ENDOGENOUS MATCHING AND MONEY*

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Abstract

We present a new framework for studying monetary economics. As in the random matching literature, agents trade bilaterally, and not through centralized markets. However, our approach is fundamentally different in the following sense: rather than assuming agents meet exogenously and at random, we determine *endogenously* who meets whom. We show how to formalize the dynamic endogenous meeting process in a tractable way, and apply the model to a variety of issues in monetary theory. Some of our results are similar to what has been found in the previous literature, while other results different significantly and in interesting ways.

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1 Introduction

This paper presents a new framework for thinking about monetary economics based on bilateral exchange. The approach is related to standard search or random matching models of money (see below for references) in that agents will be assumed to trade bilaterally and not through a centralized market. We maintain bilateral trade because, given specialization, this is a simple and natural way to generate a double coincidence problem, and this seems like a very reasonable friction upon which to build a theory of money as a medium of exchange. However, the approach here is fundamentally different from existing search models of money in the following sense: rather than assuming agents meet exogenously and at random, we determine *endogenously* who meets whom and when.

The idea is related in spirit to the literature on matching in cooperative equilibrium models going back to Gale and Shapley (1962). As in that framework, agents are matched at each point in time subject to a stability condition that says, basically, no two agents prefer to be with each other rather than with their current partners. However, the previous literature in this vein is essentially static, and because we are interested in monetary economics we need to extend things to dynamic environments. We do this in a somewhat general way since, in principle, dynamic endogenous matching may have many other applications, even though the focus here is mainly on money. Although general, our method is still very simple, and as we show it is often much simpler than random matching for monetary applications.

One reason it seems desirable to endogenize the meeting process is that many economists seem to think random matching is unattractive or unrealistic.[1] One

[1] For example, as Howitt (2000, p.1) puts it: "In contrast to what happens in search models, exchanges in actual market economies are organized by specialist traders, who mitigate search costs by providing facilities that are easy to locate. Thus when people wish to buy shoes they go to a shoe store; when hungry they go to a grocer; when desiring to sell their labor services they go to firms known to offer employment. Few people would think of planning their economic lives on the basis of random encounters."

may try to counter that random matching, while unrealistic, like many abstractions, is merely a tractable way to generate a double coincidence problem, and it is not crucial to the basic message. To see if the results – or, more precisely, to see *which* results – from previous monetary models based on bilateral exchange depend on random matching, we develop a method to make the meeting pattern endogenous. It turns out that when one does so, some of the results in the previous literature do go through, while some others change significantly and in interesting ways.

The rest of the paper can be summarized as follows. Section 2 introduces the general framework and defines our notion of equilibrium. We then explore its implications in four different environments. Section 3 uses the model to study the relationship between money and memory in supporting efficient allocations, as discussed by Kocherlakota (1998) in a variety of models. Section 4 develops a simple equilibrium model of monetary exchange similar to the one in Kiyotaki and Wright (1993), but with endogenous matching instead of random search. Section 5 presents a version of the model with prices determined through bargaining, as in Shi (1995) or Trejos and Wright (1995), but with endogenous rather than random matching. Section 6 considers the emergence of commodity money, as in Kiyotaki and Wright (1989) and Aiyagari and Wallace (1991), but again using our endogenous matching concept.

As we said above, sometimes in these applications the results are similar to what has been found in other models, but sometimes the results are quite different. More generally, we conclude that the idea of building an theory of monetary exchange based on double coincidence problems (as well as some other frictions to be made explicit) can be pursued using models of bilateral matching, without necessarily using random matching.[2]

[2] In terms of related work, in addition to Gale and Shapley (1962), see Roth and Sotomayor (1988) for an extensive discussion of cooperative matching models. Also relevant is the recent directed search literature in macro-labor, including Moen (1997), Acemoglu and Shimer (1999), Lagos (2000), and Burdett et al. (2001), where workers get to visit a firm of their

2 Endogenous Matching

Time is discrete: $t = 0, 1, \dots T$, where T may be ∞. Let A be a set of agents, where A is arbitrary for now. Agents are characterized by their type, describing, e.g., their tastes or technology. Later, there will also be state variables, describing, e.g., which goods they are holding, but for now they carry no inventories. In our framework agents are restricted to meet and trade bilaterally; however, in contrast to the standard matching literature, rather than assuming they meet exogenously and at random we will determine endogenously who meets whom. For now one should think of agents choosing to meet particular individuals. Later, we also consider models where they get to choose a certain type but not necessarily a particular individual – say, they draw an individual at random from the set of agents of that type.[3]

Bilateral matching can always be described by an *assignment rule* $\Psi = \{\psi_t\}$, where at every date t, $\psi_t : A \to A$ is a bijection that assigns to every individual a partner. As a convention, an agent could be assigned to himself, $\psi_t(i) = i$, meaning he is in autarchy that period. We need to impose some consistency conditions. First, $\psi_t[\psi_t(i)] = i$, or, in other words, you are your partner's partner, which is what it means to match bilaterally. Second, in the case where A is a continuum, ψ_t must be measure-preserving in the usual sense (e.g., see Kaneko and Wooders [1986] or Cole, Mailath and Postlewaite [1998] for discussions). Clearly, at every date t, ψ_t induces a *partition* θ_t of A into subsets, or *coalitions*, of size 1 or 2. Let Θ be the set of partitions consisting of all such coalitions.

choosing rather than sampling at random, but the models and the issues in those papers are very different from what we need to discuss monetary economics. Matsui and Shimizu (2001) present a monetary model similar in spirit to the directed search literature, and we discuss below how it compares to our work. In monetary theory there are of course many papers that discuss bilateral trade predating search models; rather than listing them here we refer the reader to the survey by Ostroy and Starr (1980).

[3] For example, one may always know where to find a taxi – at the taxi stand, say – but there can still be a negligible probability of getting the same driver more than once.

Given the current partition the instantaneous payoff of i is denoted $w_t^i(\theta_t)$. Thus, agents generally have preferences over the entire partition, although in applications w_t^i will often depend only on $\psi_t(i)$. Indeed, although for now we define payoffs on θ_t – you care directly about who is matched with whom – later we will explicitly introduce, in addition to matching considerations, decisions about production, exchange and consumption within matches, and it will be these variables rather than partners per se that generate utility. A history at t is a list $h_t = (\theta_0, \theta_1 ..., \theta_{t-1})$, and H_t is the set of all possible histories at t. Define a *matching rule* by the sequence $\Phi = \{\Phi_t\}$ where $\Phi_t : H_t \to \Theta$ gives the current partition as a function of history. Then for any Φ, the lifetime utility of agent i at history h_t is given recursively by

$$v_t^i(h_t) = w_t^i [\Phi_t(h_t)] + \beta v_{t+1}^i(h_{t+1}), \tag{1}$$

where h_{t+1} is constructed by updating h_t using $\theta_t = \Phi_t(h_t)$, and $\beta \in (0, 1)$ is a discount factor. Note that we could randomize over matching rules – indeed, pure random matching can be thought of as a special case – which means that agents need to form expectations, but otherwise one can interpret (1) as deterministic.

We say that Φ is an *equilibrium* matching rule if for every t and h_t, no coalition C consisting of 1 or 2 agents can do better by deviating in the following sense: an individual can deviate by matching with himself rather than as prescribed by Φ; and a pair can deviate by matching with each other rather than as prescribed by Φ. When we say C does better we mean that $v_t^i(h_t)$ increases for all $i \in C$, taking as given two things. First, we take as given that at date t, every $j \notin C$ matches according to Φ, unless j is abandoned by $i \in C$, by which we mean that Φ called for i to match with j but i deviated by not matching with j, in which case j stays in autarchy that period. Second, agents take as given that from date $t + 1$ on, matching will be determined by Φ, given the history induced by the deviation.

Several comments are in order. First, we emphasize that for Φ to be an equilibrium it must be immune to bilateral as well as unilateral deviations, but not double deviations in the sense of two bilateral coalitions deviating at the same time.[4] This issue typically does not come up in existing cooperative equilibrium matching models, since as long as one's payoff depends only on one's partner, a coalition contemplating a deviation does not care what other agents do. In a dynamic model, however, even if we assume w_t^i depends only on $\psi_t(i)$, *future* payoffs generally will depend on the entire partition θ_t. We also emphasize that to check if Φ is an equilibrium we only check one-period deviations; i.e., a coalition considering a deviation at t takes as given behavior from $t+1$ on, in that future matches are determined by Φ from the history h_{t+1} induced by the deviation.[5] Finally, we mention the special case of a *memoryless* equilibrium, by which we mean that Φ_t does not depend on history h_t (although it could depend on state variables); this will play an important role below.

We demonstrate how things work in an example with three agents, $A = \{1, 2, 3\}$. The set Θ of feasible partitions contains: autarchy, denoted $a = [\{1\}, \{2\}, \{3\}]$; 1 and 2 matched, $[\{1, 2\}, \{3\}]$; 2 and 3 matched, $[\{2, 3\}, \{1\}]$; and 3 and 1 matched, $[\{3, 1\}, \{2\}]$. Preferences are as follows: for each agent i, $w_t^i = u > 0$ if i is matched with $i+1$, $w_t^i = -c < 0$ if i is matched with $i+2$, and $w_t^i = 0$ if i is matched with himself that period, where addition is mod 3.

