On the Chain-Multiplier Effect of Consumption Demand in a Multi-Stage-Fabrication Economy with Inventories

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Abstract

Due to lack of models that can feature both output- and input-inventories simultaneously, several well known puzzles pertaining to inventory fluctuations and the business cycle have not been well explained by dynamic optimization theory. By presenting a general equilibrium, multi-stage production model of inventories with separate decisions to order, use, and stock input materials and to produce, sell, and store finished output, this paper offers not only a model of input-output inventories but also a neoclassical perspective on the theory of aggregate demand. It shows that due to production/delivery lags, firms opt to hold both output- and input-inventories so as to guard against demand uncertainty at all stages of production. As a result, not only is production more volatile than sales but also is input-ordering more volatile than input-usages, giving rise to a chain-multiplier mechanism that propagates and amplifies demand shocks at downstream towards upstream via input-output linkages. This multiplier effect induced by precautionary inventory investment at each production stage can explain several long-standing puzzles of the business cycle documented in the inventory literature.

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1. Introduction

Given the sheer magnitude of volatility in inventory investment and its disproportionately large contribution to output fluctuations, it is fair to say that understanding inventory behavior constitutes the ultimate frontier of business cycle research.[1] There are three long-standing puzzles associated with inventory behavior: 1) output production is more variable than output sales – suggesting that inventory investment is procyclical with respect to sales; 2) input orders are more variable than input usages; and 3) input inventories – defined here as raw materials and work-in-progress – are essentially uncorrelated with, yet far more volatile than output inventories.[2] While there is no shortage in supply of explanations for the first puzzle, the literature on resolving the second and third puzzle is remarkably thin, despite the fact that more than two thirds of total inventory investment in the US is contributed by input inventories (e.g., see Blinder and Maccini, 1991).

The three prominent features of production and inventory behavior are puzzling because we usually think that a firm has not only an incentive to smooth production in order to reduce production costs when facing uncertainty in output sales, but also an incentive to smooth input production/orders when factor-demand/input-usage is also uncertain. Furthermore, because output inventories are far more liquid and hence far more effective than input inventories in meeting uncertain finished-goods demand, it is apparently not profit maximizing for firms to keep most inventories in the form of raw materials and work-in-progress rather than in the form of finished output.

This paper conjectures that time lags involved in production/delivery at each stage of a input-output chain of production could be the prime culprit in explaining these puzzles. In any economy the ultimate goal of production is always to meet final consumption demand, regardless how many intermediate stages there are or how "round-about" the production process is. Because of production/delivery lags, however, output must be produced in advance before demand uncertainty is resolved and input must be ordered in advance before production takes place. Hence, to avoid stockouts, it is conceivable that profit-seeking firms opt to hold inventories at each stage of the production process in order to fully

[1] Inventory investment accounts for only less than one percent of GNP in mean yet the drop in inventory investment has accounted for 87 percent of the drop in GNP during the average postwar recession in the United States (e.g., see Blinder and Maccini, 1991).

[2] E.g., see Blinder (1986), Feldstein and Auerbach (1976), and Humphreys, Maccini and Schuh (2001).

guard against demand uncertainty. This may result in a precautionary motive for inventory investment in both output and input. Precautionary inventory accumulation may cause production to be more volatile than sales on the output side, and orders to be more volatile than usage on the input side. Hence, uncertainty in final demand at the downstream may be transmitted backwards to upstream industries and during the transmission the uncertainty in demand may be even magnified repeatedly at each stage of the production chain, as the input demand of one firm is also the output sales of another firm.

