$

The herd of non-withdrawals happens before stage $N+1$ if $P_L\left(p_n^U\right) > \underline{p}$, or $p_n^U > P_H\left(\underline{p}\right) = \overline{p}$. At the end of stage N, if $p_N^U > \widetilde{p}_N = \underline{p}$, the uninformed and the previously informed consumers will wait at stage $N+1$ if they are patient. So a herd of no witdrawal occurs at stage $N+1$, if $p_N^U > \underline{p}$. Similarly, herd of withdrawals occurs at stage n if $p_n^U \leq \widetilde{p}_n$.

The equilibrium has the following outcome: the informed consumers follow the productivity signals until herd of withdrawals or herd of non-withdrawals begins. The uninformed watch the queue. If too many consumers line up in the queue, the uninformed go to the bank all at once, and the bank fails. If not many people are in the line, the uninformed will wait.

The initial condition that the uninformed consumers are willing to wait before any information is revealed is that $w_0\left(p_0\right) > \alpha v_1 + (1-\alpha)\, u_1$. Otherwise, no consumer is willing to wait at the beginning of stage 1, and there is no reason for a bank to provide a contract that induces people to withdraw right after they deposit. Herds of withdrawals and no withdrawal take place depending on the random process of the signal realizations. If the demand deposit contract offers a very low c^1 such that even though the first informed consumer gets a low signal, the expected utility is not below the threshold, then the first consumer does not withdraw, and nobody expects to infer any information the decision of the first informed consumer. The second informed consumer faces the same problem and makes the same decision. Therefore, the contract is run-proof. But yet suppose such a contract is not provided. The first informed consumer withdraws if a low signal is received but waits otherwise according to the equilibrium. By not getting in the line, the first consumer fully reveals that he gets a high signal. Due to the fact that the signals are of the same quality, the second consumer's belief if he receives a low signal is updated back to p_0. Assume that $p_0\overline{u}_2 + (1-p_0)\,\underline{u}_2 > u_1$. The second consumer will not withdraw even a low signal is received. Other consumers can not infer more information after the first consumer makes the decision not to withdraw. Hence, the herd of non-withdrawals starts. It is not guaranteed that $p_0\overline{u}_2 + (1 - p_0)\underline{u}_2 > u_1$, though $w_0(p_0) > \alpha v_1 + (1 - \alpha) u_1$. However, if this is the case, after the first consumer stays, herd of non-withdrawals starts.

By proposition 1, there are sequences of the cutoff beliefs for the uninformed and the newly informed consumers. The uninformed consumers' cutoff beliefs are $(\tilde{p}_0, \tilde{p}_1, \tilde{p}_2 ..., \tilde{p}_N)$, and the newly informed consumers' are $(\underline{p}, \underline{p}, \underline{p}, ..., \underline{p})$. On the equilibrium path, the previously informed consumers have the same belief as the uninformed, and they withdraw if the uninformed withdraw. Thus, the threshold beliefs of the previously informed consumers are the same sequence as the uninformed's. Also note that $\tilde{p}_n$ is decreasing in n, and $\tilde{p}_N = \underline{p}$.

The equilibrium proved in proposition 1 is not unique. For example, there can be equilibria in which at the first few stages, the informed consumers do not withdraw according to the productivity signals though $p_n < \underline{p}$. But from stage m $(1 < m \leq N)$ on, consumers adopt the strategies described in proposition 1. Because $w_n^U(p_n^U)$ changes with the strategies adopted, it remains difficult to exhaust all the possible equilibria. However, as the purpose of this paper is to illustrate how people make withdrawal decisions while observing the withdrawal history, the assumption is that consumers only play the equilibrium strategies in proposition 1 in the post deposit game.

2.3 Predeposit Game

Once the equilibrium in the post deposit game is proved, the optimal contract can be calculated since the outcome of any contract can be described by the equilibrium. Questions remain are whether the optimal contract allows for herd of withdrawals, and if so, whether it is individually rational, or it is accepted by the consumers ex ante. Although the equilibrium in the post deposit game is proved, the stage of which herd of withdrawals occurs depends not only on the parameters, but also on the random process of which the signals are sent. For some realizations of the random process, herd of withdrawals does not happen before the last stage, yet all consumers withdraw at the last stage until all the available information on the productivity is revealed. In either case, the bank fails because it is unable to meet the payment requirements. Calculating the probability of bank run is messy. In this section, a pre-deposit game for $N = 2$ is calculated, and the following two questions are answered: (1) knowing the probability of bank run in any possible situation, what is the optimal demand deposit contract in the simple case? (2) Is that contract better than autarky?

In autarky, consumers invest in the production individually. Suppose the investing in the production and the recall of the investment are not observable. In such a case, a consumer can not infer any information by watching the decision of others. An individual consumer's expected utility is

$$\alpha v(1) + (1-\alpha)\left[p_0 u\left(\overline{R}\right) + (1-p_0)\, u(1)\right].$$

In the static bank run model, a feasible contract should at least satisfy the participation incentive compatibility constraint, which says given all other patient consumers do not withdraw the deposits, an individual patient consumer prefers to wait. In the dynamic setup, bank run could happen at any stage of the game, but a feasible contract should at least give consumers the incentive to wait before anyone gets a signal. The participation incentive compatibility constraint is

$$w_0(p_0) > \alpha v_1 + (1-\alpha)\, u_1. \tag{13}$$

The participation incentive compatibility constraint in the traditional Diamond-Dybvig model is a special case here with $N = 0$ and $p = 1$.

A run-proof contract in our model falls into one of the two cases:

(1)

$$P_L(p_0)\, \overline{u}_2 + (1 - P_L(p_0))\, \underline{u}_2 > u_1. \tag{14}$$

That is, the uninformed consumers never update beliefs, and the informed consumers do not withdraw deposits though a low signal is received. Note that a run proof contract in the traditional Diamond-Dybvig set-up satisfying $c^1 \leq 1$ is also a run-proof contract here.

(2)

$$P_L(p_0)\, \overline{u}_2 + (1 - P_L(p_0))\, \underline{u}_2 \leq u_1 \tag{15}$$

and

$$P_{\tilde{L}}^{n}(p_0) > \tilde{p}_n \ \forall\ 0 \leq n \leq N. \tag{16}$$

That is, the consumers keep updating beliefs if low signals are received. However, even if every informed consumer lines up, an uninformed consumer's belief will not be below the cutoff value. By proposition 1, since $\tilde{p}_n$ is decreasing in n, we only need $P_{\tilde{L}}^{N}(p_0) > \tilde{p}_N = \underline{p}$. So (16) can be simplified to

$$\begin{aligned} P_{\tilde{L}}^{N}(p_0) &> \underline{p} \text{ or} \\ P_{\tilde{L}}^{N}(p_0)\, \overline{u}_2 + \left(1 - P_{\tilde{L}}^{N}(p_0)\right) \underline{u}_2 &> u_1 \end{aligned} \tag{16'}$$

In the first case, a run-proof contract provides low payment in period 1, making the payment in period 2 so high that the expected utility from $P_L(p_0)$ is higher than u_1. The payments to the first period withdrawals in the second case is higher than that in the first case. However, if there are too few number of informative stages, the belief of the uninformed consumers will not be lowered below the threshold. Therefore, even though there is a line in front of the bank, the rest of the consumers are still willing to wait.

For the run-proof contract, $w_0(p_0) = \alpha v_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\underline{u}_2\right].$ The best run-proof contract solves

$$\begin{aligned} &\max_{c^1} w_0(p_0) = \alpha v_1 + (1-\alpha)\left[p_0 \overline{u}_2 + (1-p_0)\underline{u}_2\right] \\ &s.t. \quad (13) \text{ and } (14) \qquad \text{or} \\ &\qquad\quad (13), (15)-(16) \end{aligned}$$

For a run-admitting contract, whether bank run occurs depends on the realizations of the signals. The probability of bank run can be calculated by listing all the possible paths of the random processes given the banking contract. Here I calculate the probability of each possible case for $N = 2$.

A Case of $N = 2$

A run-admitting contract should at least satisfy (13) and the following.

$$P_{\widetilde{L}}^2(p_0)\overline{u}_2 + \left(1 - P_{\widetilde{L}}^2(p_0)\right)\underline{u}_2 \quad \leq \quad u_1, \tag{17}$$

$$P_H^2(p_0)\overline{u}_2 + \left(1 - P_H^2(p_0)\right)\underline{u}_2 \quad > \quad u_1, \tag{18}$$

which imply

$$\begin{aligned} w_2\left(P_{\widetilde{L}}^2(p_0)\right) &\leq \alpha v_1 + (1-\alpha) u_1 \text{ and} \\ w_2\left(P_H^2(p_0)\right) &> \alpha v_1 + (1-\alpha) u_1. \end{aligned}$$

The feasible contract also implies $w_1(P_H(p_0)) > \alpha v_1 + (1-\alpha) u_1$. However, the form of $w_1(P_H(p_0))$ depends on other parameters, making it difficult to write it out generally. I list the conditions for all of the possible outcomes after each newly informed consumer's decision is observed first.

1. Herd of withdrawals occurs if the first informed consumer withdraws.

$$w_1\left(P_{\widetilde{L}}(p_0)\right) \leq \alpha v_1 + (1-\alpha) u_1$$

2. herd of non-withdrawals occurs if the first informed consumer wait.

$$\begin{aligned} P_L P_H(p_0)\overline{u}_2 + (1 - P_L P_H(p_0))\underline{u}_2 \quad &= \\ p_0 \overline{u}_2 + (1-p_0)\underline{u}_2 \quad &> \quad u_1. \end{aligned} \tag{19}$$

3. Herd of withdrawals does not occur if the first informed consumer withdraws. The second consumer follows the signal as $P_L P_{\widetilde{L}}(p_0)\overline{u}_2 + \left(1 - P_L P_{\widetilde{L}}(p_0)\right)\underline{u}_2 \leq u_1$, guaranteed by (15). The uninformed

consumers withdraw if the second consumer withdraws, and they wait if the second consumer waits.

$$\begin{aligned} w_1\left(P_{\widetilde{L}}\left(p_0\right)\right) &> \alpha v_1+(1-\alpha)\, u_1 \\ w_2\left(P_H P_{\widetilde{L}}\left(p_0\right)\right) &= \alpha v_1+(1-\alpha)\left[P_H P_{\widetilde{L}}\left(p_0\right) \overline{u}_2+\left(1-P_H P_{\widetilde{L}}\left(p_0\right)\right) \underline{u}_2\right] \\ &> \alpha v_1+(1-\alpha)\, u_1 \qquad (20) \end{aligned}$$

4. herd of non-withdrawals does not occur if the first informed consumer waits. The second consumer follows the signal. The uninformed consumers withdraw if the second consumer withdraws, and they wait if the second consumer waits.

$$\begin{aligned} p_0 \overline{u}_2+\left(1-p_0\right) \underline{u}_2 &\leq u_1 \qquad (21) \\ w_2\left(P_{\widetilde{L}} P_H\left(p_0\right)\right) &\leq \alpha v_1+(1-\alpha)\, u_1 \end{aligned}$$

which implies

$$\alpha v_1+(1-\alpha)\left[P_H P_{\widetilde{L}}\left(p_0\right) \overline{u}_2+\left(1-P_H P_{\widetilde{L}}\left(p_0\right)\right) \underline{u}_2\right] \leq \alpha v_1+(1-\alpha)\, u_1 \qquad (22)$$

5. herd of non-withdrawals does not occur if the first informed consumer waits. The second consumer follows the signal. The uninformed wait regardless of the second consumer's decision. i.e. $(20)-(21)$.

The combinations of the above 5 constitute descriptions of equilibrium given the contract.

1. Combine 1 and 2. If the first informed consumer does not withdraw, herd of non-withdrawals begins. If the first informed consumer withdraws, herd of withdrawals begins.