[4] To illustrate, consider a static example with $A = \{1, 2, 3, 4\}$. Suppose we want to check if there is an equilibrium prescribing the partition $\theta = [\{1, 2\}, \{3, 4\}]$. We need to check potential individual deviations, including agent 1 abandonning 2 for autarchy, e.g., leading to $[\{1\}, \{2\}, \{3, 4\}]$. Our solution concept also requires that we check bilateral deviations, including agents 1 and 3 abandoning their current partners for each other, e.g., leading to $[\{1, 3\}, \{2\}, \{4\}]$. However, we do not need to check the double deviation leading to $[\{1, 3\}, \{2, 4\}]$.

[5] As is standard, for unilateral deviations this is without loss of generality: by the unimprovability principle of dynamic programming it suffices to check all one shot deviations to show that a decision rule is optimal. There is no such principle for coalitions with more than 1 agent, however, and our restriction to one time deviations can be substantive. In fact, by checking all 1 period deviations we are actually checking all k period deviations for any $k < \infty$, since every k period deviation is a 1 period deviation after some history, but there are examples where the results could change if we allowed infinite deviations (see below). In any case, for most of what we say in this paper the results do not hinge on this, but the analysis is easier if we only have to check one shot deviations.

Thus, 1 wants to match with 2, but 2 does not like 1 and wants to match with 3, etc., which yields a double coincidence problem at every date. This leads immediately to the following result: for any T, permanent autarchy is always an equilibrium, and if $T < \infty$ it is the only equilibrium.[6]

Although there may be many equilibria when $T = \infty$, we want to focus on a particular one that happens to be efficient. Thus, consider the rule Φ^e that prescribes $\theta_0^e = [\{1, 2\}, \{3\}]$, $\theta_1^e = [\{2, 3\}, \{1\}]$, $\theta_2^e = [\{3, 1\}, \{2\}]$, and then cycles back to $\theta_3^e = \theta_0^e$, and so on along the equilibrium path; and at any history h_t off the equilibrium path, we trigger to permanent autarchy, $\theta^e(h_t) = a$.[7] Since Φ^e implies that payoffs are periodic, along the equilibrium path, we can let v_j^i be the lifetime utility of agent i when it is j's turn to match with the agent he likes, and write (1) as

$$v_i^i = u + \beta v_{i+1}^i,\ v_{i+1}^i = \beta v_{i+2}^i,\ v_{i+2}^i = -c + \beta v_i^i. \tag{2}$$

As long as $v_j^i \geq 0$ for all i, j no coalition wants to deviate from Φ^e, given we assume $c \leq \beta u$.[8] Hence, Φ^e is an equilibrium. We will call such an outcome *active*, which means we are not in autarchy at any date along the equilibrium path; clearly, in this example, this means it is also efficient, in the obvious sense.

Before pursuing applications we need to generalize things to include state variables, since we may want to allow agents to carry inventories, and to include decision variables describing behavior other than matching, since we sometimes want to explicitly allow for trade, production and consumption. Let z_t^i be the

[6] If there is a finite date T, any coalition deviating from a must be of the form $C = \{i, i+1\}$ for some i, which makes $i+1$ worse off; then use backward induction. If $T = \infty$, again at any date a deviating coalition must be of the form $C = \{i, i+1\}$, which makes $i+1$ worse off in the period, and no better off in terms of continuation value as long as he takes as given that $\Phi_t(h_t) = a$ from every h_t for all future t.

[7] Although our solution concept is in the spirit of cooperative equilibrium theory, as we said in the Introduction, we think that using terminology like "triggers" and "the equilibrium path" from noncooperative game theory faciliates the discussion and should not lead to confusion.

[8] The binding incentive condition is to get i to match with the person he does not like: $v_{i+2}^i = -c + \beta v_i^i \geq 0$. Solving, we have $v_{i+2}^i = (\beta u - c)/(1 - \beta^3) \geq 0$, and so $c \leq \beta u$ implies no coalition can do better than Φ^e along the equilibrium path. As permanent autarchy is always an equilibrium, there are no profitable deviations off the equilibrium path either.

individual state. The aggregate state is generally given by Z_t, which specifies z_t^i for every i. Let σ_t^i be an individual decision variable, constrained to lie in a set which generally could depend on the state, and which we write as $\Sigma^i(Z_t)$. Let σ_t specify σ_t^i for every i. Let $Y_t = (\theta_t, Z_t, \sigma_t)$, and denote instantaneous payoffs by $w_t^i(Y_t)$ and the law of motion for the state by $Z_{t+1} = f(Y_t)$. Now a history at t is given by $h_t = (Y_0, Y_1, ..., Y_{t-1}, Z_t)$, and H_t is the set of possible histories. Matching is again described by $\Phi = \{\Phi_t\}$ where $\Phi_t : H_t \rightarrow \Theta$, and now we also have for each agent i a decision rule $\Gamma^i = \{\Gamma_t^i\}$ where $\Gamma_t^i : H_t \times \Theta \rightarrow \Sigma^i$. Let Γ specify the profile of individual decision rules, Γ^i.

For any Φ and Γ, lifetime utility of agent i at t in history h_t is now given by

$$v_t^i(h_t) = w_t^i\left[\Phi_t(h_t), Z_t, \sigma_t\right] + \beta v_{t+1}^i(h_{t+1}), \tag{3}$$

where σ_t is determined from the decision rules $\Gamma_t^i(h_t, \theta_t)$ and h_{t+1} is constructed in the obvious way. The generalized notion of equilibrium is a pair (Φ, Γ) such that for every t and h_t, no coalition C consisting of 1 or 2 agents can do better by deviating either by matching differently than as prescribed by Φ (as in the previous definition) or by taking a decision different than that prescribed by Γ. As above, when we say C does better we take as given two things. First, agents take as given that at date t, every $j \notin C$ matches according to Φ, unless j is abandoned by $i \in C$. Second, agents take as given that from date $t+1$ on, matches and decisions are determined by (Φ, Γ) given the history h_{t+1} induced by the deviation.[9]

[9] Although we will usually not be quite so pedantic with notation in the applications that follow, it seems useful to have a formal definition that allows state and decision variables. Also, we mention here how things may be different if one allows infinite deviations. Consider an example with 2 agents whose preferences from matching alternate over time. That is, $w_t^i(\{1\},\{2\}) = 0$ for all t and i, while $w_t^i(\{1,2\}) = u$ if t is even and $w_t^i(\{1,2\}) = -c$ if t is odd for agent $i = 1$, and just the opposite for $i = 2$. Permanent autarchy is an equilibrium because, without commitment, no alternative finite deviation can be sustained: there will always be a last period where the agents are matched and one prefers to be alone. However, deviating from autarchy to matching *forever* can be sustained for large β. Hence, for large β autarchy is immune to all finite but not infinite deviations.

3 Money and Memory

We now illustrate how the model works in applications to monetary economics. In this section we begin by exploring the relationship between money and memory in supporting efficient allocations, motivated by Kotcherlakota (1998), who studied the issue in several models of monetary exchange (see also Kotcherlakota and Wallace [1998], Wallace [2001], and Araujo[2001]). While it is common in this literature to assume a continuum of agents, we begin with a finite population. While not crucial for our point, having a finite set of agents actually makes some things easier here, and also allows us to derive some additional implications regarding alternative notions of memory. We begin with the case of *complete public memory* : the entire history h_t is known by everyone.

Let $T = \infty$ and assume there are N indivisible and nonstorable goods. Each agent $i \in \{1, 2, ..., N\}$ produces only good i and consumes only good $i + 1$ (mod N). When agent i produces he receives instantaneous disutility $-c$ and when he consumes he receives utility u, where $c \leq \beta u$. While later we will want to distinguish between matching and other decisions, like producing or trading, for now we identify a match between i and j with the following outcome: if $j = i + 1$ then j produces and i consumes; if $j = i - 1$ then i produces and j consumes; and for any other i and j there is no production or consumption. As we assume for now that goods are nonstorable, they must be produced and consumed simultaneously, and given that there is no money in the model as of yet, agents carry no inventories. Hence, for now there is no state variable.

With $N = 3$, this environment is equivalent to the example in the previous section. Hence, by the same reasoning, one equilibrium is permanent autarchy, and another is given by the rule Φ^e that prescribes $\theta_0^e = [\{1, 2\}, \{3\}]$, $\theta_1^e = [\{2, 3\}, \{1\}]$, $\theta_2^e = [\{3, 1\}, \{2\}]$, $\theta_3^e = \theta_0^e$, and so on along the equilibrium path, supported by a trigger to permanent autarchy. The generalization to N types is easy. If N is any even number, say, we can support a symmetric active outcome where every agent consumes and produces every second period (i alternates between matching with $i+1$ and with $i-1$) for exactly the same parameter values, $c \leq \beta u$ (if N is odd one agent must sit out one out of every N periods, as we saw with $N = 3$). In any case, the binding constraint is always to get agent i to match with (produce for) $i-1$, which we can do by allowing him to consume one period later, given $c \leq \beta u$. These outcomes are clearly efficient.