I set up a general-equilibrium, rational-expectations, multi-stage-fabrication model of production to put this conjecture to scrutiny. I show that with production (or delivery) lags at each stage of production, uncertainty in demand at the downstream indeed generates a motive for holding both output and input inventories by all upstream firms in order to avoid possible stockouts. Such a motive is sufficient for explaining the existence of both final-goods inventories and intermediate-goods inventories at all stages of production observed in the actual economy. Furthermore, if shocks to final consumption demand at the downstream are serially correlated (i.e., they are forecastable to certain degree), then production/orders become more variable than sales/usages, which may cause the volatility of economic activities to increase towards upstream firms. Consequently, the variance in consumption demand is not only propagated backwards to upstream industries via input-output linkages, but also amplified repeatedly along the production chain. This explains why in the US the variance of production at upstream industries is usually about 50 percent larger than the variance of production at downstream industries.[3]

It has long been well known that input inventories are far larger and more volatile than finished goods inventories (e.g., see Feldstein and Auerbach, 1976). Despite this, yet the vast bulk of theoretical work in the inventory literature has focused almost exclusively on finished goods inventories (e.g., see Abel, 1985; Amihud and Mendelson, 1983; Bils and Kahn, 2000; Blinder, 1986; Blinder and Maccini, 1991; Kahn, 1987, 1992; Karlin and Carr, 1962; Maccini and Zabel,

[3]For example, according to monthly US production data for manufacturing sector (1960:01 - 1995:12), the variance of growth rate of production for raw materials is 0.00014 and that for intermediate goods is 0.00009, implying a variance ratio of 1.6. Similarly, according to monthly US data for housing sector (1968:01 - 1996:08), the variance of housing starts in total units is 130321 and the variance of housing completion in total units is 80770, implying a variance ratio of 1.6.

1996; Reagan, 1982; and Scheinkman and Schechtman, 1983; among others).[4] As a consequence of neglecting input inventories, the role of inventories in amplifying and propagating the business cycle has not been well understood.

This paper is a first step towards understanding and modeling input- and output-inventory behavior simultaneously in general equilibrium. By extending the partial equilibrium, single-stage fabrication model of Kahn (1987) to a general equilibrium, multi-stage fabrication model with separate decisions to order, use, and stock input materials and to produce, sell, and store finished goods, it provides a theoretical foundation for the important empirical works of Feldstein and Auerbach (1976), Ramey (1989), and especially Humphreys, Maccini and Schuh (2001). In doing so, it offers not only a model of input-output inventories but also a new perspective on the theory of aggregate demand and the propagation of the business cycle. The neoclassical theory reveals not only that small disturbances to final demand can generate potentially large aggregate fluctuations via a chain-multiplier induced by precautionary inventory investment, but also that the multiplier mechanism reflects optimal responses of a perfectly functioning market economy towards exogenous uncertainty. Hence, although inventory investment appears to be highly destabilizing, government interventions attempting to reduce the volatility of aggregate output via distortionary monetary or fiscal policies may prove counter-productive, unless they are able to target specifically on reducing the demand uncertainty and lags involved in firms' production decisions.[5]

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 derives equilibrium decision rules. Section 4 analyzes the empirical implications of the model for inventory fluctuations and the business cycle. Section 5 concludes the paper.

[4] This is also true for much empirical work on inventories, most notably, see Blanchard (1983), Eichenbaum (1984, 1989), Ramey (1991) and West (1986).

[5] This, however, by no means implies that in the real world all government interventions are necessarily undesirable since market failures may exacerbate the neoclassical multiplier effect of inventories. But the neoclassical theory does show the potential danger of policy interventions based solely on output volatilities.