The probability of bank run is

$$\sigma_1=1-\pi\left(p_0\right).$$

The conditions for the outcome are $(13),(17)-(19)$ and

$$w_1\left(P_{\widetilde{L}}\left(p_0\right)\right) \leq \alpha v_1+(1-\alpha)\, u_1$$

which requires

$$\begin{aligned} &\pi\left(P_{\widetilde{L}}\left(p_0\right)\right)\left\{\alpha v_1+(1-\alpha)\left[P_H P_{\widetilde{L}}\left(p_0\right) \overline{u}_2+\left(1-P_H P_{\widetilde{L}}\left(p_0\right)\right) \underline{u}_2\right]\right\}+ \\ &+\left(1-\pi\left(P_{\widetilde{L}}\left(p_0\right)\right)\right)\left[\min \left\{\tfrac{1}{c^1}, 1\right\}\left(\alpha v_1+(1-\alpha)\, u_1\right)\right]<\alpha v_1+(1-\alpha)\, u_1 \end{aligned} \qquad (23)$$

$w_2\left(P_H P_{\widetilde{L}}\left(p_0\right)\right)=\alpha v_1+(1-\alpha)\left[P_H P_{\widetilde{L}}\left(p_0\right) \overline{u}_2+\left(1-P_H P_{\widetilde{L}}\left(p_0\right)\right) \underline{u}_2\right]>\alpha v_1+(1-\alpha)\, u_1$ is guaranteed by (19).

The participation incentive constraint:

$$w_0(p_0) = \pi(p_0)\, w_1(P_H(p_0)) + (1-\pi(p_0))\, w_1\left(P_{\widetilde{L}}(p_0)\right) > \alpha v_1 + (1-\alpha)\, u_1 \tag{24}$$

where

$$w_1(P_H(p_0)) = \alpha v_1 + (1-\alpha)\left[P_H(p_0)\,\overline{u}_2 + (1-P_H(p_0))\,\underline{u}_2\right] > \alpha v_1 + (1-\alpha)\, u_1.$$

is guaranteed by (19).

The ex ante expected utility maximization problem is

$$\begin{aligned}&\max_{c^1} w_0(p_0)\\ &s.t.\,(13),\ \ (17)-(19),\ \ (23)-(24).\end{aligned}$$

2. Combine 2 and 3. If the first informed consumer waits, herd of non-withdrawals begins. If the first informed consumer withdraws and the second also withdraws, then bank run happens. If the first withdraws and the second waits, the uninformed wait.

The probability of bank run is

$$\sigma_2 = (1-\pi(p_0))\left(1-\pi\left(P_{\widetilde{L}}(p_0)\right)\right).$$

The conditions for the outcome are (13), $(17)-(19)$ and

$$\begin{aligned} w_1\left(P_{\widetilde{L}}(p_0)\right) &= \pi\left(P_{\widetilde{L}}(p_0)\right)\left\{\alpha v_1 + (1-\alpha)\left[P_H P_{\widetilde{L}}(p_0)\,\overline{u}_2 + \left(1-P_H P_{\widetilde{L}}(p_0)\right)\underline{u}_2\right]\right\} + \\ &\quad + \left(1-\pi\left(P_{\widetilde{L}}(p_0)\right)\right)\left[\min\left\{\tfrac{1}{c^1},1\right\}\left(\alpha v_1 + (1-\alpha)\, u_1\right)\right] \\ &> \alpha v_1 + (1-\alpha)\, u_1 \end{aligned} \tag{25}$$

The participation incentive constraint:

$$w_0(p_0) = \pi(p_0)\, w_1(P_H(p_0)) + (1-\pi(p_0))\, w_1\left(P_{\widetilde{L}}(p_0)\right) > u_1 \tag{26}$$

where

$$\begin{aligned} w_1(P_H(p_0)) &= P_H(p_0)\left(\alpha v_1 + (1-\alpha)\,\overline{u}_2\right) + (1-P_H(p_0))\left(\alpha v_1 + (1-\alpha)\,\underline{u}_2\right) \\ &> \alpha v_1 + (1-\alpha)\, u_1. \end{aligned}$$

is guaranteed by (19).

The ex ante expected utility maximization problem is

$$\begin{aligned}&\max_{c^1} w_0(p_0)\\ &s.t.\,(13),(17)-(19),(25)-(26).\end{aligned}$$

3. Combine 1 and 4. Herd of withdrawals starts if the first informed consumer withdraws. If the first informed consumer waits, then the second consumer still follows the signal. The uninformed consumers wait if the second consumer waits, and they withdraw if the second consumer withdraws.

The probability of bank run is

$$\sigma_3 = 1 - \pi(p_0) + \pi(p_0)(1 - \pi(P_H(p_0)))$$

The conditions for the outcome are (13), $(17)-(18)$, and $(21)-(22)$.

The participation incentive constraint:

$$w_0(p_0) = \pi(p_0)\, w_1(P_H(p_0)) + (1-\pi(p_0))\, w_1\left(P_{\widetilde{L}}(p_0)\right) > \alpha v_1 + (1-\alpha) u_1 \tag{27}$$

where

$$\begin{aligned} w_1(P_H(p_0)) &= \pi(P_H(p_0))\left\{\alpha v_1 + (1-\alpha)\left[P_H^2(p_0)\,\overline{u}_2 + \left(1-P_H^2(p_0)\right)\underline{u}_2\right]\right\} + \\ &\quad + (1-\pi(P_H(p_0)))\left[\min\left\{\tfrac{1}{c^1},1\right\}(\alpha v_1 + (1-\alpha)u_1)\right] \\ &> \alpha v_1 + (1-\alpha)u_1 \end{aligned} \tag{28}$$

and

$$w_1\left(P_{\widetilde{L}}(p_0)\right) = \min\left\{\tfrac{1}{c^1},1\right\}(\alpha v_1 + (1-\alpha)u_1)$$

is guaranteed by (22).

The ex ante expected utility maximization problem is

$$\begin{aligned} &\max_{c^1} w_0(p_0) \\ &s.t.\ (13), (17)-(18), (21)-(22), (27)-(28). \end{aligned}$$

4. Combine 1 and 5. Herd of withdrawals starts if the first informed consumer withdraws. If the first informed consumer waits, then the second consumer still follows the signal, but the uninformed do not withdraw regardless of the second consumer's decision.

The probability of bank run is σ_1.

The conditions for the outcome are $(13), (17)-(18), (21)-(22)$ and (24).

The participation incentive constraint:

$$w_0(p_0) = \pi(p_0)\, w_1(P_H(p_0)) + (1-\pi(p_0))\, w_1\left(P_{\widetilde{L}}(p_0)\right) > \alpha v_1 + (1-\alpha) u_1 \tag{29}$$

where

$$w_1\left(P_{\widetilde{L}}(p)\right) = \min\left\{\tfrac{1}{c^1},1\right\}(\alpha v_1 + (1-\alpha)u_1)$$

is implied by (24), and

$$w_1\left(P_H\left(p_0\right)\right)=\alpha v_1+(1-\alpha)\left[P_H\left(p_0\right)\overline{u}_2+\left(1-P_H\left(p_0\right)\right)\underline{u}_2\right]>\alpha v_1+(1-\alpha)\,u_1$$

is guaranteed by (21).

The ex ante expected utility maximization problem is

$$\begin{aligned}&\max_{c^1} w_0\left(p_0\right)\\&s.t.\,(13),(17)-(18),(21)-(22),(24),(29).\end{aligned}$$

5. Combine 3 and 5. herd of non-withdrawals does not occur if the first informed consumer waits. The second consumer still follows the signal, but the uninformed do not withdraw regardless the second consumer's decision. Herd of withdrawals does not occur after the first consumer withdraws. If the first and the second informed consumers withdraw, then bank run happens.

The probability of bank run is σ_2.

The conditions are $(13),(17)-(18),(20)-(21)$, and (25).

The participation incentive constraint:

$$w_0\left(p_0\right)=\pi\left(p_0\right)w_1\left(P_H\left(p_0\right)\right)+\left(1-\pi\left(p_0\right)\right)w_1\left(P_{\widetilde{L}}\left(p_0\right)\right)>\alpha v_1+(1-\alpha)\,u_1 \tag{30}$$

where

$$w_1\left(P_H\left(p_0\right)\right)=\alpha v_1+(1-\alpha)\left[P_H\left(p_0\right)\overline{u}_2+\left(1-P_H\left(p_0\right)\right)\underline{u}_2\right]>\alpha v_1+(1-\alpha)\,u_1$$

by (20) and

$$\begin{aligned}w_1\left(P_{\widetilde{L}}\left(p_0\right)\right) &= \pi\left(P_{\widetilde{L}}\left(p_0\right)\right)\left\{\alpha v_1+(1-\alpha)\left[P_H P_{\widetilde{L}}\left(p_0\right)\overline{u}_2+\left(1-P_H P_{\widetilde{L}}\left(p_0\right)\right)\underline{u}_2\right]\right\}+\\&\quad+\left(1-\pi\left(P_{\widetilde{L}}\left(p_0\right)\right)\right)\left[\min\left\{\tfrac{1}{c^1},1\right\}\left(\alpha v_1+(1-\alpha)\,u_1\right)\right]\\&> \alpha v_1+(1-\alpha)\,u_1\end{aligned}$$

by (25).

The ex ante expected utility maximization problem is

$$\begin{aligned}&\max_{c^1} w_0\left(p_0\right)\\&s.t.\,(13),(17)-(18),(20)-(21),(25),(30).\end{aligned}$$

An example

An example of $N=2$ will be shown here.

The utility functions take the form $u\left(c\right)=\frac{\left(c+b\right)^{1-\gamma}-b^{1-\gamma}}{1-\gamma}$, $v\left(c\right)=\frac{A\left[\left(c+b\right)^{1-\gamma}-b^{1-\gamma}\right]}{1-\gamma}$, $A=10$, $b=0.001$, $\gamma=1.01$.

The risky production is as follows: $\overline{R}=1.05$, $\underline{R}=1$, $p=0.99$.

The accuracy of the signal: $q=0.9$.

The proportion of impatient consumers: $\alpha=0.05$.

The calculated $\sigma_1=0.1169$, $\sigma_2=0.0199$, and $\sigma_3=0.2140$.

The expected utility for autarky, best run-proof contract, and run-admitting contract in each of the five cases are listed in table 1.

	σ	c^1	$w_0\left(p_0\right)$
Autarky	0	1.0000	7.8445
Best run-proof contract in case 1	0	1.0121	7.8455
Best run-proof contract in case 2	0	1.0290	7.8470
Best run-admitting contract in case 1	0.1169	1.0431	7.8426
Best run-admitting contract in case 2	0.0199	1.0399*	7.8473*
Best run-admitting contract in case 3	0.2140	N.A.	N.A.
Best run-admitting contract in case 4	0.1169	N.A.	N.A.
Best run-admitting contract in case 5	0.0199	N.A.	N.A.

Table 1: An Example of the Optimal Contract

A run-proof contract is not the best in this case for two reasons. First, since c^1 is low, a run-proof contract does not provide enough liquidity if a consumer turns out to be impatient. Consumption is not smoothed enough. Second, a run-proof contract does not induce consumers to reveal the signals they received, hence the economy can not benefit from the available information on productivity. In the model economy, if the productivity is in the low state, liquidating assets in period 1 or 2 does not make any difference. However, the expected utility updated by a high signal increases as the probability of getting high productivity is raised. On the other hand, since the productivity signal is imperfect and the inference of a low signal is imperfect, there is a possibility that the herd of withdrawals occurs due to the wrong signals or due to too many impatient consumers receiving signals in the first few signals. However, the benefit from obtaining correct signals to liquidate assets overcomes the loss from the bank run. The run-admitting contract in case 1 is undesirable, as the probability of bank run is too high.

The probability of bank run is increasing in c^1 for two reasons. First, if c^1 is high, it is more tempting to withdraw for both uninformed and newly informed consumers. Second, since the newly informed consumer is more likely to withdraw if a low signal is received, the uninformed lower the belief if the newly informed withdraws, making withdrawals from the bank more attractive. However, a higher c^1 helps smooth consumption across types. An optimal contract balances consumption smoothness and the probability of bank run.

Literature on herd behavior notes that herd is not always socially optimal. If signals are public, then the statistics are more accurate, and people can make better choice. The information released after herd begins is not observable and is not used. The statistics based on a few signals has lower quality. However, given the features of the economy, collecting productivity information before giving out payments is not feasible because the impatient need to consume right away. When a contract is designed, it also takes into account how the information is used efficiently. A lower c^1 takes more stages to finally achieves herds, but diminishes the function of the bank as a liquidity insurance provider.