Consider now the opposite extreme of *no memory*: for whatever reason, agents simply do not know h_t.[10] Formally, we simply impose that matching is memoryless in the sense of the previous section: Φ_t cannot depend on h_t. With no state variable this implies the unique equilibrium is autarchy, as i will never match with $i-1$ at t if the future cannot depend on θ_t. However, we now introduce money in the form of M intrinsically useless, indivisible, storable objects (e.g., coins). We can initially distribute money to agents as we choose (e.g., we can give one coin to each of M agents, give them all to agent 1, etc.). The state variable for i is his money inventory, $z_t^i \in \{0, 1..., M\}$, and everyone sees all inventories through the aggregate state $Z_t = (z_t^1, ...z_t^N)$. A necessary condition for i to match with $i-1$ is that money change hands. We now show the efficient outcome can be supported using monetary exchange.

Begin with $N = 3$ and $M = 1$, and at any date t, if $z_t^i = 1$ let Φ prescribe that i match with $i+1$ and swap his unit of money for i's consumption good. The potentially binding deviation is for $i+1$ to stay in autarchy that period, which means he does not have to produce but also does not change his state. If $\beta \geq c/u$, this makes him no better off. Hence, monetary exchange is an equilibrium and replicates what we achieved using memory. The result generalizes easily to

[10] In random matching models a lack of memory can be motivated by assuming a large number of agents, since what is actually relevant turns out to be the probability of meeting someone that you met before, or that has met someone you met before, etc. In this section we prefer to work with a finite number of agents and simply rule out knowledge of h_t by brute force. To the extent that this is an issue, one could motivate the assumption by saying that every period agents reproduce offspring who are indentical except that they have no knowledge of the family history (Anderlini and Lagunoff [2001] formalize this). In any case, we move to models with large numbers of agents in the next section.

any number of agents as long as M is set appropriately. For instance, if N is even, we can set $M = N/2$, give one unit of money to every second agent, and support the outcome where i consumes every second period.[11] Although money and memory are *conceptually* different, we conclude here that they are *perfect substitutes*, in the sense that the same condition $\beta \geq c/u$ implies the same outcome can be supported using one or the other.

We summarize these findings in the following proposition, the proof of which follows directly from the above discussion.

Proposition 1 *In the endogenous matching model we have the following: with complete memory we can support a symmetric active pattern of exchange that is efficient (e.g., if N is even, each agent i consumes every second period) as an equilibrium iff $\beta \geq c/u$; with money but no memory the same outcome can be supported under the same condition, given that M is set appropriately.*

By way of comparison, suppose agents are forced exogenously to meet at random, but otherwise the environment is identical. The binding incentive condition is that i should agree to match with (produce for) $i-1$ when they happen to meet, with a trigger to autarchy. This incentive condition holds iff $\beta \geq c/\left[c + \frac{\alpha}{N-1}(u-c)\right] \equiv \beta^r$, where α is the probability of meeting anyone on a given date and each meeting is a random draw from the population. The condition $\beta \geq \beta^r$ is obviously more difficult to satisfy than $\beta \geq c/u$ if α is small. But even if $\alpha = 1$, so that you meet someone with certainty each period, $\beta \geq \beta^r$ is still more difficult to satisfy than $\beta \geq c/u$ because you often meet the *wrong type*. Indeed, for this reason, $\beta \geq \beta^r$ becomes harder to sustain as N increases, while in our endogenous matching model the relevant condition $\beta \geq c/u$ does not depend on N at all.

[11] There are also many active equilibria that are not symmetric. For example, for large β we can use memory to support the outcome where agent 1 matches with 2 for two periods in a row, then 2 matches with 3 for one period, then 3 with 1 for one period, and then repeat the pattern. To support this with no memory, give agent 1 two units of money and have him spend one at a time, while agents 2 and 3 always spend two at a time.

Now assume no memory but introduce money in the random matching model. With $N = 3$ and $M = 1$, monetary exchange is an equilibrium iff $\beta \geq \beta^r$, but clearly monetary exchange implies lower payoffs than the allocation we sustained using memory. There are two issues to emphasize. First, random matching makes the incentive conditions more stringent. Second, with random matching, even if monetary exchange is an equilibrium, agents sometimes *run out of money* and hence cannot consume in a meeting where they would like to and where they could consume if we had recourse to memory. In our endogenous matching model, such meetings do not occur – an agent only matches with the producer of the good he likes when he has cash. This is why money is a perfect substitute for memory in our model but not the random matching model.[12]

We also emphasize that these problems in random matching models get worse as N gets bigger since there is a greater chance of meeting the wrong type. Moreover, with random matching, the analysis is extremely complicated for general N and M, because one has to keep track of the stochastic evolution of the state (letting the set of agents get large does not resolve this difficulty). Hence, although random matching is often motivated as a simplifying abstraction, it turns out the endogenous matching model is much more tractable: some extra work needs to be done to determine who matches with whom, but once this is sorted out, the outcome is very simple.

To close this section we briefly consider some implications of assuming there is *private* but not public memory (the comparison to Kandori's [1992] social norm concept should be clear from what follows). Assume the complete history h_t is not known by any given agent, but he does know his own experience. Consider as a candidate equilibrium the outcome that is identical to the efficient

[12]Note that money *is* a perfect substitute for memory in the overlapping generations and turnpike models considered by Kocherlakota. Intutively, in those models as in our model, transaction patterns are nonrandom but periodic, although here this is derived and not assumed.

Φ described above except that after an agent only triggers to autarchy after he *directly* observes any deviation. Let $N = 3$ and consider agent 2 at $t = 0$, when Φ prescribes that he matches with 1, and suppose he deviates to autarchy that period. Agent 1 observes this but agent 3 does not; as far as 3 knows, at $t = 1$ we are still on the equilibrium path, and so he matches with 2. Only at $t = 2$ will agent 3 know there has been a deviation when agent 1 refuses to match with him, at which point permanent autarchy sets in.

There are two relevant incentive conditions. First, it must be the case that agent 1 makes the deviation by 2 known to 3 at $t = 2$ by refusing to match, which is always true given $c > 0$. Second, one can check agent 2 will not deviate at $t = 0$ iff $\beta^4 u \geq c$. This is a stronger condition than needed with public memory, $\beta u \geq c$. Intuitively, when actions are not all publicly observable, deviations trigger chains of punishments that get back to a deviator only eventually. However, notice that the monetary equilibrium described above still exists iff $\beta u \geq c$. Hence, for some parameters, money supports outcomes that private memory cannot. Moreover, private memory becomes less useful as the number of agents grows, but existence conditions for monetary equilibria are independent of N.[13]

4 Monetary Equilibria

In this section we study monetary equilibria in a different model.[14] There is now a continuum of agents, $A = [0, 1]$, equally divided into N types. There are N indivisible and non-storable goods, and each type i produces good i and consumes good $i + 1$ (mod N). Production costs $c > 0$, consumption yields $u > c$, and the discount rate is written $\beta = 1/(1 + r)$. In contrast to the

[13] Araujo (2001) makes a similar point in a random matching model, but the analysis there is much more complicated precisely because matching is random. Hence, endogenizing the meeting process agains simplifies things a lot.

[14] Aside from technical differences, like the number of agents, the main distinction from the previous section is the following: there we were interested in the extent to which money could support certain efficient outcomes; here we will describe all equilibria in a certain class. In particular, when we introduced money earlier we introduced the right amount, and below we consider equilibria for arbitrary M.

previous section, here we assume each agent can store at most one indivisible unit of money, and $M \in [0, 1]$ units are distributed uniformly across agents. The state variable for agent i is his money inventory, $z_t^i \in \{0, 1\}$. We only consider symmetric outcomes here, which implies the fraction of type i with $z_t^i = 1$ will equal M for all i and t. Also, let $\gamma \geq 0$ be a per period storage cost of holding money; if $\gamma = 0$ this reduces to the pure fiat money case.

If agents met randomly this would be basically the same as the environment in Kiyotaki and Wright (1993), but we will use endogenous matching. To be precise, agents get to choose the type i and money holdings z of people they meet, but rather than a particular individual they only get to draw at random from the set of agents who are type i with inventory z. This implies that if you meet someone at t, the probability that you meet him again later, the probability that you meet someone who will meet him later, and so on, are all 0. This implies deviant behavior cannot be punished, and the model is effectively memoryless, so matching rules Φ cannot depend on history. Hence, the only possible trades involve agents of type i giving money to agents of type $i+1$ in exchange for goods, and along the equilibrium path the only interesting matches are between type i agents with $z^i = 1$ and type $i+1$ agents with $z^{i+1} = 0$.