2. A General Equilibrium Model of Input and Output Inventories

Consider an economy similar to that described by Humphreys, Maccini, and Schuh (2001) in which most firms produce goods in stages. A typical firm orders input materials from an upstream supplier and uses them to produce intermediate goods, which are combined with other factor inputs to produce finished goods. Many firms sell their finished goods to downstream firms as input materials. Figure 1 presents a schematic illustration of the model (a near replication of Humphreys, Maccini, and Schuh, 2001). The diagram focuses on flows through the multi-stage-production process employed by a representative agent (or planner) to transform raw materials (or labor) into intermediate goods inventories (or work-in-progress) and then further into output inventories (finished goods) to meet final demand of consumer (or another downstream firm). More precisely, at the beginning of a period, say $t-2$, based on expected future demand for finished goods (c) the agent orders raw materials (n) in order to produce intermediate goods (y_2). Since production takes one period, y_2 is not available until period $t-1$. In this next period, based on updated information about final demand (c) the agent decides how much of the intermediate goods to be kept as inventories (s_2) and how much of them (in the amount m) to be combined with other factor inputs (such as l) to produce finished goods (y_1). Again due to time lags in production, y_1 is not available until period t. In this period, the uncertainty in final demand for finished goods (c) is resolved, the agent then decides on how much of the finished goods to be kept as output inventories (s_1) and how much to be consumed (c).

This model can be easily decentralized or mapped to a competitive market economy in which consumers choose labor supply and final demand given market prices for output and factor payments, and downstream firms choose orders, usages, and storages of intermediated goods, as well as production and inventory investment in finished goods given expected prices of finished goods and input materials. Upstream suppliers choose production of intermediate goods and demand for raw materials given expected output price for intermediate goods and input price for raw materials. More stages of production can be added easily if needed.

In this paper I adopt the representative agent version of the model (by the welfare theorem, competitive-market interpretations may be applied also to enhance understanding) in which a social planner chooses consumption plan (c), supply of raw materials and labor (n, l), input usages for intermediate goods (m), and inventory holdings for final and intermediate goods (s_1, s_2) to solve

$$\max_{\{n_t\}} E_{-2} \left\{ \max_{\{m_t, l_t, s_{2t}\}} E_{-1} \left\{ \max_{\{c_t, s_{1t}\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\theta_t u(c_t) - w l_t - a n_t \right] \right\} \right\} \right\}$$

subject to resource constraints in the finished goods market and the intermediate goods market (denoting δ_j as the depreciation rate for inventory type j):

$$\begin{aligned} c_t + s_{1t} &\leq y_{1t} + (1 - \delta_1) s_{1t-1} \\ m_t + s_{2t} &\leq y_{2t} + (1 - \delta_2) s_{2t-1} \end{aligned}$$

production technologies for finished goods and intermediate goods:

$$\begin{aligned} y_{1t} &= f(m_t, l_t) \\ y_{2t} &= g(n_t) \end{aligned}$$

and nonnegativity constraints on final-goods inventories and intermediate-goods inventories:

$$\begin{aligned} s_{1t} &\geq 0 \\ s_{2t} &\geq 0. \end{aligned}$$

The operator E_j in the objective function denotes expectation based on information available in period j, and θ in the utility function represents random shifts in consumption demand (preference shocks) and it is assumed to follows a stationary AR(1) process:

$$\theta_t = \gamma + \rho \theta_{t-1} + \varepsilon_t,$$

where ε is *i.i.d* with zero mean.

To ensure that θ represents genuine demand shocks that affect consumption goods demand only, linear utility functions in terms of the costs of leisure (labor supply), $wl + an$, are adopted following Hansen's (1985) indivisible labor model, so that the marginal cost of labor supply is constant and hence θ does not shift labor supply curves. Consequently, equilibrium employment in this model is determined solely by labor demand which responds to changes in output demand or output prices.

In order to yield simple closed-form solutions, it is also assumed that the utility function on consumption goods is quadratic,[6]

$$\theta u(c) = \theta c - \frac{1}{2}c^2,$$

and that the two production functions satisfy constant returns to scale:

$$g(n) = n$$

$$f(m, l) = \min\{m, l\}.$$

That is, one unit of raw materials (labor n) can be transformed into one unit of intermediate goods at the marginal cost of a, and one unit of intermediate goods (m) can be combined with one unit of labor (l) to be transformed into one unit of consumption goods at the marginal cost of $w + \lambda^m$, where λ^m denotes the shadow price (Lagrangian multiplier) of intermediate goods. The shadow price (Lagrangian multiplier) of final goods is simply the marginal utility of consumption, $\lambda^c = \theta - c$. For simplicity without loss of generality, I assume that the depreciation rates for both types of inventory goods are zero, $\delta_1 = \delta_2 = 0$. Notice that in equilibrium both forms of inventories may coexist, because without knowing precisely the final demand of consumers the planner may opt not to transform all the intermediate goods inventories into finished goods inventories if the expected demand for finished goods next period is low. This can postpone the labor cost of producing finished goods.