As noted in the literature on fundamental driven bank runs, suspension of convertibility can not prevent the patient consumers to withdraw if there is information on weak fundamentals. It is true in my paper too. The patient consumers want to withdraw because the next period expected utility is below the payoff if they withdraw immediately. However, the sequential service constraint is crucial. So far, the bank is restricted to provide a demand deposit contract that prevails in the real banking sector. The bank, which updates the information on productivity as an uninformed consumer, can offer a contract contingent on its updated beliefs. At least it can allocate 1 unit of consumption goods to each consumer when bank run is predicted at stage $N + 1$. Furthermore, the bank might want to provide a contract that would induce the informed consumers to reveal the signal by changing the payments at each stage. The optimal contract in broader banking mechanisms is generally hard to compute. The question remains: in a broader class of banking mechanisms, is the run proof contract the optimal, or does it allow the consumers to follow the signals? The following section discusses a two-consumer, two-stage model that allows the bank to provide a contract depending on the observation of the withdrawals. In some economies, a run-admitting contract that allows queueing in front of the bank is optimal and ex ante acceptable.

3 A More General Banking Mechanism with Two Consumers

This section provides a model of two consumers and two informative stages to illustrate that the perfect Bayesian equilibrium in the previous section exists within a broad class of banking mechanisms. By a broad class of banking mechanism, I mean a banking mechanism that allows for the bank to give contingent payments on withdrawal history.

A deadline is still imposed here. Therefore, the first consumer can not change his decision after he observes the second consumer's decision. The second consumer has no motivation to withdraw before he learns the signal, as he is the last consumer at $t = 1$. But, if the second consumer withdraws regardless of the signal he gets, it is equivalent to the herd of withdrawals in the previous sections. The sequence of timing is the same as in the previous sections.

To simplify the calculation, let $u(c) = v(c)$ for $c \geq 0$. To make things more interesting, let $\underline{R} < 1$. The bank offers a contract specifying the payments to withdrawals at each stage at $t = 1$ depending on the number of withdrawals that have been made, and payments to withdrawals at $t = 2$ depending on the number of withdrawals at $t = 1$ and the realization of the production. Let $u(c_1(1))$, $u(c_1(0,1))$ and $u(c_1(1,1))$ denote the payments at stage 1, at stage 2 if consumer 1 does not withdraw, and at stage 2 if consumer 1 withdraws, respectively. Let $u(c_2(R, x_1, x_2))$, $x_i = 0$ or 1, $i = 1, 2$, denote the payment at $t = 2$. Note that $u(c_2(R, x_1, x_2))$ is increasing in R.

Because the bank's contract is contingent on the number of withdrawals made, the bank always meets the payment demands. If a consumer is paid less than another at the same period, the bank does not default on the payments, since the situation is described in the contract. For such a contract, bank run happens if the consumers withdraw the deposits but do not need to consume immediately. The consumers prefer to withdraw and put the money in a safety vault at home. If a run-proof contract is provided, consumers do not withdraw unless they are impatient.

Because the investment in production is rigid but can be recalled at any time in period 1, the bank first invests all deposits in the production and adjusts the portfolio at the end of stage 2 with the productivity information inferred from the withdrawal decisions of the consumers. I will first assume a banking contract and describe the equilibrium. Then, I will check the contract given consumers' strategies. The equilibrium involves three players: the bank, consumer 1 and consumer 2. I start with the post deposit game, then I check the expected utilities in autarky and with a bank to complete the predeposit game.

3.1 Post Deposit Game

When the consumers get the signal, they first update the belief of productivity by Bayes' rule as follows:

$$p_n^{I,S_n} = \begin{cases} P_L(p_{n-1}^U), & \text{if } S_n = L; \\ P_H(p_{n-1}^U), & \text{if } S_n = H. \end{cases} \tag{31}$$

$n = 1, 2$.

Using backward induction, at stage 2, the second consumer compares the utility from withdrawing deposits immediately with the utility from waiting. In other words, he compares $u(c_2(R, x_1, 0))$ with

$u\left(c_1\left(x_1,1\right)\right)$. Consumer 2's expected utility, if he is patient at stage 2, and depending on his belief and on whether the first consumer withdraws or not, is

$$w_2\left(p_2^{I,S_2},x_1\right)=\max\left\{p_2^{I,S_2}u\left(c_2\left(\overline{R},x_1,0\right)\right)+\left(1-p_2^{I,S_2}\right)u\left(c_2\left(\underline{R},x_1,0\right)\right),u\left(c_1\left(x_1,1\right)\right)\right\} \tag{32}$$

$w_2\left(p_2^{I,S_2},x_1\right)$ is strictly increasing in p_2^{I,S_2}. Denote consumer 2's cutoff belief given x_1 by $\widetilde{p}_{2,x_1}$. $\widetilde{p}_{2,x_1}$ satisfies

$$w_2\left(p_2^{I,S_2},x_1\right)\begin{cases} >u\left(c_1\left(x_1,1\right)\right), & \text{if } p_2^{I,S_2}>\widetilde{p}_{2,x_2};\\ =u\left(c_1\left(x_1,1\right)\right), & \text{if } p_2^{I,S_2}\leq\widetilde{p}_{2,x_2}.\end{cases} \tag{33}$$

Consumer 2's strategy is

$$x_2=\begin{cases} 1, & \text{if impatient or } p_2^{I,S_2}\leq\widetilde{p}_{2,x_2};\\ 0, & \text{otherwise.}\end{cases} \tag{34}$$

Knowing consumer 2's strategies and the signal he will get, consumer 1 knows how consumer 2's decision corresponds to his decision. Suppose $w_2\left(p_2^{I,H},0\right)>u\left(c_1\left(0,1\right)\right)\geq w_2\left(p_2^{I,L},0\right)$. Consumer 2 reacts differently to high and low signals if consumer 1 waits. Consumer 1 infers some information from consumer 2's decision, although he has no chance to reverse the decision. In other situations, i.e. $w_2\left(p_2^{I,H},0\right)>w_2\left(p_2^{I,L},0\right)>u\left(c_1\left(0,1\right)\right)$ or $u\left(c_1\left(0,1\right)\right)\geq w_2\left(p_2^{I,H},0\right)>w_2\left(p_2^{I,L},0\right)$, consumer 2's decision is invariant to the signals but may still depend on the consumption type. When 1 makes a decision, he will take 2's possible reaction into account and compare the expected utility from waiting to the utility of getting paid right now.

Consumer 1's expected utility if he is patient is as follows:

(1) if $w_2\left(p_2^{I,H},0\right)>u\left(c_1\left(0,1\right)\right)\geq w_2\left(p_2^{I,L},0\right)$,

$$\begin{aligned} w_1\left(p_1^{I,S_1}\right) &= \max\{\pi\left(P_{S_1}\left(p_0\right)\right)\left[P_H P_{S_1}\left(p_0\right)u\left(c_2\left(\overline{R},0,0\right)\right)+\left(1-P_H P_{S_1}\left(p_0\right)\right)u\left(c_2\left(\underline{R},0,0\right)\right)\right] \\ &\quad +\left(1-\pi\left(P_{S_1}\left(p_0\right)\right)\right)\left[P_{\widetilde{L}}P_{S_1}\left(p_0\right)u\left(c_2\left(\overline{R},0,1\right)\right)+\left(1-P_{\widetilde{L}}P_{S_1}\left(p_0\right)\right)u\left(c_2\left(\underline{R},0,1\right)\right)\right], \\ &\quad u\left(c_1\left(1\right)\right)\}\end{aligned} \tag{35-1}$$

(2) if $w_2\left(p_2^{I,H},0\right)>w_2\left(p_2^{I,L},0\right)>u\left(c_1\left(0,1\right)\right)$,

$$\begin{aligned} w_1\left(p_1^{I,S_1}\right) &= \max\{(1-\alpha)\left[P_{S_1}\left(p_0\right)u\left(c_2\left(\overline{R},0,0\right)\right)+\left(1-P_{S_1}\left(p_0\right)\right)u\left(c_2\left(\underline{R},0,0\right)\right)\right]+ \\ &\quad +\alpha\left[P_{S_1}\left(p_0\right)u\left(c_2\left(\overline{R},0,1\right)\right)+\left(1-P_{S_1}\left(p_0\right)\right)u\left(c_2\left(\underline{R},0,1\right)\right)\right],u\left(c_1\left(1\right)\right)\}\end{aligned} \tag{35-2}$$

(3) if $u\left(c_1\left(0,1\right)\right)\geq w_2\left(p_2^{I,H},0\right)>w_2\left(p_2^{I,L},0\right)$,

$$w_1\left(p_1^{I,S_1}\right)=\max\{P_{S_1}\left(p_0\right)u\left(c_2\left(\overline{R},0,1\right)\right)+\left(1-P_{S_1}\left(p_0\right)\right)u\left(c_2\left(\underline{R},0,1\right)\right),u\left(c_1\left(1\right)\right)\} \tag{35-3}$$

$w_1\left(p_1^{I,S_1}\right)$ is also strictly increasing in p_1^{I,S_1} in each case. Given consumer 2's strategies, consumer 1's expected utility falls into one of the three cases. The cutoff beliefs for consumer 1 at stage 1, denoted by $\widetilde{p}_1$, is defined as

$$w_1\left(p_1^{I,S_1}\right)\begin{cases} > u\left(c_1\left(1\right)\right), & \text{if } p_1^{I,S_1} > \widetilde{p}_1; \\ = u\left(c_1\left(1\right)\right), & \text{if } p_1^{I,S_1} \leq \widetilde{p}_1. \end{cases} \tag{36}$$

Consumer 1's strategy is

$$x_1 = \begin{cases} 1, & \text{if impatient or } p_1^{I,S_1} \leq \widetilde{p}_1; \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

The belief of consumer 2 as an uninformed consumer at the end of stage 1 is

$$p_1^U = \begin{cases} P_{\widetilde{L}}(p_0), & \text{if } x_1 = 1,\ w_1\left(p_1^{I,L}\right) \leq u\left(c_1\left(1\right)\right) \text{ and } w_1\left(p_1^{I,H}\right) > u\left(c_1\left(1\right)\right); \\ P_H(p_0), & \text{if } x_1 = 0,\ w_1\left(p_1^{I,L}\right) \leq u\left(c_1\left(1\right)\right) \text{ and } w_1\left(p_1^{I,H}\right) > u\left(c_1\left(1\right)\right); \\ p \in [0,1], & \text{if } x_1 = 0 \text{ and } w_1\left(p_1^{I,H}\right) \leq u\left(c_1\left(1\right)\right); \\ p_0, & \text{otherwise.} \end{cases} \tag{38}$$

p is the off equilibrium path belief. p can be any number between 0 and 1. The bank's belief of productivity is updated as if the bank were an uninformed consumer. At the end of stage 2, the bank's belief is

$$p_2^U = \begin{cases} P_{\widetilde{L}}(p_1^U), & \text{if } x_2 = 1,\ w_2\left(p_2^{I,L}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right) \text{ and } w_2\left(p_2^{I,H}, x_1\right) > u\left(c_1\left(x_1, 1\right)\right); \\ P_H(p_1^U), & \text{if } x_2 = 0,\ w_2\left(p_2^{I,L}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right) \text{ and } w_2\left(p_2^{I,H}, x_1\right) > u\left(c_1\left(x_1, 1\right)\right); \\ p' \in [0,1], & \text{if } x_2 = 0 \text{ and } w_2\left(p_2^{I,H}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right); \\ p_1^U, & \text{otherwise.} \end{cases} \tag{39}$$

p' is the belief off the equilibrium path. The belief of a previously informed consumer, i.e. consumer 1's belief at the end of stage 2, is

$$p_2^{I,S_1} = \begin{cases} P_{\widetilde{L}}(p_1^{I,S_1}), & \text{if } x_2 = 1,\ w_2\left(p_2^{I,L}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right) \text{ and } w_2\left(p_2^{I,H}, x_1\right) > u\left(c_1\left(x_1, 1\right)\right); \\ P_H(p_1^{I,S_1}), & \text{if } x_2 = 0,\ w_2\left(p_2^{I,L}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right) \text{ and } w_2\left(p_2^{I,H}, x_1\right) > u\left(c_1\left(x_1, 1\right)\right); \\ p'' \in [0,1], & \text{if } x_2 = 0 \text{ and } w_2\left(p_2^{I,H}, x_1\right) \leq u\left(c_1\left(x_1, 1\right)\right); \\ p_1^{I,S_1}, & \text{otherwise.} \end{cases} \tag{40}$$

Similarly, p'' is the belief off the equilibrium path.