Therefore we look for active equilibria where each period Φ_t specifies the following: if $M < 1/2$ then every type i agent with money meets a type $i+1$ agent without money while exactly $\frac{M}{1-M}$ of the type $i+1$ agents without money (drawn at random) meet a type i agent with money; and if $M > 1/2$ then every type $i+1$ agent without money meets a type i agent with money while exactly $\frac{1-M}{M}$ of the type i agents with money (drawn at random) meet a type $i+1$ agent without money. The probabilities of agents with and without money meeting someone each period are $a_1^e = \min\left\{\frac{1-M}{M}, 1\right\}$ and $a_0^e = \min\left\{\frac{M}{1-M}, 1\right\}$ (the superscript e stands for "endogenous" matching). In what follows we denote this matching rule by $\hat{\Phi}$ and describe it by saying *the long side of the market is* *rationed*.[15]

In this model we want to distinguish explicitly between matching and trading. Thus, using the notation of Section 2, we assume that in any meeting the following happens: first one agent (chosen arbitrarily) and then the other announces $\sigma \in Z = \{yes, no\}$; if both announce $\sigma = yes$ they trade and otherwise they part company. Since we know the only possible trades occur when i with money meets $i+1$ without money, we need only specify what happens in these matches. Moreover, it is a dominant strategy for the agent with money to announce $\sigma^i = yes$, so we can focus on the decision of the other agent σ^{i+1}. Allowing for mixed strategies, we let $\pi = \Pr(\sigma^{i+1} = yes)$.

Let V_z be the value function of an agent with z units of money. Given the matching rule $\hat{\Phi}$ described above – i.e., the long side of the market is rationed – the value functions satisfy the standard dynamic programming equations:

$$rV_1 = a_1^e \pi(u + V_0 - V_1) - \gamma \quad (4)$$

$$rV_0 = a_0^e \pi(-c + V_1 - V_0). \quad (5)$$

The best response condition for π is: $\pi = 1$ if $-c + V_1 - V_0 > 0$; $\pi = 0$ if $-c + V_1 - V_0 < 0$; and $\pi \in (0,1)$ implies $-c + V_1 - V_0 = 0$. For an equilibrium, all we need is that no agent or pair wants to deviate from Φ and that π satisfy this best response condition.[16]

Clearly there are no profitable deviations from $\hat{\Phi}$: although some agents on the long side of the market are rationed, they cannot find anyone on the other side who prefers them over their incumbent partner. It is now a routine matter to solve (4) and (5) for the value functions and determine the parameter values

[15] In Matsui and Shimizu (2001), agents get to go to particular market places (as long as they are open, which is endogenous) where they can only meet agents on the other side of the same markets. If markets are costless it is efficient to have lots of them open, naturally, since this makes it easier to meet the right type. As in our framework, even with a sufficiently rich set of markets in their model some agents will be rationed in the meeting process if M is not set correctly (although the analogue of M is endogenous in their model).

[16] If we allow free disposal then we also have to check $V_1 \geq 0$; this will be implicit in what follows.

for which the best response condition holds. Hence, following result is stated without proof.

Proposition 2 *Consider equilibria in the endogenous matching model with* $\hat{\Phi}$ *as described above. Then there is a critical value* $\overline{\gamma}^e = a_1^e(u-c) - rc$ *such that the following is true:* $\gamma < \overline{\gamma}^e$ *implies there are three equilibria,* $\pi = 0$, $\pi = 1$, *and* $\pi = \frac{rc+\gamma}{a_1^e(u-c)} \in (0,1)$*; and* $\gamma > \overline{\gamma}^e$ *implies the only equilibrium is* $\pi = 0$.

By way of comparison, consider the random matching version of the model. The probability a type i agent with money meets a type $i+1$ without money is $a_1^r = \frac{\alpha}{N-1}(1-M)$ and the probability a type $i+1$ agent without money meets a type i with money is $a_0^r = \frac{\alpha}{N-1}M$, where α is the probability of meeting anyone each period (the superscript r stands for "random" matching). An equilibrium now has to satisfy only the best response condition for π. Thus, the set of equilibria is qualitatively the same as described in the above proposition – just change the arrival rates from a_z^e to a_z^r. Note that, as in the previous section, it is harder for money to be valued with random matching, since $\overline{\gamma}^r < \overline{\gamma}^e$. Also, note that it again becomes more difficult to support a monetary equilibrium as N grows with random matching, but not with endogenous matching, since $\overline{\gamma}^r$ is increasing in N while $\overline{\gamma}^e$ is independent of N.

To the extent that one finds very simple models like the one in Kiyotaki and Wright (1993) useful, this shows how we can build a similar model with endogenous rather than random matching. To provide just one application, we compare the models in terms of their predictions about *sunspot equilibria*. Assume there is a Markov process for a variable $s_t \in \{m, n\}$, which switches according to the exogenous probabilities $\Pr(s' = n \mid s = m) = \lambda_{mn}$ and $\Pr(s' = m \mid s = n) = \lambda_{nm}$ (the letters m and n refer to "monetary" and "nonmonetary"). Consider equilibria where $\hat{\Phi}$ is exactly as above, but now, even though nothing fundamental depends on s, we have $\pi_m = 1$ and $\pi_n = 0$. One can show that such an equilibrium exists in our model in the region of

$(\lambda_{nm}, \lambda_{mn})$ space bounded by[17]

$$\lambda_{mn} = \frac{a_1^e(u-c) - rc}{r[a_1^e(u-c)+c]}(r+\lambda_{nm})$$

$$\lambda_{mn} = \frac{[a_1^e(1+r)(u-c) - rc(1-a_1^e-a_0^e)]\lambda_{nm} - rc(a_1^e+a_0^e+r)}{rc(1-a_1^e-a_0^e)}.$$

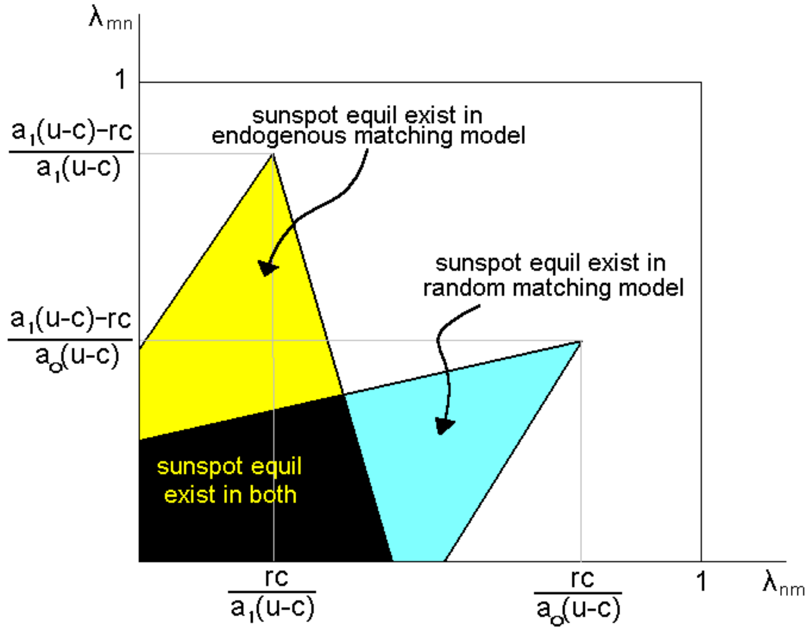

Figure 1: Regions of Sunspot Equilibria

In the random matching model, sunspot equilibria exist in the region defined by the same conditions after changing a_z^e to a_z^r. As Figure 1 shows, the model

[17] For simplicity here we set $\gamma = 0$. Then, following the procedure in Wright (1997), let V_z^s be the value function of an agent with z units of money in state s. This imples:

$$\begin{aligned} rV_1^m &= (1-\lambda_{mn})a_1^e(u+V_0^m - V_1^m) + \lambda_{mn}(V_1^n - V_1^m) \\ rV_1^n &= \lambda_{nm}a_1^e(u+V_0^m - V_1^n) + \lambda_{nm}(1-a_1^e)(V_1^m - V_1^n) \\ rV_0^m &= (1-\lambda_{mn})a_0^e(-c+V_1^m - V_0^m) + \lambda_{mn}(V_0^n - V_0^m) \\ rV_0^n &= \lambda_n m a_0^e(-c+V_1^m - V_0^n) + \lambda_{nm}(1-a_0^e)(V_0^m - V_0^n). \end{aligned}$$

For $\pi_m = 1$ and $\pi_n = 0$ to be an equilibrium, we require $V_1^m - V_0^m - c \geq 0$ and $V_1^n - V_0^n - c \leq 0$. Straightforward algebra implies that these conditions are satisfied under the conditions gien in the text.

is more likely to generate sunspot equilibria with random matching than with endogenous matching when $\lambda_{nm}/\lambda_{mn}$ is large and less likely when $\lambda_{nm}/\lambda_{mn}$ is small. One might have guessed that sunspot equilibria are less likely in our model – i.e., eliminating randomness in matching might reduce the set of parameters for which sunspots can matter – but this is not the case. Intuitively, as money is easier to spend and hence more valuable with endogenous matching, it is easier to satisfy the condition for $\pi_m = 1$ and harder to satisfy the condition for $\pi_n = 0$; so we need λ_{nm} smaller or λ_{mn} larger for sunspot equilibria to exist with endogenous matching. Hence, not only can one build an endogenous matching version of the baseline monetary search model, in general, we think that comparisons across the models can be interesting.