The first order conditions for optimal plans for the variables $\{c, s_1, m, s_2, n\}$ are given respectively by (note: cost minimization by downstream firms implies $l = m$):

$$\theta_t - c_t = \lambda_t^c \tag{1}$$

$$\lambda_t^c = \beta E_t \lambda_{t+1}^c + \pi_t^c \tag{2}$$

$$w + \lambda_t^m = E_{t-1}\lambda_t^c \tag{3}$$

$$\lambda_t^m = \beta E_{t-1}\lambda_{t+1}^m + \pi_t^m \tag{4}$$

$$a = E_{t-2}\lambda_t^m \tag{5}$$

plus two market clearing conditions,

$$c_t + s_{1t} = y_{1t} + s_{1t-1} \tag{6}$$

[6] This assumption is not necessary for closed-form solutions but it simplifies expressions.

$$y_{1t} + s_{2t} = y_{2t} + s_{2t-1} \tag{7}$$

and two complementary slackness (Kuhn-Tucker) conditions associated with the nonnegativity constraints on inventories,

$$\pi_t^c s_{1t} = 0$$

$$\pi_t^m s_{2t} = 0.$$

Equation (1) says that the optimal consumption level is chosen to the point where the marginal utility of consumption equals its marginal cost (the shadow price of finished goods). Equation (2) says that the optimal inventory holding of finished goods inventories is chosen to the point such that the cost of obtaining one extra unit of inventory (the shadow price of finished goods) equals the discounted benefit of having one extra unit of inventory available next period ($\beta E_t \lambda_{t+1}^c$) plus the value gained by relaxing the slackness constraint (π^c). Equation (3) says that the optimal level of production for finished goods is chosen based on information available in period $t-1$ to the point where the marginal cost of production ($w + \lambda_t^m$) – both labor and material costs – equals the expected value of marginal product ($E_{t-1}\lambda_t^c$). Equation (4) says that the optimal level of intermediate goods inventory is chosen such that the cost of obtaining one extra unit of intermediate-goods inventory (λ^m) equals the discounted benefit of having one extra unit of such inventory available next period ($\beta E_{t-1}\lambda_{t+1}^m$) plus the value gained by relaxing the slackness constraint (π^m). Equation (5) says that the optimal level of production for intermediate goods is chosen based on information in period $t-2$ to the point so that the marginal input cost for raw materials (a) equals the expected value of marginal product ($E_{t-2}\lambda_t^m$).

3. Deriving Equilibrium Decision Rules

Given output price (λ_t^c) and the availability of finished goods ($y_{1t} + s_{1t-1}$) as well as the realized demand shocks (θ) at the beginning of period t, the planner chooses consumption (c_t) and inventory holdings (s_{1t}) to maximize utility. Consider two possible cases:

Case A_1. The demand shock is below "normal", hence the existing supply ($y_{1t} + s_{1t-1}$) is sufficient to meet the utility maximizing level of consumption demand. In this case, we have $s_{1t} \geq 0$ and $\pi_t^c = 0$. Equation (2) implies that the shadow price of finished goods is a constant, $\lambda_t^c = \bar{\lambda}$, implying that goods price is endogenously downward sticky when inventories are allowed.[7] The intuition for this result is straightforward. From the point of view of the representative agent, a low θ implies a low marginal utility of consumption. Hence it is optimal not to consume now but to hold the excess supply as inventories so as to be able to consume more in the future when θ maybe high. Translating this into the language of a competitive market economy, this implies that when demand is low, although firms can potentially dump all excess supply (inventory holdings) in the market to push down equilibrium price, this is not profit maximizing since inventories would be sold at equilibrium price below marginal cost. Hence it is optimal not to sell excess supply but to hold them into the future when market price or demand becomes high. This results in equilibrium market price being not responsive to changes in demand when demand lies below certain threshold level (as inventories effectively render the supply curve perfectly elastic), offering a plausible explanation for price stickiness (see Blinder, 1982, for more discussions and references therein on this point).