Before any information is revealed, consumers need the incentive to wait for the signals. Otherwise both consumers withdraw the deposits right after they deposit, and there would be no ground for the existence of a bank. Consumers have equal opportunity to receive the signal at stage 1. The initial expected utility, $w_0(p_0)$, should satisfy the condition that $w_0(p_0) > u(c_1(1))$. Let $\gamma(p) = pq + (1-p)(1-q)$. It is the probability of getting a high signal given the belief of p. Depending on the values of $w_1\left(p_1^{I,H}\right)$ and $w_1\left(p_1^{I,L}\right)$,

(1) if $w_1\left(p_1^{I,H}\right) > u(c_1(1)) \geq w_1\left(p_1^{I,L}\right)$,

$$\begin{aligned} w_0(p_0) &= \frac{1}{2}\left\{\pi(p_0) w_1\left(p_1^{I,H}\right) + (1-\pi(p_0)) w_1\left(p_1^{I,L}\right)\right\} + \\ &\quad \frac{1}{2}\{\pi(p_0)\alpha u(c_1(0,1)) + (1-\pi(p_0))\alpha u(c_1(1,1)) \\ &\quad +\pi(p_0)(1-\alpha)\left[\gamma(P_H(p_0)) w_2\left(P_H^2(p_0),0\right) + (1-\gamma(P_H(p_0))) w_2(p_0,0)\right] + \\ &\quad +(1-\pi(p_0))(1-\alpha)\left[\gamma\left(P_{\widetilde{L}}(p_0)\right) w_2\left(P_{\widetilde{L}} P_H(p_0),1\right) + \left(1-\gamma\left(P_{\widetilde{L}}(p_0)\right)\right) w_2\left(P_{\widetilde{L}} P_L(p_0),1\right)\right]\} \\ &> u(c_1(1)) \end{aligned} \tag{41-1}$$

(2) if $w_1\left(p_1^{I,H}\right) > w_1\left(p_1^{I,L}\right) > u(c_1(1))$,

$$\begin{aligned} w_0(p_0) &= \frac{1}{2}\left\{\alpha u(c_1(1)) + (1-\alpha)\left[\gamma(p_0) w_1\left(p_1^{I,H}\right) + (1-\gamma(p_0)) w_1\left(p_1^{I,L}\right)\right]\right\} + \\ &\quad \frac{1}{2}\{\alpha^2 u(c_1(1,1)) + (1-\alpha)\alpha u(c_1(0,1)) + \\ &\quad +\alpha(1-\alpha)[\gamma(p_0) w_2(P_H(p_0),1) + (1-\gamma(p_0)) w_2(P_L(p_0),0)] + \\ &\quad +(1-\alpha)^2[\gamma(p_0) w_2(P_H(p_0),1) + (1-\gamma(p_0)) w_2(P_L(p_0),1)]\} \\ &> u(c_1(1)) \end{aligned} \tag{41-2}$$

(3) if $u(c_1(1)) > w_1\left(p_1^{I,H}\right) > w_1\left(p_1^{I,L}\right)$,

$$\begin{aligned} w_0(p_0) &= \frac{1}{2}u(c_1(1)) + \frac{1}{2}\{\alpha u(c_1(1,1)) + (1-\alpha)[\gamma(p_0) w_2(P_H(p_0),1) + (1-\gamma(p_0)) w_2(P_L(p_0),1)]\} \\ &> u(c_1(1)) \end{aligned} \tag{41-3}$$

Proposition 2 *Given the bank's contract c, the beliefs and strategies in* $(31)-(41)$ *constitute a perfect Bayesian equilibrium in the post deposit game. The equilibrium is unique.*

Proof. The equilibrium is constructed by backward induction. The proof of existence is obvious and omitted here. Uniqueness is demonstrated by showing that the cutoff beliefs are unique at each node reached, which is true because of the monotonicity of w_1 and w_2. Therefore, the cutoff beliefs for each consumer exist at each stage above which the informed consumer waits, below which the informed withdraws. ■

The uniqueness of the equilibrium in the two-consumer, two-stage case is attributed to the facts that neither consumer can change their decision after the stage they are informed. In a more complicated model with more consumers and stages, the uniqueness is not guaranteed.

By looking at the description of the equilibrium, it is hard to tell the equilibrium outcomes immediately. We will look at the equilibrium by the strategies of the consumers case by case. Consumer 1's strategy in equilibrium is one of the following three. Let Θ_1 denote 1's set containing all the possible strategies.

$$\Theta_1 = \begin{cases} x_1 = 1 \text{ if impatient;} & x_1 = 0 \text{ otherwise.} \\ x_1 = 1 \text{ if impatient or } S_1 = L; & x_1 = 0 \text{ otherwise.} \\ x_1 = 1. & \end{cases}$$

Consumer 2's strategy set Θ_2 contains 9 possible strategies:

$$\Theta_2 = \begin{cases} x_2 = 1 \text{ if impatient;} & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } S_2 = L; & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } S_2 = L \text{ or } x_1 = 1; & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } S_2 = L \text{ or } x_1 = 0; & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } (S_2 = L \text{ and } x_1 = 1); & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } (S_2 = L \text{ and } x_1 = 0); & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } x_1 = 1; & x_2 = 0 \text{ otherwise.} \\ x_2 = 1 \text{ if impatient or } x_1 = 0; & x_2 = 0 \text{ otherwise.} \\ x_2 = 1. & \end{cases}$$

But given the contract, only one equilibrium outcome exists that consists of one strategy from each set. The equilibrium beliefs can be derived from consumers' strategies at each stage. There are $3 \times 9 = 27$ possible cases for a two-consumer, two-stage model. To find an optimal contract, we need to exhaust all possible contracts in each case. The optimal contract is "the best of the best" of the 27 cases.

In the dynamic setting, the contract that the bank offers ex ante encounters a credibility problem ex post. The bank offers a payment scheme c, which specifies 9 payments depending on a consumer's position in the line, number of withdrawals made, and the productivity. A consumer compares the payments, given different strategies, and chooses the strategy that brings the higher expected utility. When the first consumer withdraws, he gets $u(c_1(1))$, but if he waits, he gets expected utility, which is a function of $c_1(0,1)$ and some other variables. However, either $c_1(1)$ or $c_1(0,1)$ will be reached. On the path along $c_1(1)$, the second consumer either gets $c_1(1,1)$, if he withdraws immediately, or $c_2(R|1,0)$ if he waits until $t = 2$. Similarly, consumer 2 gets either $c_1(0,1)$ or $c_2(R|0,0)$ along the path of $c_1(0,1)$. Ex ante, a bank offers a contract that includes all four possible future withdrawal paths. At stage 1, after consumer 1 makes the decision, the bank faces two of the four paths. The bank can re-optimize the contract without considering the payments on the other two paths. A bank on the competitive market may only consider an ex ante optimal contract that offers a highest level of $w_0(p_0)$ and commit to it. But a bank that behaves like a central bank might want to re-optimize at each stage, thus the ex ante payment scheme might not be credible. I will discuss both the contracts with commitment and those without commitment in the following sections. The contracts without commitment are consistent with fewer equilibrium outcomes, or the set of the contracts without commitment is a subset of that with commitment.

Although there are 9 payments to be determined, the crucial variables are $c_1(1)$, $c_1(0,1)$, λ_{01}, λ_{10} and λ_{11}. Once those variables are calculated, all the 9 payments in the scheme are determined.

3.2 Bank Contracts with Commitment

An optimal bank contract with commitment maximizes the ex ante expected utility. Once c is announced, the bank does not re-optimize at stage 1 or 2. The contract maximizes $w_0(p_0)$ in each of the 27 cases subject to 6 constraints that define the case. For example, if $\{x_1 = 1$ if impatient or $S_1 = L$, $x_1 = 0$ otherwise; $x_2 = 1$ if impatient or ($S_2 = L$ and $x_1 = 1$), $x_2 = 0$ otherwise$\}$ are the equilibrium strategies, by proposition 2, we have $w_2\left(P_H P_{\widetilde{L}}(p_0), 1\right) > u(c_1(1,1))$, $w_2\left(P_H^2(p_0), 0\right) > u(c_1(0,1))$, $w_2\left(P_L P_{\widetilde{L}}(p_0), 1\right) < u(c_1(1,1))$, $w_2(p_0, 0) > u(c_1(0,1))$, $w_1(P_L(p_0)) < u(c_1(1))$ and $w_1(P_H(p_0)) > u(c_1(1))$. Given the constraints, the equilibrium beliefs in this case are

$$p_1^U = \begin{cases} P_{\widetilde{L}}(p_0), & \text{if } x_1 = 1; \\ P_H(p_0), & \text{if } x_1 = 0. \end{cases} \qquad p_2^U = \begin{cases} P_{\widetilde{L}}(p_1^U), & \text{if } x_2 = 1; \\ P_H(p_1^U), & \text{if } x_2 = 0. \end{cases}$$

$$p_n^{I,S_n} = \begin{cases} P_L(p_{n-1}^U), & \text{if } S_n = L; \\ P_H(p_{n-1}^U), & \text{if } S_n = H. \end{cases} \qquad p_2^{I,S_1} = \begin{cases} P_{\widetilde{L}}(p_1^{I,S_1}), & \text{if } x_2 = 1; \\ P_H(p_1^{I,S_1}), & \text{if } x_2 = 0. \end{cases}$$

The best contract in this case solves

$$
\begin{array}{rlr}
\max\limits_{c_1(1),c_1(0,1),\lambda_{01},\lambda_{10},\lambda_{00}} & w_0\left(p_0\right) & \\
s.t. & w_2\left(P_H P_{\tilde{L}}\left(p_0\right),1\right) > u\left(c_1\left(1,1\right)\right), & w_2\left(P_H^2\left(p_0\right),0\right) > u\left(c_1\left(0,1\right)\right), \\
& w_2\left(P_L P_{\tilde{L}}\left(p_0\right),1\right) \leq u\left(c_1\left(1,1\right)\right), & w_2\left(p_0,0\right) > u\left(c_1\left(0,1\right)\right), \\
& w_1\left(P_L\left(p_0\right)\right) \leq u\left(c_1\left(1\right)\right), & w_1\left(P_H\left(p_0\right)\right) > u\left(c_1\left(1\right)\right), \\
& w_0\left(p_0\right) > u\left(c_1\left(1\right)\right), & 0 \leq c_1\left(1\right) \leq 2, \\
& 0 \leq c_1\left(0,1\right) \leq 2, & 0 \leq \lambda_{01} \leq 2 - c_1\left(0,1\right), \\
& 0 \leq \lambda_{10} \leq 2 - c_1\left(1\right), & 0 \leq \lambda_{00} \leq 2.
\end{array}
$$

$w_2\left(\cdot\right)$, $w_1\left(\cdot\right)$ and $w_0\left(\cdot\right)$ are (32), $(35-2)$ and $(41-1)$, respectively. Solve the other 26 cases in the same way, and compare the best contract in each case, we get the optimal contract that yields the highest ex ante welfare. I have included an example in the appendix illustrating the best contract in each case. Below, I list the optimal contract and the best run-proof contract in detail for comparison. This example requires the following parameters.

A computed example:

The utility function: $u\left(c\right) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.01$, $\gamma = 1.5$.

The risky production: $\overline{R} = 1.2$, $\underline{R} = 0.8$, $p = 0.75$.

The accuracy of the signal: $q = 0.9$.

The probability of being impatient: $\alpha = 0.6$.

1. A run-proof contract: $\{x_1 = 1$ if impatient, $x_1 = 0$ otherwise; $x_2 = 1$ if impatient, $x_2 = 0$ otherwise$\}$.

$w_0\left(p_0\right)$	$c_1\left(1\right)$	$c_1\left(1,1\right)$	$c_2\left(\overline{R},1,0\right)$	$c_2\left(\underline{R},1,0\right)$
18.0227	0.9053	1.0947	1.0947	1.0947

$c_1\left(0,1\right)$	$c_2\left(\overline{R},0,0\right)$	$c_2\left(\underline{R},0,0\right)$	$c_2\left(\overline{R},0,1\right)$	$c_2\left(\underline{R},0,1\right)$
0.8795	1.1992	0.8008	1.3445	0.8964

2. The optimal contract: in this example, the optimal contract results in $\{x_1 = 1$ if impatient or $S_1 = L$, $x_1 = 0$ otherwise; $x_2 = 1$ if impatient or ($S_2 = L$ and $x_1 = 1$), $x_2 = 0$ otherwise$\}$.