5 Money and Prices

We now generalize the model in the previous section by assuming commodities are divisible, and endogenize prices by having agents bargain over the amount of the good they get for a unit of money.[18] When an agent produces q units of a good he suffers disutility $-c(q)$, and when he consumes q units of his consumption good he enjoys utility $u(q)$. Assume $c'(q) > 0$, $c''(q) \geq 0$, $\lim_{q\to 0} c'(q) = 0$, and $\lim_{q\to\infty} c'(q) = \infty$, as well as $u'(q) > 0$, $u''(q) \leq 0$, $\lim_{q\to 0} u'(q) = \infty$, and $\lim_{q\to\infty} u'(q) = 0$. Also, there exists $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. All other assumptions regarding this environment, including the structure of who produces and who consumes which goods, are identical to the previous section. Hence, we will focus on equilibria where the matching rule is given by $\hat{\Phi}$ (long side rationed).

[18] This way of endogenizing prices follows what Shi (1995) and Trejos and Wright (1995) did in the random matching model. A much more complicated approach is to allow goods and money to both be divisible, or to allow agents to hold multiple units of indivisible money; see Molico (1998), Green and Zhou (1998), Zhou (1999), and Camera and Corbae (1999).

The dynamic programming equations are now

$$rV_1 = a_1^e [u(q) + V_0 - V_1] - \gamma \quad (6)$$

$$rV_0 = a_0^e [-c(q) + V_1 - V_0], \quad (7)$$

where the arrival rates are as defined in the previous section, and we have set $\pi = 1$ (equilibria with $\pi = 0$ are not very interesting and those with $0 < \pi < 1$ do not exist in this version). To determine q, we adopt the generalized Nash (1950) bargaining solution,

$$\max_q [u(q) + V_0 - V_1]^\omega [-c(q) + V_1 - V_0]^{1-\omega} \quad (8)$$

where ω is the bargaining power of an agent with money, and the threat point of an agent with z units of money is given by V_z.[19] An equilibrium is defined as the obvious generalization of the previous section, where we add one more variable, q, and the equilibrium condition (8).

As in the previous section we can generate a random matching version of the model simply by replacing a_z^e with a_z^r. It is useful as a benchmark here to give the results for the random matching version first, where we do not have to worry about some details regarding commitment (see below). Then the construction of equilibria is straightforward: if we take the first order conditions from (8) and insert the value functions derived from solving (6) and (7), we see that a monetary equilibrium is a positive solution to $T(q) = 0$, where

$$\begin{aligned} T(q) &= \omega \{a_1^r [u(q) - c(q)] - \gamma - rc(q)\} u'(q) \\ &\quad -(1-\omega) \{a_0^r [u(q) - c(q)] + ru(q) + \gamma\} c'(q). \end{aligned}$$

The function $T(q)$ is shown in Figure 2. Note in particular that $\gamma = 0$ implies $T(0) = 0$ and $\gamma > 0$ implies $T(0) < 0$.

[19] Using the Nash solution is consistent with the spirit of our cooperative equilibrium approach, and avoids having to go into detail concerning the choice variables σ. In any case, it is well known that there is a sense in which the Nash solution is equivalent to having q determined by a game such as the one in Rubinstein (1982); see Osborne and Rubinstein (1990) for a general discussion, or Coles and Wright (1998) for a discussion in the context of monetary theory.

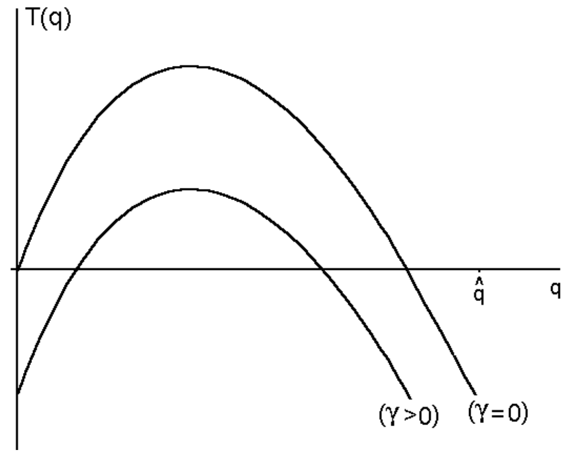

Figure 2: Equilibrium Function $T(q)$

As seen in the Figure, for $\gamma = 0$, there always exists a unique monetary equilibrium, whereas for $\gamma > 0$ there is a critical $\bar{\gamma}^r$ such that if $\gamma \in (0, \bar{\gamma}^r]$ there are an even number of monetary equilibria and if $\gamma \in (\bar{\gamma}^r, \infty)$ there are no monetary equilibria.[20] In the $\gamma = 0$ case it is straightforward to check $\partial q/\partial M < 0$ and $\partial q/\partial r < 0$, and that in the limit as $r \to 0$ we have

$$\frac{u'(q)}{c'(q)} = \frac{(1-\omega)a_0^r}{\omega a_1^r} = \frac{(1-\omega)M}{\omega(1-M)}.$$

It should be clear that $u'(q) = c'(q)$ is the efficient outcome; hence, even when $r \to 0$ deviations from efficiency occur if ω or M is not set appropriately. For example, if $\omega = 1/2$ then we need to set $M = 1/2$ for equilibrium to be efficient when $r \to 0$. This summarizes a few key features of the random matching model, and we now return to our model.

An issue that arises is, to what extent can agents commit to a value of q in the matching process, or, equivalently, when does the bargaining occur? This is

[20] See Schindler et al. (2001) for detailed proofs of this and other claims for the random matching model; related statements made below for the endogenous matching model can be proved using exactly the same kind of arguments.

not an issue with random matching, because agents can only bargain with someone they meet after they meet them, but here (speaking heuristically) agents can try to match with any agent in A at a point in time, and we need to specify if they know q before we determine meetings. One possibility is that after they match agents exit the meeting process and then bargain over q (and are not allowed to re-match with anyone else the same period). Alternatively, we can assume q and meetings are determined jointly. We call the latter case matching with commitment and the former matching without commitment, since no commitment implies that agents cannot agree to q before they trade – it can be "renegotiated" once they exit the matching process.

On the one hand, with no commitment our model is qualitatively the same as the random matching model except for different arrival rates. In particular, q is still given by (8) and equilibria are given by solutions to $T(q) = 0$ with a_z^e substituting for a_z^r. Thus, for $\gamma = 0$ there exists a unique monetary equilibrium, whereas for $\gamma > 0$ there is a critical $\overline{\gamma}^e$ such that if $\gamma \in (0, \overline{\gamma}^e]$ then there exists an even number of monetary equilibria and if $\gamma \in (\overline{\gamma}^e, \infty)$ then there exists no monetary equilibrium. It is easy to show $\overline{\gamma}^e > \overline{\gamma}^r$, so again it is easier to support monetary equilibria with endogenous matching. In the $\gamma = 0$ case we also have $\partial q/\partial M < 0$ and $\partial q/\partial r < 0$, and as $r \to 0$

$$\frac{u'(q)}{c'(q)} = \frac{(1-\omega)\min\left(1, \frac{M}{1-M}\right)}{\omega \min\left(1, \frac{1-M}{M}\right)} = \frac{(1-\omega)M}{\omega(1-M)}.$$

If r is negligible, q is the same in the two models, but for $r > 0$, q is necessarily higher with endogenous matching.[21]

[21] Let $T^r(q)$ and $T^e(q)$ denote the T functions in the two models. At any q such that $T^r(q) = 0$, we can substitute

$$c'(q) = \frac{\omega\left[a_1^r\,(u(q) - c(q)) - \gamma - rc(q)\right]u'(q)}{(1-\omega)\left[a_0^r\,(u(q) - c(q)) + \gamma + ru(q)\right]},$$

into $T^r(q) - T^e(q)$ to show that it is equal in sign to:

$$\begin{aligned} D \quad = \quad & (a_1^r - a_1^e)\,[a_0^r\,(u(q) - c(q)) + ru(q) + \gamma] \\ & - (a_0^r - a_0^e)\,[a_1^r\,(u(q) - c(q)) - rc(q) - \gamma] \end{aligned}$$

On the other hand, with commitment our model is fundamentally different. Consider first the case $M \leq 1/2$, so that $a_1^e = 1$ and $a_0^e = M/(1-M)$. Now an agent with money has all the bargaining power: if he is not getting all the gains from trade in a match, he could deviate and re-match with an unmatched agent without money at a higher q, making himself and his new partner strictly better off. In equilibrium, therefore, agents without money get no surplus: $V_1 - V_0 - c(q) = 0$. Combining this with (6) and (7), q solves

$$S(q) = u(q) - \gamma - (1+r)\,c(q) = 0. \tag{9}$$

By the same reasoning, for $\gamma = 0$ there exists a unique monetary equilibrium, whereas for $\gamma > 0$ there is a critical $\hat{\gamma}^e$ such that if $\gamma \in (0, \hat{\gamma}^e]$ there are an even number of monetary equilibria and if $\gamma \in (\hat{\gamma}^e, \infty)$ there are no monetary equilibria.