Equation (1) then implies that the optimal consumption level is given by

$$c_t = \theta_t - \bar{\lambda},$$

and equation (6) implies that the optimal level of inventory holdings is given by

$$s_{1t} = y_{1t} + s_{1t-1} - \theta_t + \bar{\lambda}.$$

Since $s_{1t} \geq 0$, the threshold preference shock determining that the economy is in case A_1 is then given by $y_{1t} + s_{1t-1} - \theta_t + \bar{\lambda} \geq 0$ or equivalently, by

$$\theta_t \leq y_{1t} + s_{1t-1} + \bar{\lambda}.$$

Case B_1. The demand shock is above "normal", and the existing goods supply $(y_{1t} + s_{1t-1})$ falls short in meeting the consumption demand. In this case we have $s_{1t} = 0$ and $\pi_t^c > 0$. The optimal consumption policy is then to consume the entire stock given by the right-hand side of equation (6),

$$c_t = y_{1t} + s_{1t-1}.$$

Notice that the probability distribution of case A_1 and case B_1 (i.e., the probability of stocking out in period t) depends not only on demand shocks but also

[7] see Samuelson (1971) and Deaton and Laroque (1996) for a similar result in a slightly different context.

on the supply (production) level of finished goods (y_1) determined in period $t-1$. To choose y_{1t} optimally in period $t-1$ without knowing the demand shock in period t, equation (3) shows that y_{1t} should be chosen such that the marginal cost of production (labor cost plus the shadow price of intermediate goods, $w + \lambda_t^m$) equals the expected next period value of the output (the shadow price), $E_{t-1}\lambda_t^c$. Denoting $[\underline{\alpha}, \bar{\alpha}]$ as the support of the innovation in demand (ε) and denoting

$$z_t \equiv E_{t-1}s_{1t} = y_{1t} + s_{1t-1} - \left(E_{t-1}\theta_t - \bar{\lambda}\right)$$

as the optimal target level of inventory holdings such that the firm will stock out only if $\varepsilon \geq z$, then the expected price for finished goods can be expanded into two terms,

$$E_{t-1}\lambda_t^c = \int_{\underline{\alpha}}^{z} \bar{\lambda} f(\varepsilon)d\varepsilon + \int_{z}^{\bar{\alpha}} \left(\theta_t - y_{1t} - s_{1t-1}\right) f(\varepsilon)d\varepsilon,$$

where $\lambda^c = \bar{\lambda}$ with probability $\int_{\underline{\alpha}}^{z} f(\varepsilon)d\varepsilon$ if there is no stockout (i.e., case A_1), and $\lambda^c = \theta - c$ with probability $\int_{z}^{\bar{\alpha}} f(\varepsilon)d\varepsilon$ if there is a stockout (i.e., case B_1). Note in the latter case optimal consumption is determined by $c_t = y_{1t} + s_{1t-1}$. Given the law of motion for the preference shocks, $\theta_t = E_{t-1}\theta_t + \varepsilon_t$, equation (3) then becomes