$w_0\left(p_0\right)$	$c_1\left(1\right)$	$c_1\left(1,1\right)$	$c_2\left(\overline{R},1,0\right)$	$c_2\left(\underline{R},1,0\right)$
18.0546	1.0064	0.9936	1.1924	0.7949

$c_1\left(0,1\right)$	$c_2\left(\overline{R},0,0\right)$	$c_2\left(\underline{R},0,0\right)$	$c_2\left(\overline{R},0,1\right)$	$c_2\left(\underline{R},0,1\right)$
1.0296	1.2000	0.8000	1.1644	0.7763

The optimal banking contract resulting in $\{x_1 = 1$ if impatient or $S_1 = L$, $x_1 = 0$ otherwise; $x_2 = 1$ if impatient or ($x_1 = 1$ and $S_2 = L$), $x_2 = 0$ otherwise$\}$ yields higher expected utility than the autarky. There are several reasons for the role of a bank in this example. First, a bank contract provides smooth consumptions across consumption types even in the presence of uncertainty regarding the number of impatient consumers. Second, since the withdrawals of the deposits are observable, but the portfolio adjustment in autarky is not, a line in front of the bank provides more information on productivity than in autarky. A run-proof contract is not desirable because it discourages consumers to reveal the information on productivity. This information is valuable because if the productivity is low, it is socially optimal to interrupt the production, and, thus, to prevent a crisis in progress. There exists positive probability that consumers withdraw deposits because they get wrong signals or the consumers ahead are impatient and decide to withdraw. In this situation, bank run is undesirable. However, this adverse side of information externality in herding is overcome by the positives.

A run-proof contract gives too little to the consumer who withdraws first in order to encourage him to wait. In this example, a run proof contract pays less than 1 unit of the consumption good to the first consumer who withdraws in period 1. By doing so, even if one consumer withdraws, the other one will be left with enough funds in the last period, thus he has no incentive to withdraw even if a low signal is received. The contract does not provide enough risk-sharing between consumers, although it completely eliminates bank run. A good contract balances between the consumption risk-sharing and the use of information. In particular, if the signal is perfect, we have the following proposition.

Proposition 3 *For* $q = 1$*, the optimal contract has* $\lambda_{10}^{*} = 0$*.*

Proof. The proposition says if consumer 1 withdraws and gets $c_1(1)$, the bank leaves funds in production if consumer 2 does not withdraw.

First, given consumer 1 withdraws, it is optimal to leave the money in the production when $R = \overline{R}$, while if $R = \underline{R}$, it is better to disrupt the production to prevent the loss in progress. If consumer 2 withdraws, he gets $2 - c_1(1)$. If he waits, he gets $(2 - c_1(1) - \lambda_{10})\overline{R} + \lambda_{10} \geq 2 - c_1(1)$ if $R = \overline{R}$, or gets $(2 - c_1(1) - \lambda_{10})\underline{R} + \lambda_{10} \leq 2 - c_1(1)$ if $R = \underline{R}$. He weakly prefers to wait if the productivity is high, and withdraws otherwise. By putting zero funds to storage, the bank induces the consumer to follow the signal as a dominant strategy as $(2 - c_1(1))\overline{R} > 2 - c_1(1)$ and $(2 - c_1(1))\underline{R} < 2 - c_1(1)$. The contract grabs all benefits from high productivity and prevents losses if the productivity is low. Therefore, $\lambda_{10}^{*} = 0$. ■

3.3 Bank Contracts without Commitment

Contracts without commitment allow the bank to re-optimize at each stage given the information the consumers reveal. Backward induction is employed to calculate the optimal contract. The way I calculate the contract without commitment is as follows. First, assume a pair of equilibrium strategies in one of the 27 cases. Given the strategies, calculate the beliefs at each node reached. Then optimize the contract by backward induction using the assumed beliefs at each node. Last, check the solution to the optimization problem. If the solution satisfies the conditions for the assumed equilibrium strategies, that is, $(31)-(41)$, then there exists a bank contract without commitment in this case. Otherwise, there does not exist such a contract.

Let $p_2^U(x_1, x_2)$ be the bank's belief at stage 2 given the withdrawal decision of consumers 1 and 2. At the end of stage 2, the bank maximizes the following expected utility function if consumer 1 waits and consumer 2 withdraws:

$$
\begin{aligned}
w^2(0,1) &= \max_{\lambda_{01}, c_1(0,1)} u(c_1(0,1)) + p_2^U(0,1)\, u(c_2(\overline{R},0,1)) + \left(1 - p_2^U(0,1)\right) u(c_2(\underline{R},0,1)) \\
&\quad s.t. \qquad c_1(0,1) \geq 0 \\
&\qquad\qquad\qquad \lambda_{01} \geq 0 \\
&\qquad\qquad c_1(0,1) + \lambda_{01} \leq 2
\end{aligned}
$$

$w^2(0,1)$ denotes the maximized welfare at stage 2 given the withdrawal history of $(0,1)$. If consumer 1 withdraws but consumer 2 waits, then the bank solves λ_{10} as a function of $c_1(1)$. The maximization problem is as follows:

$$
\begin{aligned}
w^2(1,0) &= \max_{\lambda_{10}} p_2^U(1,0)\, u(c_2(\overline{R},1,0)) + \left(1 - p_2^U(1,0)\right) u(c_2(\underline{R},1,0)) \\
&\quad s.t. \quad 0 \leq \lambda_{10} \leq 2 - c_1(1)
\end{aligned}
$$

The solutions to $c_2(\overline{R},1,0)$ and $c_2(\underline{R},1,0)$ are functions of $c_1(1)$.

If both consumers wait, then the bank solves:

$$
\begin{aligned}
w^2(0,0) &= \max_{\lambda_{00}} 2p_2^U(0,0)\, u(\overline{R},0,0) + 2\left(1 - p_2^U(0,0)\right) u(\underline{R},0,0) \\
&\quad s.t. \quad 0 \leq \lambda_{00} \leq 2
\end{aligned}
$$

While at stage 1, the bank solves the following problem if 1 withdraws given the updated belief. $p_1^U(1)$ denotes the bank's belief at stage 1 given that 1 withdraws, while $\eta\left(p_1^U(1)\right)$ represents the bank's forecast of the probability that the second consumer will wait at stage 2. $\eta\left(p_1^U(1)\right)$ is decided by the presumed equilibrium strategies:

$$\begin{aligned} w^1(1) &= \max_{c_1(1)} u(c_1(1)) + \eta\left(p_1^U(1)\right) w^2(1,0) + \left(1-\eta\left(p_1^U(1)\right)\right) u(2-c_1(1)) \\ & \quad s.t. \quad 0 \le c_1(1) \le 2 \end{aligned}$$

The solutions to $c_1(1)$, $c_1(0,1)$, λ_{01}, λ_{10} and λ_{00} are functions of p_2^U or p_1^U, which, in turn, are determined by the consumers' equilibrium strategies. The dynamic game is now played between three players. Assume an arbitrary pair of strategies from Θ_1 and Θ_2. The solutions to the optimization problems do not always give the right values consistent with the strategies.

In solving the contracts with commitment, there are also four paths: the path along $c_1(1)$ and $c_2(R|1,0)$, the pathe along $c_1(1)$ and $c_1(1,1)$, the path along $c_1(0,1)$ and $c_2(R|0,1)$, and the path along $c_2(R|0,0)$. The objective function of the optimization problem of $w_0(p_0)$ is separable regarding the paths except that it is constrained by the 6 conditions for adopting the presumed strategies. In solving the contracts without commitment, four paths are solved separately without constraint. If the solutions are consistent with the presumed strategies, it is automatically the solution to the problem with commitment. Therefore, the set of contracts without commitment is a subset of those with commitment.

The following proposition states the optimal contract without commitment is also the optimal contract with commitment, while the reserve is not true.

Proposition 4 *Given the presumed equilibrium strategies, in each case, the solution to the best contract without commitment is also the solution to the best contract with commitment. The reverse is not true.*

Proof. PNC solves 4 maximization problems without constraint. Check the solution by the definition of the presumed strategies.

PC solves a maximization problem subject to the definition of the presumed equilibrium strategies. The objective function of PC is a weighted average of the four objective functions of PNC. ■

However, proposition 4 does not imply that an optimal contract without commitment is also the optimal contract with commitment. It is possible that the optimal contract, or "the best of the best", with commitment is not credible.

I continue the example of an optimal contract without commitment in the appendix. As expected, most of them are not valid given bank's response. The contract that results in $\{x_1 = 1$ if impatient or $S_1 = L$, $x_1 = 0$ otherwise; $x_2 = 1$ if impatient or ($x_1 = 1$ and $S_2 = L$), $x_2 = 0$ otherwise$\}$ still works here.

3.4 Predeposit Game

In autarky, an impatient consumer consumes 1 unit of goods. Since each consumer has the opportunity to receive the signal, each adjusts his portfolio with the updated information at the end of stage 2. The payment at $t = 2$, denoted by $c_2 (S_1, R)$, depends on the state and the signal a consumer receives. Let λ_{S_i} be the amount of goods saved in storage given signal S_i to consumer i. If a signal is received, a patient consumer maximizes the following problem by choosing the amount in storage:

$$\begin{aligned} w\left(P_{S_i}(p_0)\right) &= \max_{\lambda_{S_i}} P_{S_i}(p_0)\, u\left((1-\lambda_{S_i})\overline{R} + \lambda_{S_i}\right) + \left(1 - P_{S_i}(p_0)\right) u\left((1-\lambda_{S_i})\underline{R} + \lambda_{S_i}\right) \\ & \quad s.t. \quad 0 \leq \lambda_{S_i} \leq 1 \end{aligned}$$

The ex ante expected utility is

$$w^{AUT}(p_0) = \alpha u(1) + (1-\alpha)\left[\gamma(p_0)\, w\left(P_H(p_0)\right) + \left(1-\gamma(p_0)\right) w\left(P_L(p_0)\right)\right]$$

Let us continue the example in the post deposit game. In autarky, $\lambda_H = 0$ and $\lambda_L = 1$. The payments in periods 1 and 2 are,

$$c_1 = 1, \quad c_2\left(H, \overline{R}\right) = 1.2, \quad c_2\left(H, \underline{R}\right) = 0.8, \quad c_2\left(L, \overline{R}\right) = 1, \quad c_2\left(L, \underline{R}\right) = 1.$$

The ex ante expected utility for each consumer is $w^{AUT}(p_0) = 18.0540$, which is lower than what an optimal contract in the post deposit game can achieve. Therefore, the optimal contract that allows for bank run is individually rational.

4 Conclusion

This paper provides a dynamic model for studying bank run. In an economy with uncertainty in production, a line in front of a bank carries information on the production status. In my model, information on fundamentals is released stage by stage, and consumers make withdrawal decisions according to the observation of withdrawal history as well as his own information on the economy. Given any simple demand deposit contract, there is a perfect Bayesian equilibrium in which consumers withdraw deposits if their beliefs are below the threshold, and wait otherwise. In some economies, the simple demand deposit contract allowing for herd of withdrawals is the optimal because it achieves higher risk-sharing among consumers and interrupts inefficient production in progress.

The paper also discusses a more complicated contract, assuming the bank can provide a contract conditional on the withdrawal history it observes. The perfect Bayesian equilibrium still exists. In some economies, the optimal contract is not run-proof.

In which economy should the bank provide a run-proof contract and in which economy should the bank offer a run-admitting one? It depends on the quality of the signals and also on how strongly the consumers desire smooth consumption. In an economy with accurate information on production, a consumer's decision is more convincing to the uninformed consumers, thus a contract should encourage the consumers to respond differently to different signals. A run-admitting contract provides higher consumption in the first period, but a healthy bank can fail if the signals are wrong, or if too many consumers need to consume at the first few stages and the line misleads the uninformed consumers. Consumer attitudes toward risk-sharing have two effects. The more risk averse the consumers, the more consumption goods they prefer in the first period, yet the more the consumers dislike bank runs. An optimal contract balances between the bank's function of consumption insurance and the use of information.

In this paper, bank has no information on productivity, which is not quite true in reality. In a more complicated model in which the bank receives signals on productivity, there arise problems such as how to eliminate bank's moral hazard problem due to the information asymmetry between the bank and the consumers, and how the bank reduces the probability of bank run due to the wrong signals. This can be the extensions to the paper.

Is there a banking mechanism achieving a more efficient allocation? It is possible. An efficient banking mechanism should allow the bank to provide a contract depending on the withdrawal history, and also eliminate asymmetric information between the bank and the consumers. Thus, the bank's objective function is always the same as the consumers' objective function. To find a more efficient mechanism is another extension of this paper, and more policy implications can be derived from the finding of such a mechanism.