Thus, in the case $M < 1/2$, the existence results for our endogenous matching model with commitment are similar to those we found in the other models, although one can show that with commitment q is higher, and also that with commitment q is independent of M since M does not appear in (9). In the other case $M > 1/2$, however, things are quite different: with commitment and $M > 1/2$ there exists no monetary equilibrium. The reason is simply that when agents without money are on the short side of the market they get all the gains to trade: any agent without money producing $q > 0$ can deviate with an unmatched agent holding money at a lower q and make both better off. This bids q down to 0.

The following Proposition summarizes the results for our model (results for the random matching version are the same as for our model without commitment with $\overline{\gamma}^r$ replacing $\overline{\gamma}^e$). Figure 3 shows q in each of the three different models as

$$= \quad ru(q)\,(a_1^r - a_1^e) + rc(q)\,(a_0^r - a_0^e) + \gamma\,[a_0^r - a_0^e + a_1^r - a_1^e] < 0$$

(the last equality uses $a_0^e a_1^r = a_1^e a_0^r$). Hence, at any q such that $T^r(q) = 0$, we have $T^e(q) > 0$, which implies the equilibrium q is lower in the random matching model.

a function of M (assuming $\gamma = 0$): q^e is for endogenous and q^r is for random matching, and in the former case we show the outcome both with commitment and without commitment.

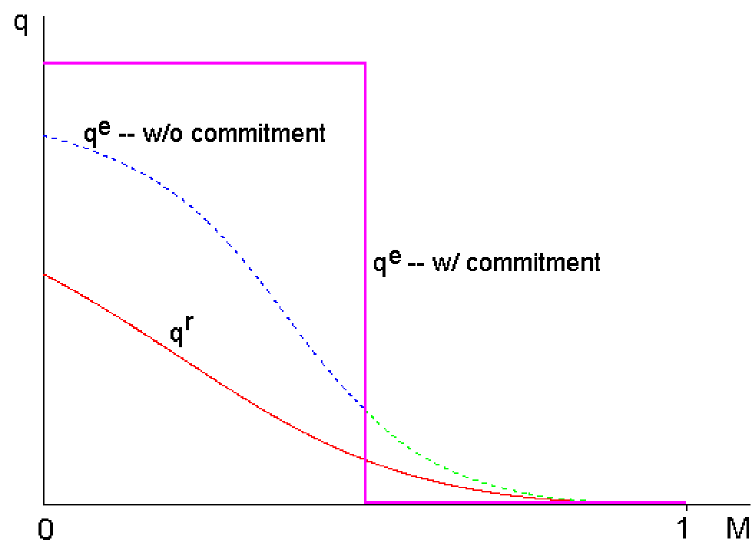

Figure 3: Equilibrium q in the Different Models

Proposition 3 *Consider the endogenous matching model with $\hat{\Phi}$ as described above. With commitment and $M > 1/2$, there is no monetary equilibrium. With commitment and $M < 1/2$, if $\gamma = 0$ there is a unique monetary equilibrium, and if $\gamma > 0$ there is a $\hat{\gamma}^e$ such that $\gamma < \hat{\gamma}^e$ implies there exist an even number of monetary equilibria and $\gamma > \hat{\gamma}^e$ implies there exist no monetary equilibria. Without commitment, for any M, if $\gamma = 0$ there is a unique monetary equilibrium, and if $\gamma > 0$ there is a $\overline{\gamma}^e$ such that $\gamma < \overline{\gamma}^e$ implies there exist an even number of monetary equilibria and $\gamma > \overline{\gamma}^e$ implies there exist no monetary equilibria.*

6 Commodity Money

In this section we use our framework to explore the endogenous determination of which objects play the role of commodity money, an issue dealt with using random search in Kiyotaki and Wright (1989). The environment has a $[0,1]$ continuum of agents with equal proportions of 3 types, and 3 indivisible goods, where now type i agents consume good i and produce good $i+1$ (mod 3), and each agent derives utility $u > 0$ from consumption and 0 disutility from production. Unlike the previous sections, now goods are *storable*, one unit at a time, and the cost per period of storing good i is given by c_i where $c_3 > c_2 > c_1 > 0$. In this model, there is no object designated *ex ante* as money, although since they are storable any of the consumption goods could in principle be used as a medium of exchange. The objective is to see which one(s) emerge as commodity money.

As above, an agent can choose the type he meets but not the exact individual, and we again identify matching with trading: when we say i matches with j we mean they meet and swap inventories. Assume parameters are such that agents consume and produce whenever they get their consumption good, or, eqivalently, that the participation constraints are not binding (this is always true for large u). The state variable is described as follows. Any inventory distribution at time t can be summarized by listing the fractions of type i agents holding good j at t, say $P_t(i,j)$. In fact, since any agent of type i who gets good i immediately consumes it and produces good $i+1$, we have $P_t(i,i) = 0$, and the state is completely characterized by $p_t = (p_{1t}, p_{2t}, p_{3t})$ where $p_{it} = P_t(i, i+1)$ is the fraction of type i holding their production good $i+1$.

We are interested in active (trade at every date) equilibria, with stationary, memoryless, deterministic and symmetric matching rules. Thus, the partition is a time- and history-invariant function of the state, $\theta_t = \Phi(p_t)$, with the property that agents of the same type all match the same way. This implies that the state variable will be such that each p_{it} is either 1 or 0, since either all type i agents or no type i agents hold good $i+1$; thus p_t lies in the 8 element set $\Delta = \{(1,1,1),(1,1,0)...(0,0,0)\}$. Any matching rule in the relevant class maps each $p \in \Delta$ into a partition $\theta = \Phi(p)$ which generates (via trade, consumption and production) a new state $p' \in \Delta$, which generates a new partition $\theta' = \Phi(p')$, and so on. This yields a path through Δ, which clearly has to *cycle*, because Δ is finite.

Supposing, for instance, that we start at $p = (1,1,1)$, we have three (active, symmetric) choices for $\theta = \Phi(p)$: either $\theta = [\{1,2\},\{3\}]$, $\theta = [\{2,3\},\{1\}]$, or $\theta = [\{3,1\},\{2\}]$. As 4 shows, each of these generates a new state, p'; e.g. $\theta = [\{2,3\},\{1\}]$ implies $p' = (1,0,1)$ (type 2 acquires good 1 and stores it, while type 3 acquires good 3, consumes it, and produces a new unit of good 1). From $p' = (1,0,1)$, since 2 and 3 have the same good, there are only two relevant choices: $\theta' = [\{1,2\},\{3\}]$ or $\theta' = [\{3,1\},\{2\}]$. The former leads back to where we started, $p'' = (1,1,1) = p$, while the latter leads to a new state from which can continue the process. Any possible matching rule Φ in the relevant class is represented by one of the paths in Figure 4, all of which cycle in anywhere from 2 to 7 periods.[22]

It is convenient to introduce the notation $p_t \overset{i,j}{\rightarrow} p_{t+1}$ to indicate that a matching rule prescribes in state p_t type i agents meet type j agents, and the state transits to p_{t+1}. Thus, the 2 period cycle discussed in the previous paragraph can be represented by

$$(1,1,1) \overset{2,3}{\rightarrow} (1,0,1) \overset{1,2}{\rightarrow} (1,1,1), \tag{10}$$

which is the path on the far right in Figure 4. As we said, any Φ in the relevant class generates some such cycle, and to determine if it is an equilibrium we need

[22] There are at most 7 period cycles, even though Δ contains 8 elements, because there is no way to get to $p = (0,0,0)$ from any other p using a relevant Φ. Note also that all paths go through $p = (1,1,1)$, except the 6 period cycles starting from $(0,1,1)$, $(1,1,0)$, and $(1,0,1)$ in the figure.

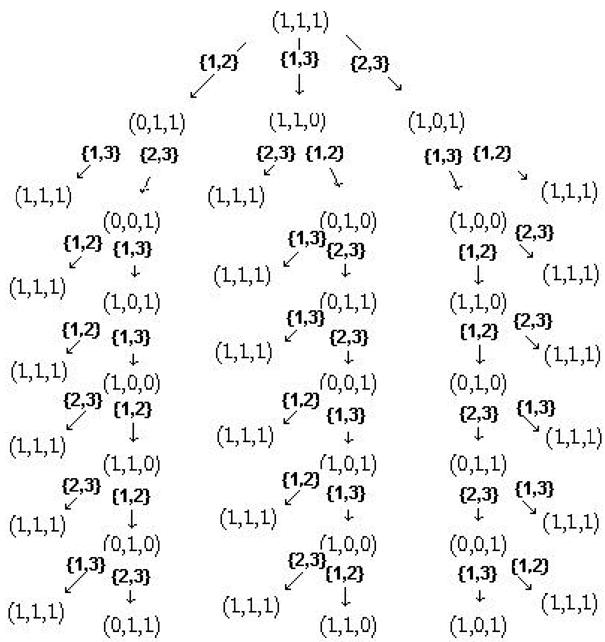

Figure 4: Evolution of p

to check, at each point of this cycle, if any individual or pair have a profitable deviation from matching as prescribed by Φ.