5 Appendix

5.1 Proofs of lemmas 1, 2 and 3

Lemma 1 *$w_n(p_n)$ is increasing in p_n on $[0,1]$. For $\underline{p} < 0$, $w_n(p_n) = \alpha v_1 + (1-\alpha)[p_n\overline{u}_2 + (1-p_n)\underline{u}_2] > \alpha v_1 + (1-\alpha)u_1$ for $p_n \in [0,1]$. For $\underline{p} > 1$, $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$ on $p_n \in [0,1]$. For $\underline{p} \in [0,1]$, $w_N(p_N) = \alpha v_1 + (1-\alpha)[p_N\overline{u}_2 + (1-p_N)\underline{u}_2]$ on $(\underline{p},1]$ and $w_N(p_N) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$ on $p_N \in [0,\underline{p}]$. At stage $0 \leq n \leq N-1$, there exist $\tilde{p}_n$ and $\hat{p}_n$ $(0 < \tilde{p}_n \leq \hat{p}_n \leq \bar{p})$ such that $w_n(p_n) = \alpha v_1 + (1-\alpha)[p_n\overline{u}_2 + (1-p_n)\underline{u}_2] > \alpha v_1 + (1-\alpha)u_1$ for $p_n \in (\hat{p}_n, 1]$; $w_n(p_n)$ is increasing on $(\tilde{p}_n, p_n^2]$ with $\alpha v_1 + (1-\alpha)u_1 < w_n(p_n) < \alpha v_1 + (1-\alpha)[p_n\overline{u}_2 + (1-p_n)\underline{u}_2]$; $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$ for $p_n \in [0,\tilde{p}_n]$. $\hat{p}_n$ and $\tilde{p}_n$ are decreasing in n.*

Proof. If $\underline{p} < 0$, $c^1 \leq 1$. $w_N(p_N) = \alpha v_1 + (1-\alpha)[p_N\overline{u}_2 + (1-p_N)\underline{u}_2] > \alpha v_1 + (1-\alpha)u_1$ on $[0,1]$. Since $0 \leq P_L(p_{N-1}) < p_{N-1}$, $P_L(p_{N-1}) > \underline{p}$ for $p_{N-1} \geq 0$. So $w_{N-1}(p_{N-1}) = \alpha v_1 + (1-\alpha)[p_{N-1}\overline{u}_2 + (1-p_{N-1})\underline{u}_2]$ for $p_{N-1} \in [0,1]$. By induction, $w_n(p_n) = \alpha v_1 + (1-\alpha)[p_n\overline{u}_2 + (1-p_n)\underline{u}_2] > \alpha v_1 + (1-\alpha)u_1$ for $p_n \in [0,1]$.

Same argument applies to $\underline{p} > 1$. $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$ on $p_n \in [0,1]$.

If $\underline{p} \in [0,1]$, first we have $c^1 > 1$. It is obvious that $w_N(p_N)$ is increasing in p_N. $w_N(p_N) = p_N\overline{u}_2 + (1-p_N)\underline{u}_2$ on $(\underline{p},1]$ and $w_N(p_N) = \frac{1}{c^1}u_1$ on $[0,\underline{p}]$ by definition of $w_N(p_N)$. $0 < \tilde{p}_N = \hat{p}_N = \underline{p} < \bar{p}$. Prove the rest by induction.

At stage $n = N-1$, we begin with high p_{N-1} to low p_{N-1}.Since $P_L(p)$ and $P_H(p)$ are strictly increasing in p, and the support of both functions are $[0,1]$, we can divide cases according to the value of $P_H(p)$ and $P_L(p)$.

(I) For $P_L(p_{N-1}) > \underline{p}$, $w_{N-1}(p_{N-1}) = \alpha v_1 + (1-\alpha)[p_{N-1}\overline{u}_2 + (1-p_{N-1})\underline{u}_2]$.

If $P_L(p_{N-1}) \leq \underline{p}$, let us look at $\pi(p_{N-1})w_N(P_H(p_{N-1})) + (1-\pi(p_{N-1}))w_N\left(P_{\widetilde{L}}(p_{N-1})\right)$.

(II) For $P_H(p_{N-1}) > P_{\widetilde{L}}(p_{N-1}) > \underline{p}$: $w_N(P_H(p_{N-1})) = \alpha v_1 + (1-\alpha)[P_H(p_{N-1})\overline{u}_2 + (1-P_H(p_{N-1}))\underline{u}_2]$, and $w_N\left(P_{\widetilde{L}}(p_{N-1})\right) = \alpha v_1 + (1-\alpha)\left[P_{\widetilde{L}}(p_{N-1})\overline{u}_2 + \left(1-P_{\widetilde{L}}(p_{N-1})\right)\underline{u}_2\right]$,

$$w_{N-1}(p_{N-1}) = \alpha v_1 + (1-\alpha)[p_{N-1}\overline{u}_2 + (1-p_{N-1})\underline{u}_2].$$

$\hat{p}_{N-1}$ is the probability which such that $P_{\widetilde{L}}(\hat{p}_{N-1}) = \underline{p}$. Since $P_{\widetilde{L}}(p_{N-1}) > P_L(p_{N-1})$, $\hat{p}_{N-1} < \bar{p}$. As $P_{\widetilde{L}}(p) < p$ for $p \in (0,1)$, $\hat{p}_{N-1} > \underline{p} = \hat{p}_N$.

(III) For $P_H(p_{N-1}) > \underline{p}$, $P_{\widetilde{L}}(p_{N-1}) \leq \underline{p}$, and $\alpha v_1 + (1-\alpha)\left[P_{\widetilde{L}}(p_{N-1})\overline{u}_2 + \left(1-P_{\widetilde{L}}(p_{N-1})\right)\underline{u}_2\right] > \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$:

(i) if $\pi(p_{N-1})$ is high:

$$\begin{aligned}
w_N\left(P_{\widetilde{L}}(p_{N-1})\right) &= \frac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right] \\
&< \alpha v_1 + (1-\alpha)\left[P_{\widetilde{L}}(p_{N-1})\,\overline{u}_2 + \left(1 - P_{\widetilde{L}}(p_{N-1})\right)\underline{u}_2\right].
\end{aligned}$$

$$\begin{aligned}
w_{N-1}(p_{N-1}) &= \pi(p_{N-1})\,w_N\left(P_H(p_{N-1})\right) + (1-\pi(p_{N-1}))w_N\left(P_{\widetilde{L}}(p_{N-1})\right) \\
&= \pi(p_{N-1})\left\{\alpha v_1 + (1-\alpha)\left[P_H(p_{N-1})\,\overline{u}_2 + (1-P_H(p_{N-1}))\,\underline{u}_2\right]\right\} + \\
&\quad + (1-\pi(p_{N-1}))\tfrac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right] \\
&> \alpha v_1 + (1-\alpha)\,u_1 \text{ and} \\
w_{N-1}(p_{N-1}) &< \alpha v_1 + (1-\alpha)\left[p_{N-1}\overline{u}_2 + (1-p_{N-1})\,\underline{u}_2\right].
\end{aligned}$$

(ii) if $\pi(p_{N-1})$ is low:

$$\begin{aligned}
& \pi(p_{N-1})\,w_N\left(P_H(p_{N-1})\right) + (1-\pi(p_{N-1}))w_N\left(P_{\widetilde{L}}(p_{N-1})\right) \\
&= \pi(p_{N-1})\left\{\alpha v_1 + (1-\alpha)\left[P_H(p_{N-1})\,\overline{u}_2 + (1-P_H(p_{N-1}))\,\underline{u}_2\right]\right\} + \\
&\quad + (1-\pi(p_{N-1}))\tfrac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right] \\
&\leq \alpha v_1 + (1-\alpha)\,u_1. \text{ Hence,} \\
w_{N-1}(p_{N-1}) &= \tfrac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right].
\end{aligned}$$

Since $w_N(P_H(p_{N-1}))$ and $\pi(p_{N-1})$ are increasing, $w_{N-1}(p_{N-1})$ is also increasing on this interval. If there is a p_{N-1} in the interval for case (III) below which $w_{N-1}(p_{N-1}) = \frac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right]$, above which $w_{N-1}(p_{N-1}) > \alpha v_1 + (1-\alpha)\,u_1$, this p_{N-1} is $\hat{p}_{N-1}$. In this interval, $w_{N-1}(p_{N-1}) < \alpha v_1 + (1-\alpha)\left[p_{N-1}\overline{u}_2 + (1-p_{N-1})\,\underline{u}_2\right]$, so $w_{N-1}(\hat{p}_{N-1}) < \alpha v_1 + (1-\alpha)\left[\hat{p}_{N-1}\overline{u}_2 + (1-\hat{p}_{N-1})\,\underline{u}_2\right]$.

(IV) For $P_H(p_{N-1}) > \underline{p}$, $P_{\widetilde{L}}(p_{N-1}) \leq \underline{p}$, and $\alpha v_1 + (1-\alpha)\left[P_{\widetilde{L}}(p_{N-1})\,\overline{u}_2 + \left(1-P_{\widetilde{L}}(p_{N-1})\right)\underline{u}_2\right] \leq \frac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right]$:

(i) if $\pi(p_{N-1})$ is high:

$$\begin{aligned}
w_{N-1}(p_{N-1}) &= \pi(p_{N-1})\,w_N\left(P_H(p_{N-1})\right) + (1-\pi(p_{N-1}))w_N\left(P_{\widetilde{L}}(p_{N-1})\right) \\
&= \pi(p_{N-1})\left\{\alpha v_1 + (1-\alpha)\left[P_H(p_{N-1})\,\overline{u}_2 + (1-P_H(p_{N-1}))\,\underline{u}_2\right]\right\} + \\
&\quad + (1-\pi(p_{N-1}))\tfrac{1}{c^1}\left[\alpha v_1 + (1-\alpha)\,u_1\right] \\
&> \alpha v_1 + (1-\alpha)\,u_1
\end{aligned}$$

(ii) if $\pi\left(p_{N-1}\right)$ is low:

$$\begin{aligned} & \pi\left(p_{N-1}\right) w_N\left(P_H\left(p_{N-1}\right)\right)+\left(1-\pi\left(p_{N-1}\right)\right) w_N\left(P_{\widetilde{L}}\left(p_{N-1}\right)\right) \\ = & \pi\left(p_{N-1}\right)\left\{\alpha v_1+(1-\alpha)\left[P_H\left(p_{N-1}\right) \overline{u}_2+\left(1-P_H\left(p_{N-1}\right)\right) \underline{u}_2\right]\right\}+ \\ & +\left(1-\pi\left(p_{N-1}\right)\right) \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] \\ \leq & \alpha v_1+(1-\alpha) u_1 \text{. Hence, } \\ w_{N-1}\left(p_{N-1}\right) = & \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] . \end{aligned}$$

If there exists a cutoff probability in the interval of case (IV) below which $w_{N-1}\left(p_{N-1}\right)=\frac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right]$, above which $w_{N-1}\left(p_{N-1}\right)>\alpha v_1+(1-\alpha) u_1$, this p_{N-1} is $\hat{p}_{N-1}$.It is possible that $w_{N-1}\left(p_{N-1}\right)=\frac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right]$ for all p_{N-1} in this interval.

Note that if $P_{\widetilde{L}}\left(p_{N-1}\right)=\underline{p}$,

$$\begin{gathered} \pi\left(p_{N-1}\right)\left\{\alpha v_1+(1-\alpha)\left[P_H\left(p_{N-1}\right) \overline{u}_2+\left(1-P_H\left(p_{N-1}\right)\right) \underline{u}_2\right]\right\}+ \\ +\left(1-\pi\left(p_{N-1}\right)\right) \frac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] \leq \alpha v_1+(1-\alpha) u_1 \end{gathered}$$

as $\pi\left(p_{N-1}\right)$ is too low, $\tilde{p}_{N-1}=\hat{p}_{N-1}$.

(V) For $P_H\left(p_{N-1}\right) \leq \underline{p}$, $P_{\widetilde{L}}\left(p_{N-1}\right) \leq \underline{p}$, and $\alpha v_1+(1-\alpha)\left[P_{\widetilde{L}}\left(p_{N-1}\right) \overline{u}_2+\left(1-P_{\widetilde{L}}\left(p_{N-1}\right)\right) \underline{u}_2\right] \leq \frac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right]$

$$\begin{aligned} & \pi\left(p_{N-1}\right) w_N\left(P_H\left(p_{N-1}\right)\right)+\left(1-\pi\left(p_{N-1}\right)\right) w_N\left(P_{\widetilde{L}}\left(p_{N-1}\right)\right) \\ = & \pi\left(p_{N-1}\right) \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right]+\left(1-\pi\left(p_{N-1}\right)\right) \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] \\ = & \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] . \end{aligned}$$

Note that

$$\begin{aligned} & \lim_{P_H\left(p_{N-1}\right) \rightarrow^{+} \underline{p}}\left[\pi\left(p_{N-1}\right) w_N\left(P_H\left(p_{N-1}\right)\right)+\left(1-\pi\left(p_{N-1}\right)\right) w_N\left(P_{\widetilde{L}}\left(p_{N-1}\right)\right)\right] \\ = & \pi\left(p_{N-1}\right)\left[\alpha v_1+(1-\alpha) u_1\right]+\left(1-\pi\left(p_{N-1}\right)\right) \tfrac{1}{c^1}\left[\alpha v_1+(1-\alpha) u_1\right] \\ < & \alpha v_1+(1-\alpha) u_1 . \end{aligned}$$

$\tilde{p}_{N-1}$ must be in the interval such that $P_H\left(\tilde{p}_{N-1}\right)>\underline{p}$, $P_{\widetilde{L}}\left(\tilde{p}_{N-1}\right) \leq \underline{p}$. As $P_H\left(\underline{p}\right)>\underline{p}$ and $P_{\widetilde{L}}\left(\underline{p}\right)<\underline{p}$, by the continuity and monotonicity of $P_H\left(\underline{p}\right)$ and $P_{\widetilde{L}}\left(\underline{p}\right)$, $\tilde{p}_{N-1}$ must be less than $\underline{p}$ and satisfies $P_H\left(\tilde{p}_{N-1}\right)>\underline{p}$, $P_{\widetilde{L}}\left(\tilde{p}_{N-1}\right) \leq \underline{p}$.

Assume it is true for stage n. There exist $\tilde{p}_n$ and $\hat{p}_n$ $\left(0<\tilde{p}_n \leq \hat{p}_n \leq \bar{p}\right)$ such that $w_n\left(p_n\right)=\alpha v_1+(1-\alpha)\left[p_n \overline{u}_2+\left(1-p_n\right) \underline{u}_2\right]>\alpha v_1+(1-\alpha) u_1$ for $p_n \in\left(\hat{p}_n, 1\right]$; $w_n\left(p_n\right)$ is increasing on $\left(\tilde{p}_n, \hat{p}_n\right]$, with

$w_n(\tilde{p}_n) \geq \frac{1}{c^1}[\alpha v_1 + (1-\alpha) u_1]$ and $\alpha v_1 + (1-\alpha) u_1 < w_n(\hat{p}_n) < \alpha v_1 + (1-\alpha)[\hat{p}_n \overline{u}_2 + (1-\hat{p}_n) \underline{u}_2]$; $w_n(p_n) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha) u_1]$ for $p_n \in [0, \tilde{p}_n]$.

Let us look at stage $n-1$.

(I) For $P_L(p_{n-1}) > \underline{p}$, $w_{n-1}(p_{n-1}) = \alpha v_1 + (1-\alpha)[p_{n-1}\overline{u}_2 + (1-p_{n-1}) \underline{u}_2]$.

For $P_L(p_{n-1}) \leq \underline{p}$, let us look at $\pi(p_{n-1}) w_n(P_H(p_{n-1})) + (1-\pi(p_{n-1})) w_n\left(P_{\widetilde{L}}(p_{n-1})\right)$.

(II) $P_H(p_{n-1}) > P_{\widetilde{L}}(p_{n-1}) > \hat{p}_n$, $w_{n-1}(p_{n-1}) = \alpha v_1 + (1-\alpha)[p_{n-1}\overline{u}_2 + (1-p_{n-1}) \underline{u}_2] > \alpha v_1 + (1-\alpha) u_1$.

(III) $P_H(p_{n-1}) > \hat{p}_n$ and $\tilde{p}_n < P_{\widetilde{L}}(p_{n-1}) \leq \hat{p}_n$:

$$w_n(P_H(p_{N-1})) = \alpha v_1 + (1-\alpha)[P_H(p_{n-1})\overline{u}_2 + (1-P_H(p_{n-1})) \underline{u}_2]$$

and

$$\alpha v_1 + (1-\alpha) u_1 < w_n\left(P_{\widetilde{L}}(p_{n-1})\right) < \alpha v_1 + (1-\alpha)[\hat{p}_n \overline{u}_2 + (1-\hat{p}_n) \underline{u}_2],$$

$$\begin{aligned}
w_{n-1}(p_{n-1}) &= \pi(p_{n-1})\{\alpha v_1 + (1-\alpha)[P_H(p_{n-1})\overline{u}_2 + (1-P_H(p_{n-1})) \underline{u}_2]\} + \\
&\quad +(1-\pi(p_{n-1})) w_n\left(P_{\widetilde{L}}(p_{n-1})\right) \\
&> \pi(p_{n-1})\{\alpha v_1 + (1-\alpha)[P_H(p_{n-1})\overline{u}_2 + (1-P_H(p_{n-1})) \underline{u}_2]\} + \\
&\quad +(1-\pi(p_{n-1}))[\alpha v_1 + (1-\alpha) u_1] \\
&> \alpha v_1 + (1-\alpha) u_1
\end{aligned}$$

At $P_{\widetilde{L}}(p_{n-1}) = \hat{p}_n$ and $P_H(p_{n-1}) > \hat{p}_n$,

$$w_{n-1}(p_{n-1}) < \alpha v_1 + (1-\alpha)[p_{n-1}\overline{u}_2 + (1-p_{n-1}) \underline{u}_2]$$

Hence $P_{\widetilde{L}}(\hat{p}_{n-1}) = \hat{p}_n$. Since $P_{\widetilde{L}}(p) < p$, $\hat{p}_{n-1} \geq \hat{p}_n$.

(IV) $\tilde{p}_n < P_H(p_{n-1}) \leq \hat{p}_n$ and $\tilde{p}_n < P_{\widetilde{L}}(p_{n-1}) \leq \hat{p}_n$:

$$\alpha v_1 + (1-\alpha) u_1 < w_n(P_H(p_{n-1})) < \alpha v_1 + (1-\alpha)[\hat{p}_n \overline{u}_2 + (1-\hat{p}_n) \underline{u}_2]$$

and

$$\alpha v_1 + (1-\alpha) u_1 < w_n\left(P_{\widetilde{L}}(p_{n-1})\right) < \alpha v_1 + (1-\alpha)[\hat{p}_n \overline{u}_2 + (1-\hat{p}_n) \underline{u}_2],$$

$$\begin{aligned}
w_{n-1}(p_{n-1}) &= \pi(p_{n-1}) w_n(P_H(p_{n-1})) + (1-\pi(p_{n-1})) w_n\left(P_{\widetilde{L}}(p_{n-1})\right) \\
&> \alpha v_1 + (1-\alpha) u_1, \text{ and} \\
w_{n-1}(p_{n-1}) &< \alpha v_1 + (1-\alpha)[\hat{p}_n \overline{u}_2 + (1-\hat{p}_n) \underline{u}_2].
\end{aligned}$$

(V) $P_H(p_{n-1}) > \tilde{p}_n$ and $P_{\widetilde{L}}(p_{n-1}) \leq \tilde{p}_n$: $w_n(P_H(p_{n-1})) > \alpha v_1 + (1-\alpha)u_1$ and $w_n\left(P_{\widetilde{L}}(p_{n-1})\right) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$.

(i) If $\pi(p_{n-1})$ is high,

$$\begin{aligned} w_{n-1}(p_{n-1}) &= \pi(p_{n-1})w_n(P_H(p_{n-1})) + (1-\pi(p_{n-1}))\tfrac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1] \\ &> \alpha v_1 + (1-\alpha)u_1, \text{ and} \\ w_{n-1}(p_{n-1}) &< \alpha v_1 + (1-\alpha)[\hat{p}_n\overline{u}_2 + (1-\hat{p}_n)\underline{u}_2]. \end{aligned}$$

(ii) If $\pi(p_{n-1})$ is low,

$$\begin{aligned} \pi(p_{n-1})w_n(P_H(p_{n-1})) + (1-\pi(p_{n-1}))\tfrac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1] &< \alpha v_1 + (1-\alpha)u_1, \text{ so} \\ w_{n-1}(p_{n-1}) &= \tfrac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]. \end{aligned}$$

(VI) $P_H(p_{n-1}) < \tilde{p}_n$ and $P_{\widetilde{L}}(p_{n-1}) < \tilde{p}_n$: $w_n(P_H(p_{n-1})) = w_n\left(P_{\widetilde{L}}(p_{n-1})\right) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$.

$$w_{n-1}(p_{n-1}) = \tfrac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1].$$

Since $\pi(p_{n-1})$, $w_n(P_H(p_{n-1}))$ and $w_n\left(P_{\widetilde{L}}(p_{n-1})\right)$ is increasing and $w_n(P_H(p_{n-1})) \geq w_n\left(P_{\widetilde{L}}(p_{n-1})\right)$, there is the cutoff probability above which $w_{n-1}(p_{n-1}) > \alpha v_1 + (1-\alpha)u_1$, below which $w_{n-1}(p_{n-1}) = \frac{1}{c^1}[\alpha v_1 + (1-\alpha)u_1]$. ■

Lemma 2 *If* $w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha)u_1$, $w_n^U\left(P_H\left(p_{n-1}^U\right)\right) > w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha)u_1$.

Proof. By Lemma 1, if $w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha)u_1$, we have $p_{n-1}^U \in (\tilde{p}_{n-1}^U, 1]$. Also by proposition 1, $\hat{p}_n^U \leq \hat{p}_{n-1}^U$.

If $p_{n-1}^U > \hat{p}_{n-1}^U$, $P_H\left(p_{n-1}^U\right) > p_{n-1}^U > \hat{p}_{n-1}^U \geq \hat{p}_n^U$. Hence

$$\begin{aligned} w_n^U\left(P_H\left(p_{n-1}^U\right)\right) &= \alpha v_1 + (1-\alpha)\left[P_H\left(p_{n-1}^U\right)\overline{u}_2 + \left(1 - P_H\left(p_{n-1}^U\right)\right)\underline{u}_2\right] \\ &> \alpha v_1 + (1-\alpha)\left[p_{n-1}^U\overline{u}_2 + \left(1-p_{n-1}^U\right)\underline{u}_2\right] \\ &= w_{n-1}^U\left(p_{n-1}^U\right). \end{aligned}$$

If $\tilde{p}_{n-1}^U < p_{n-1}^U \leq \hat{p}_{n-1}^U$,

$$\begin{aligned} w_{n-1}^U\left(p_{n-1}^U\right) &= \pi\left(p_{n-1}^U\right)w_n^U\left(P_H\left(p_{n-1}^U\right)\right) + (1-\pi\left(p_{n-1}^U\right))w_n^U\left(P_{\widetilde{L}}\left(p_{n-1}^U\right)\right) \\ &> \alpha v_1 + (1-\alpha)u_1. \end{aligned}$$

By Lemma 1, $w_n^U\left(P_H\left(p_{n-1}^U\right)\right) \geq w_n^U\left(P_{\widetilde{L}}\left(p_{n-1}^U\right)\right)$. Since $w_{n-1}^U\left(p_{n-1}^U\right) > \alpha v_1 + (1-\alpha)u_1$ and $w_{n-1}^U\left(p_{n-1}^U\right)$ is a weighted average of $w_n^U\left(P_H\left(p_{n-1}^U\right)\right)$ and $w_n^U\left(P_{\widetilde{L}}\left(p_{n-1}^U\right)\right)$, $w_n^U\left(P_H\left(p_{n-1}^U\right)\right) \geq w_{n-1}^U\left(p_{n-1}^U\right)$. ■

Lemma 3 If $w_n^U(p) > \alpha v_1 + (1-\alpha) u_1$, then $w_n^{I,S_r}(p) > u_1$.

Proof. Prove by induction. Show that at each stage, if $w_n^U(p) > \alpha v_1 + (1-\alpha) u_1$, $w_n^U(p) = \alpha \big[\rho_n v_1 + (1-\rho_n) \min\left\{\frac{1}{c^1}, 1\right\} v_1$ $(1-\alpha) w_n^{I,S_r}(p), \rho_n \in [0,1]$, and $w_n^{I,S_r}(p) > u_1$.