To keep things manageable here we only look for *strict* equilibria, in the following sense: we assume that a coalition will deviate from a candidate equilibrium if one agent in the coalition is strictly better off and the other agent is at least weakly better off. We will show that there is one and only one such equilibrium, and it corresponds to the meeting pattern shown in (10), where type 2 agents acquire good 1 from type 3 in one period, use it to trade with type 1 agents the next period, and then repeat the pattern.[23] We call this the

[23] We do not know if there are other equilibria that are not strict equilibria.

fundamental equilibrium since, as in Kiyotaki and Wright (1989), every exchange involves either agents trading for their consumption goods or trading a higher storage cost good for a lower storage cost good. It also that implies the lowest storage cost good 1 serves as commodity money.

Proposition 4 *In the endogenous matching commodity money model, the fundamental equilibrium exists and is unique within the class of active, deterministic, stationary, symmetric, strict equilibria.*

We sketch the argument here and provide details in the Appendix. Lemmas A.1 to A.3 establish existence by showing there are no profitable deviations from the matching pattern implied by the fundamental equilibrium. We then establish uniqueness in the relevant class by ruling out other candidates as follows. Lemma A.5 shows that in no equilibrium will types 2 and 3 match at $p = (0, 1, 1)$; Lemma A.6 shows in no equilibrium will types 1 and 2 match at $p = (1, 1, 0)$; and Lemma A.7 shows that in no equilibrium will types 3 and 1 match at $(1, 0, 1)$. As seen in Figure 4, this rules out all possibilities except for the fundamental equilibrium described by (10), and two other 2 period cycles: $(1, 1, 1) \overset{1,2}{\to} (0, 1, 1) \overset{1,3}{\to} (1, 1, 1)$ and $(1, 1, 1) \overset{1,3}{\to} (1, 1, 0) \overset{2,3}{\to} (1, 1, 1)$. Lemma A.4 rules these out directly by constructing profitable deviating coalitions.

To illustrate the construction, consider the first of the 2 cycles to be ruled out, where type 1 agents acquire good 3 from type 2 and use it next period to trade with type 3 (which this implies good 3 is the medium of exchange). Consider a deviation at $(0, 1, 1)$, where rather than staying with the equilibrium, a type 2 and type 3 agent match, implying the former gives good 3 to the latter for good 1. Note that since agents are small, this does not change the evolution of the aggregate state. The deviating type 3 agent is indifferent since he receives good 3 in any case. The type 2 agent is strictly better off in the period of the deviation, since he economizes on storage costs by paying c_1 instead of c_3, but now we need to work through the implications for the future.

Next period, the deviating type 2 finds himself holding good 1 when the aggregate state is $(1,1,1)$. In this situation, *if* he could exchange the good 1 with a type 1 agent holding good 2, he could generate a short run gain from consumption with no long term consequences, since the following period he is back on the candidate equilibrium path (i.e., holding good 3). Hence, if he can make this trade, the initial deviation is profitable. We claim that he always could make the requisite trade. The reason is that any type 1 agent, if he goes along with it, can consume that period, stay in autarchy the next period, and be back on the candidate equilibrium path the following period, which improves his payoff relative to the candidate equilibrium (see the Appendix for more details). Thus, in any equilibrium, a type 1 will necessarily agree to the requisite trade, and this is sufficient to guarantee that our type 2 agent finds the original deviation profitable.[24]

Similar arguments used throughout the Appendix allow us to conclude that the fundamental equilibrium exists and is unique in the class under consideration. By comparison, in the random matching environment in Kiyotaki and Wright (1989), a fundamental equilibrium exists iff $c_3 - c_2 \geq \frac{1}{2}\beta u$, while a so-called speculative equilibrium where both goods 1 and 3 serve as money exists iff $c_3 - c_2 \leq (\sqrt{2}-1)\beta u$. One difference is that we have existence for all parameter values, while in that version there is a range of parameters where there are no equilibria in the relevant class (although there do exist mixed strategy, asymmetric, and nonstationary equilibria – see Aiyagrai and Wallace [1991] and Kehoe et al. [1993]). Perhaps more interestingly, in our model the fundamental equilibrium is unique in the relevant class.

Intuitively, speculative behavior in Kiyotaki and Wright (1989) is due to randomness. What happens in that model is that type 1 agents trade good 2 for the higher storage cost good 3 because they know there is a good chance

[24] Note that we are not using double deviations here: the type 2 agent is simply using what he knows other agents will do in equilibrium once we are off the equilibrium path.

of meeting a type 3 agent with good 1: they are willing to sacrifice storability because for them good 3 provides an increased probability of consumption. This does not happen in our model, since agents do not meet randomly, and so we eliminate the speculative motive for holding a high storage cost object as a medium of exchange. Whether or not one thinks speculative trading in this sense is interesting, it seems important to know what drives it.[25] More generally, we conclude that it is possible to develop an endogenous matching version of the commodity money framework that has been studied previously with random matching, even though some of the results are rather different.

7 Conclusion

We have studied models based on bilateral trade that are similar in spirit to earlier search models of money, with one essential difference: we endogenize the meeting process. We showed how to apply the framework to several issues in monetary economics. The approach is tractable, and sometimes much more tractable than random matching models. While some of our substantive results are similar to what one finds in the previous literature, others are quite different. We think the approach provides new insights into a variety of questions, including the relationship between money and memory, the dependence of monetary equilibria on the number of agents, the dependence of the value of money on the nature of matching and bargaining, and the existence of speculative trading patterns, for example. We hope that endogenous matching will have many other applications in the future.

[25] A related example where randomness is critical is the result of Cavalcanti and Wallace (1999) that an inside money regime is superior to an outside money regime – a result which does not hold in a version of their model with endogenous meetings. This does not mean that such results are uninteresting, only that one has to justify the randomness somehow (see Wallace [2001]).

8 Appendix

We being with a preliminary result that simplifies the types of things we have to check, and then prove the lemmas discussed in Section 6 in order to establish the existence and uniqueness of the fundamental commodity money equilibrium equilibrium.

Lemma A.0 Given any matching rule Φ, we know the following: (a) any agent who is supposed to store his production good from t to $t+1$ is willing to deviate and trade with someone holding his consumption good; and (b) if there is any profitable deviation from Φ then there is a profitable deviation that involves storing the same good for no more than k periods where k is the length of the cycle in p implied by Φ.

Proof. Part (a) is obvious: if i trades for his consumption good he immediately consumes and produces a new unit of his production good, which increases his current payoff and leaves him with the same continuation value. Part (b) is only slightly harder. Suppose i has a profitable deviation at t that involves him acquiring good j at some date t in state p_t. Clearly this cannot involve storing good j forever, by the assumption that the participation constraints are not binding. So there has to be a date when it is traded, say t'. If $t' > t+k$, then there was a date $t'' < t'$ at which the state was the same, $p_{t'} = p_{t''}$, since p cylces in k periods. Since other agents are using stationary strategies in equilibrium he could have made the same trade at t'', and since he discounts this is better than the original deviation. ■

Lemma A.1. Consider the candidate equilibrium described by fundamental trade, $(1,1,1) \overset{2,3}{\rightarrow} (1,0,1) \overset{1,2}{\rightarrow} (1,1,1)$; there is no profitable deviation that involves type 3 accepting good 2.

Proof. We use the following notation: $V_i(p_1, p_2, p_3)$ is the value to type i in state (p_1, p_2, p_3) from following the candidate equilibrium while $V_i^D(p_1, p_2, p_3)$

is the value to a deviation. Thus, on the candidate equilibrium path type 3's payoffs are $V_3(1,1,1) = u - c_1 + \beta V_3(1,0,1)$ and $V_3(1,0,1) = -c_1 + \beta V_3(1,1,1)$.

First consider the deviation where a type 3 agent accepts good 2 in state $(1,1,1)$. In the following period his potential options are to trade for good 1 or store good 2, since there is no good 3 at $p = (1,0,1)$. Consider the former case. We do not know if he actually could trade for good 1, but if he could he is back on the equilibrium path in two periods with $V_3^D(1,1,1) - V_3(1,1,1) = -(c_2 - \beta c_1) - u < 0$. Now consider the latter case of storing good 2. There are two options for the following period when $p = (1,1,1)$: trade for good 1 or trade for good 3 (continuing to store good 2 for another period can be ignored by part b of Lemma A.0). In each case, the type 3 agent is back on the equilibrium path in three periods but worse off – e.g. if he trades for good 3, $V_3^D(1,1,1) - V_3(1,1,1) = -(1+\beta)(c_2 - c_1) - u < 0$.

Second, consider the deviation where type 3 accepts good 2 in state $(1,0,1)$. In the following period his potential options are to trade for his production good 1, trade for his consumption good 3, or store the good 2. In the former two cases, he is back on the equilibrium path in two periods but worse off – e.g. trading for his consumption good yields $V_3^D(1,0,1) - V_3(1,0,1) = -(c_2 - c_1) < 0$. In the remaining case, the only relevant option is to trade for good 1 in state $(1,0,1)$ (since there is no good 3 available in this state, and storing good 2 for another period can be ignored by part b of Lemma A.0). This again puts him back on the equilibrium path with lower utility – i.e. $V_3^D(1,0,1) - V_3(1,0,1) = -(1+\beta)(c_2 - c_1) - \beta u < 0$. ■

Lemma A.2. In the candidate equilibrium described by fundamental trade, there is no profitable deviation that involves type 1 accepting good 3.