Begin with stage N, if $w_N^U(p) > \alpha v_1 + (1-\alpha) u_1$,

$$\begin{aligned}
w_N^U(p) &= \alpha v_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\underline{u}_2\right] \\
&= \alpha v_1 + (1-\alpha) w_N^{I,S_r}(p) \\
&> \alpha v_1 + (1-\alpha) u_1, \text{ so} \\
w_N^{I,S_r}(p) &= p\overline{u}_2 + (1-p)\underline{u}_2 > u_1, \text{ and} \\
\rho_N &= 1.
\end{aligned}$$

Suppose it is true for stage n. If $w_n^U(p) > \alpha v_1 + (1-\alpha) u_1$, we have $w_n^U(p) = \alpha\left[\rho_n v_1 + (1-\rho_n)\min\left\{\frac{1}{c^1}, 1\right\} v_1\right] + (1-\alpha) w_n^{I,S_r}(p), \rho_n \in [0,1]$, and $w_n^{I,S_r}(p) > u_1$.

At stage $n-1$, suppose $w_{n-1}^U(p) > \alpha v_1 + (1-\alpha) u_1$.

If $P_L(p) > \underline{p}$, $w_{n-1}^U(p) = \alpha v_1 + (1-\alpha)\left[p\overline{u}_2 + (1-p)\underline{u}_2\right] > \alpha v_1 + (1-\alpha) u_1$. $w_n^{I,S_r}(p) = p\overline{u}_2 + (1-p)\underline{u}_2 > u_1$.

If $P_L(p) \leq \underline{p}$, $w_{n-1}^U(p) = \pi(p) w_n^U(P_H(p)) + (1-\pi(p)) w_n^U\left(P_{\widetilde{L}}(p)\right) > \alpha v_1 + (1-\alpha) u_1$. By lemma 2, $w_n^U(P_H(p)) > \alpha v_1 + (1-\alpha) u_1$. Suppose $w_n^U\left(P_{\widetilde{L}}(p)\right) > \alpha v_1 + (1-\alpha) u_1$. By assumption at stage n, $w_n^{I,S_r}(P_H(p)) > u_1$ and $w_n^{I,S_r}\left(P_{\widetilde{L}}(p)\right) > u_1$. So $w_{n-1}^{I,S_r}(p) = \pi(p) w_n^{I,S_r}(P_H(p)) + (1-\pi(p)) w_n^{I,S_r}\left(P_{\widetilde{L}}(p)\right)$ $> u_1$, and $\rho_{n-1} = 1$.

Suppose $w_n^U\left(P_{\widetilde{L}}(p)\right) < \alpha v_1 + (1-\alpha) u_1$, so $w_n^U\left(P_{\widetilde{L}}(p)\right) = \min\left\{\frac{1}{c^1}, 1\right\}\left[\alpha v_1 + (1-\alpha) u_1\right]$. If $w_n^{I,S_r}\left(P_{\widetilde{L}}(p)\right) = \min\left\{\frac{1}{c^1}, 1\right\} u_1$,

$$\begin{aligned}
w_{n-1}^U(p) &= \pi(p) w_n^U(P_H(p)) + (1-\pi(p)) w_n^U\left(P_{\widetilde{L}}(p)\right) \\
&= \pi(p) w_n^U(P_H(p)) + (1-\pi(p)) \min\left\{\frac{1}{c^1}, 1\right\}\left[\alpha v_1 + (1-\alpha) u_1\right] \\
&= \pi(p)\left\{\alpha\left[\rho_n v_1 + (1-\rho_n)\min\left\{\frac{1}{c^1}, 1\right\} v_1\right] + (1-\alpha) w_n^{I,S_r}(P_H(p))\right\} + \\
&\quad + (1-\pi(p)) \min\left\{\frac{1}{c^1}, 1\right\}\left[\alpha v_1 + (1-\alpha) u_1\right] \\
&= \alpha\left[\rho_{n-1} v_1 + (1-\rho_{n-1})\min\left\{\frac{1}{c^1}, 1\right\} v_1\right] + (1-\alpha)\left[\pi(p) w_n^{I,S_r}(P_H(p)) + (1-\pi(p)) w_n^{I,S_r}\left(P_{\widetilde{L}}(p)\right)\right] \\
&> \alpha v_1 + (1-\alpha) u_1, \text{ so} \\
w_{n-1}^{I,S_r}(p) &= \pi(p) w_n^{I,S_r}(P_H(p)) + (1-\pi(p)) w_n^{I,S_r}\left(P_{\widetilde{L}}(p)\right) \\
&> u_1.
\end{aligned}$$

■

Lemma 5 *If* $w^U_{n-1}(p) > \alpha v_1 + (1-\alpha) u_1$ *and* $p^U_n = P_H(p)$, $w^{I,S_n}_n(P_H(p)) > u_1$.

Proof. At stage $n-1$, if $w^U_{n-1}(p) > \alpha v_1 + (1-\alpha) u_1$, we have $w^U_n(P_H(p)) > \alpha v_1 + (1-\alpha) u_1$ by lemma 2. By lemma 3,

$$\begin{aligned} w^{I,S_r}_n(P_H(p)) &= \pi(P_H(p))\, w^{I,S_r}_{n+1}\left(P^2_H(p)\right) + (1-\pi(P_H(p)))w^{I,S_r}_{n+1}\left(P_{\widetilde{L}} P_H(p)\right) \\ &> u_1. \end{aligned}$$

If the previously informed consumers hold the same belief, regardless of the stage the consumer is informed, $w^{I,S_r}_n(p) = w^{I,S_{r'}}_n(p)$ for $r \neq r'$ by definition of $w^{I,S_r}_n(p)$ .We have

$$\begin{aligned} w^{I,S_r}_n(P_H(p)) &= \pi(P_H(p))\, w^{I,S_n}_{n+1}\left(P^2_H(p)\right) + (1-\pi(P_H(p)))w^{I,S_n}_{n+1}\left(P_{\widetilde{L}} P_H(p)\right) \\ &= w^{I,S_n}_n(P_H(p)) \\ &> u_1. \end{aligned}$$

■

5.2 A Computed Example of Optimal Contract in Section 3:

The utility function takes the form $u(c) = \frac{(c+b)^{1-\gamma} - b^{1-\gamma}}{1-\gamma}$, $b = 0.01$, $\gamma = 1.5$.

The risky production is as follows: $\overline{R} = 1.2$, $\underline{R} = 0.8$, $p = 0.75$.

The accuracy of the signal: $q = 0.9$.

The proportion of impatient consumers: $\alpha = 0.6$.

In autarky, $w = 18.0540$.

The listed are the best contracts with commitment in each of the 27 cases. NC denotes a contract feasible without commitment.

$x_1 = 1$ iff impatient.	$c_1(1)$	$c_1(1,1)$	$c_1(0,1)$	$c_2\left(\overline{R},0,0\right)$	$c_2\left(\overline{R},0,1\right)$	$c_2\left(\overline{R},1,0\right)$	$c_2\left(\underline{R},0,0\right)$	$c_2\left(\underline{R},0,1\right)$	$c_2\left(\underline{R},1,0\right)$	$w_0(p_0)$
$x_2 = 1$ iff impatient or:										
$x_2 = 1$ iff impatient;	0.9053	1.0947	0.8795	1.1992	1.3445	1.0947	0.8008	0.8964	1.0947	18.0227
$S_2 = L$;	0.9733	1.0267	0.9473	1.2000	1.1866	1.2321	0.8000	0.9189	0.8214	18.0440
$S_2 = L$ or $x_1 = 1$;	1.0039	0.9961	0.9987	1.2000	1.0116	0.9961	0.8000	0.9909	0.9961	18.0280
$S_2 = L$ or $x_1 = 0$;	0.9040	1.0960	1.0000	1.0000	1.1637	1.3153	1.0000	0.8363	0.8768	18.0309
$S_2 = L$ and $x_1 = 1$;	0.9417	1.0583	0.8792	1.1997	1.3422	1.2700	0.8003	0.8994	0.8467	18.0380
$S_2 = L$ and $x_1 = 0$;	0.8016	1.1984	0.9939	1.2000	1.2073	1.1984	0.8000	0.8049	1.1984	18.0151
$x_1 = 1$;	0.9989	1.0011	0.8541	1.1191	1.3166	1.0011	0.8809	0.9752	1.0011	18.0204
$x_1 = 0$;	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	18.0099
$x_2 = 1$.	0.6637	1.3363	1.0001	1.0000	1.1998	1.3363	1.0000	0.7999	1.3363	17.9700

$x_1 = 1$ iff imp or $S_1 = L$;	$c_1(1)$	$c_1(1,1)$	$c_1(0,1)$	$c_2\left(\overline{R},0,0\right)$	$c_2\left(\overline{R},0,1\right)$	$c_2\left(\overline{R},1,0\right)$	$c_2\left(\underline{R},0,0\right)$	$c_2\left(\underline{R},0,1\right)$	$c_2\left(\underline{R},1,0\right)$	$w_0(p_0)$
$x_2 = 1$ iff impatient or:										
$x_2 = 1$ iff impatient;	1.0240	0.9760	1.0258	1.2000	1.1690	0.9760	0.8000	0.7793	0.9760	18.0406
$S_2 = L$;	1.0064	0.9936	1.0760	1.2000	1.1088	1.1923	0.8000	0.7392	0.7949	18.0538
$S_2 = L$ or $x_1 = 1$;	0.7938	1.2062	1.1210	1.1931	1.0403	1.2062	0.8069	0.7178	1.2062	18.0146
$S_2 = L$ or $x_1 = 0$;	1.0065	0.9935	1.0272	1.0274	1.1673	1.1923	0.9726	0.7782	0.7948	18.0459
$S_2 = L$ and $x_1 = 1$; (NC)	1.0064	0.9936	1.0296	1.2000	1.1644	1.1924	0.8000	0.7763	0.7949	18.0546**
$S_2 = L$ and $x_1 = 0$;	0.8407	1.1593	1.0760	1.2000	1.1088	1.1593	0.8000	0.7392	1.1593	18.0264
$x_1 = 1$;	0.8380	1.1620	1.0760	1.2000	1.1088	1.1620	0.8000	0.7392	1.1620	18.0265
$x_1 = 0$;	0.9984	1.0016	1.0373	1.0000	1.1553	1.0016	1.0000	0.7702	1.0016	18.0321
$x_2 = 1$.	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	18.0099

The best contract for the equilibrium strategy $x_1 = 1$ is not necessarily unique in each case, because the beliefs off equilibrium path are arbitrary. The followings are the examples.

$x_1 = 1$; $x_2 = 1$ iff impatient or:	$c_1(1)$	$c_1(1,1)$	$c_1(0,1)$	$c_2\left(\overline{R},0,0\right)$	$c_2\left(\overline{R},0,1\right)$	$c_2\left(\overline{R},1,0\right)$	$c_2\left(\underline{R},0,0\right)$	$c_2\left(\underline{R},0,1\right)$	$c_2\left(\underline{R},1,0\right)$	$w_0(p_0)$
$x_2 = 1$ iff impatient;	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	18.0099
$S_2 = L$;	1.0074	0.9926	1.1597	1.1917	0.9268	1.1911	0.8083	0.7539	0.7941	18.0320
$S_2 = L$ or $x_1 = 1$;	1.0000	1.0000	1.2000	1.2000	0.9302	1.0000	0.8000	0.6698	1.0000	18.0099
$S_2 = L$ or $x_1 = 0$; (NC)	1.0074	0.9926	2.0000	1.2000	0.0000	1.1911	0.8000	0.0000	0.7941	18.0320
$S_2 = L$ and $x_1 = 1$;	1.0074	0.9926	1.0000	1.0000	1.0130	1.1911	1.0000	0.9870	0.7941	18.0320
$S_2 = L$ and $x_1 = 0$;	0.9999	1.0001	1.1846	1.1965	0.8918	1.0001	0.8035	0.7390	1.0001	18.0099
$x_1 = 1$;	1.0000	1.0000	1.0659	1.0659	0.9343	1.0000	0.9341	0.9339	1.0000	18.0099
$x_1 = 0$;	0.8468	1.1532	2.0000	1.2000	0.0000	1.1532	0.8000	0.0000	1.1532	17.9925
$x_2 = 1$. (NC)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	18.0099

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