Proof. First, consider the deviation where type 1 accepts good 3 in state $(1,1,1)$. In the following period his potential options are to trade for good 1, trade for good 2, or store good 3. In the former two cases he is back on the equilibrium path in two periods but worse off – e.g. trading for good 1 yields $V_1^D(1,1,1) - V_1(1,1,1) = -(c_3 - c_2) < 0$. In the latter case, there are two potential options in $(1,1,1)$: trade for good 1 or trade for good 2 (continuing to store good 3 can be ignored by part b of Lemma A.0). In both cases, he is back on the equilibrium path in three periods but worse off – e.g. trading for good 1 yields $V_1^D(1,1,1) - V_1(1,1,1) = -(1+\beta)(c_3 - c_2) - \beta(1-\beta)u < 0$.

Second, consider the deviation where a type 1 accepts good 3 in state $(1,0,1)$. In the following period he could either trade for good 1, trade for good 2, or store good 3. In the former two cases, he is back on the equilibrium path in two periods but worse off – e.g. trading for good 1 yields $V_1^D(1,0,1) - V_1(1,0,1) = -(c_3 - c_2) - (1-\beta)u < 0$. In the last case, he could potentially trade for good 1 or good 2 in state $(1,0,1)$ (continuing to store good 3 can again be ignored). This again puts him back on the equilibrium path with lower utility – e.g. trading for good 1 yields $V_1^D(1,0,1) - V_1(1,0,1) = -(1+\beta)(c_3 - c_2) - u < 0$. ■

Lemma A.3. In the candidate equilibrium described by fundamental trade, there are no profitable deviations by agents of types 2 and 3 in state $(1,1,1)$ or by agents of types 1 and 2 in state $(1,0,1)$.

Proof. First, consider the state $p = (1,1,1)$. One set of possible deviations is by pairs. But type 1 will not agree to match with type 2 by Lemma A.2, and type 3 will not agree to match with type 1 by Lemma A.1. The other possible deviations in $(1,1,1)$ are unilateral deviations to autarchy by type 2 or 3 (type 1 is already in autarchy in this state). Suppose a type 2 deviates to autarchy. This means he enters $(1,0,1)$ holding good 3 with three potential options: store good 3 one more period, trade with a type 1, or trade with a type 3. If he store good 3 one more period he will be back on the equilibrium path with lower utility – i.e. $V_2^D(1,1,1) - V_2(1,1,1) = -(c_3 - c_1) - \beta u < 0$. By Lemma A.2, type 1 agents will not trade for good 3 at $(1,0,1)$, so the second possibility cannot occur.

The last case results in the deviating type 2 holding good 1 in state $(1,1,1)$. He then has two options. First, he can store good 1 another period, which puts him back on the equilibrium path with a lower payoff. Second, he can trade good 1 to a type 1 and consume, entering state $(1,0,1)$ holding his production good. This would put him back to the two options he had in $(1,0,1)$ with an overall lower payoff. All other deviations by type 2 are ruled out by part b of Lemma A.0. Hence, type 2 will not deviate to autarchy. Clearly type 3 will not deviate to autarchy since $V_3^D(1,1,1) - V_3(1,1,1) = -u < 0$. This exhausts all possible unilateral as well as bilateral deviations in this state.

Now consider possible deviations in $(1,0,1)$. A type 1 cannot match with type 3 by Lemma A.1, and types 2 and 3 are holding the same good, so we need only consider deviations to autarchy. It is not in the interest of type 1 to deviate to autarchy since $V_1^D(1,0,1) - V_1(1,0,1) = -u < 0$. Suppose a type 2 chooses autarchy, which means he enters $(1,1,1)$ holding good 1 with two potential options: store it for another period or trade with a type 1 (type 3's are already holding good 1). In the former case, $V_2^D(1,0,1) - V_2(1,0,1) = -c_1 - (u - c_3) < 0$. In the latter case, type 2 enters $(1,0,1)$ holding good 3. But this is the state that we showed above was suboptimal for a deviating type 2. Hence, provided along the deviation path to that state the type 2 is worse off, then he is worse off overall. But along the deviation path starting in autarchy, the deviating type 2 receives $-c_1 + \beta(u - c_3)$ while along the equilibrium path starting in $(1,0,1)$ he receives $u - c_3 - \beta c_1$, so he is indeed worse off. This exhausts all possible deviations in $(1,0,1)$. ■

Lemma A.4. Consider the candidates described by $(1,1,1) \overset{1,2}{\to} (0,1,1) \overset{1,3}{\to} (1,1,1)$ and by $(1,1,1) \overset{1,3}{\to} (1,1,0) \overset{2,3}{\to} (1,1,1)$. These are not equilibria.

Proof. In the first case, along the candidate equilibrium path type 2 agents are in autarchy in state $(0,1,1)$. Consider a deviation by a type 2 and 3 in this state. Since the type 3 agent consumes in either case he is willing to go along, and we only have to show the type 2 is better off by acquiring good 1. He would be better off for sure if he could get good 2 from a type 1 next period at $(1,1,1)$, since then $V_2^D(0,1,1) - V_2(0,1,1) = c_3 - c_1 > 0$. But we claim that in any equilibrium he can get good 2 from type 1 in exchange for good 1 at $(1,1,1)$, by part a of Lemma A.0. Thus, the deviation in question is profitable for type 2.

Now consider the second case, and suppose a type 3 deviates and matches with type 2 at state $(1,1,1)$. The type 3 can always stay in autarchy in $(1,1,0)$, which implies $V_3^D(1,1,1) - V_3(1,1,1) = (1-\beta)u + c_2 - c_1 > 0$, so it is profitable for him. The type 2 deviator, if he could match with a type 1 in state $(1,1,0)$, would realize $V_2^D(1,1,1) - V_2(1,1,1) = c_3 - c_1 > 0$. But again we know that in any equilibrium type 1 agrees to such a trade by part a of Lemma A.0. Hence type 2 agrees to the deviation with the type 3. ■

Lemma A.5. There is no equilibrium in which type 2 agents match with type 3 agents in state $(0,1,1)$.

Proof. Suppose type 2 and 3 agents match in $(0,1,1)$, which results in $(0,0,1)$ the following period. At $(0,0,1)$, since 2 and 3 have the same good, there are two active partitions, $[\{1,2\},\{3\}]$ and $[\{1,3\},\{2\}]$. The former implies $(1,1,1)$ next period while the latter implies $(1,0,1)$. The important thing to note is that both cases along the candidate equilibrium path result in type 1 holding his production good two periods after $(0,1,1)$. Consider, then, the following deviation. At $(0,1,1)$, a type 1 holding good 3 matches with a type 3 holding good 1. By part a of Lemma A.0, type 3 accepts. The deviating type 1 can consume at $(0,1,1)$, stay in autarchy at $(0,0,1)$, and return to the candidate equilibrium path $(1,1,1)$, which yields $V_1^D(0,1,1) - V_1(0,1,1) = (1-\beta)u + c_3 - c_2 > 0$. ■

Lemma A.6. There is no equilibrium in which type 1 agents match with type 2 agents in state $(1,1,0)$.

Proof. Suppose type 1 and 2 agents match in $(1,1,0)$, which results in $(0,1,0)$ the following period, where type 1 have accepted a higher storage cost good. At $(0,1,0)$ there are two active partitions: $[\{1,3\},\{2\}]$ and $[\{2,3\},\{1\}]$. In the case of $[\{1,3\},\{2\}]$, a type 1 agent could deviate to autarchy for two periods at $(1,1,0)$, obtaining $V_1^D(1,1,0) - V_1(1,1,0) = c_3 - c_2 > 0$. In the case of $[\{2,3\},\{1\}]$, a type 1 agent could deviate to autarchy at $(1,1,0)$ and trade good 2 to a type 2 agent in state $(0,1,0)$ in any equilibrium by part a of Lemma A.0. This yields $V_1^D(1,1,0) - V_1(0,1,1) = c_3 - c_2 > 0$. ■

Lemma A.7. There is no equilibrium in which type 3 agents match with type 1 agents in state $(1,0,1)$.

Proof. Suppose type 3 and 1 agents match in $(1,0,1)$, which results in $(1,0,0)$ the following period, where type 3 have accepted a higher storage cost good. At $(1,0,0)$ there are two active partitions: $[\{1,2\},\{3\}]$ and $[\{2,3\},\{1\}]$. In the case of $[\{1,2\},\{3\}]$, a type 3 agent could deviate to autarchy at $(1,0,1)$ and then trade good 1 with a type 1 agent at $(1,0,0)$ by part a of Lemma A.0. This puts him back on the candidate equilibrium path with $V_3^D(1,0,1) - V_3(1,0,1) = c_2 - c_1 > 0$. In the case of $[\{2,3\},\{1\}]$, a type 3 agent can deviate to autarchy for two periods at $(1,0,1)$, realizing $V_3^D(1,0,1) - V_3(1,0,1) = c_2 - c_1 > 0$. ■

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