$$\begin{aligned} w + \lambda_t^m &= \bar{\lambda} \int_{\underline{\alpha}}^{z} f(\varepsilon)d\varepsilon + \underbrace{\left(E_{t-1}\theta_t - y_{1t} - s_{1t-1}\right)}_{\bar{\lambda}-z} \int_{z}^{\bar{\alpha}} f(\varepsilon)d\varepsilon + \int_{z}^{\bar{\alpha}} \varepsilon_t f(\varepsilon)d\varepsilon \qquad (8) \\ &= \bar{\lambda} - z \int_{z}^{\bar{\alpha}} f(\varepsilon)d\varepsilon + \int_{z}^{\bar{\alpha}} \varepsilon_t f(\varepsilon)d\varepsilon \\ &\equiv \Gamma(z_t), \end{aligned}$$

where $\Gamma()$ is a nonlinear but monotonically decreasing function of z and its shape is influenced by the probability distribution function $f(\varepsilon)$. Since z is proportional to y_1 by definition, the higher y_1, the less likely there is a stockout in period t, given expected demand $E_{t-1}\theta_t$. Thus equation (8) implicitly determines the optimal level of final goods production, y_{1t}, in period $t-1$.

Turning to the left-hand side of equation (8), the price of intermediate goods (λ^m) depends on the tightness of the intermediate goods market. Hence z_t is time dependent if λ_t^m is. There are two cases to consider for the possible values of λ^m:

Case A_2. The demand for intermediate goods (y_1) is below "normal". In this case we have $s_{2t} \geq 0$ and $\pi_t^m = 0$. Hence equation (4) implies that intermediate goods price is a constant,

$$\lambda_t^m = \beta a,$$

where the right side is derived based on equation (5). The rationale for this endogenously constant (sticky) price when demand is below a threshold level is similar to that discussed above regarding the finished goods price λ^c. The interpretation for βa is, in case there is no stockout, the firm gets to save on the marginal cost of production next period, and the discounted value for this is precisely βa. In this case, equation (8) implies that the optimal target inventory level z_t based on period $t-1$ information is also a constant: $z_t = k$, where k solves $\Gamma(k) = w + \beta a$. This implies that the optimal demand for intermediate goods (y_1) is determined by the equation,

$$k = y_{1t} + s_{1t-1} - E_{t-1}\theta_t + \bar{\lambda},$$

or equivalently,

$$y_{1t} = \left(E_{t-1}\theta_t - \bar{\lambda}\right) + (k - s_{1t-1}). \tag{9}$$

Hence, the optimal demand of intermediate goods by a downstream (finished-goods producing) firm is characterized by a policy that specifies a constant target level for finished-goods inventory holdings ($k = E_{t-1}s_{1t}$) or a target level of inventory investment ($k - s_{1t-1}$), such that orders (purchases) of intermediate goods move one-for-one with expected consumption demand ($E_{t-1}\theta_t - \bar{\lambda}$) given the target inventory investment level ($k - s_{1t-1}$), provided that we are in case A_2 (i.e., provided there is no stockout in the intermediate goods market: $s_{2t} \geq 0$). Using equation (7), the requirement $s_{2t} \geq 0$ implies that the threshold level of expected demand that determines the probability of stockout in the intermediate goods market is given by,

$$y_{2t} + s_{2t-1} - y_{1t} \geq 0,$$

or equivalently, by

$$E_{t-1}\theta_t - \bar{\lambda} \leq y_{2t} + s_{2t-1} - (k - s_{1t-1}),$$

where we have substituted out y_{1t} using the optimal policy (9). That is, if optimal expected demand for finished goods (which equals consumption demand, $E_{t-1}c_t = E_{t-1}\theta_t - \bar{\lambda}$, plus inventory investment demand, $k - s_{1t-1}$) is less than the potential supply of finished goods (which equals the total supply of intermediate goods, $y_{2t} + s_{2t-1}$), then some intermediate goods should be kept as inventories (i.e.,

$s_{2t} \geq 0$) and the optimal volume of orders (usages) for intermediate goods is characterized by the policy (9). Since demand equals supply in equilibrium, policy (9) is also the policy for production of finished goods ($y_1 = f(m, l)$) given expected demand for finished goods.

Case B_2. The demand for intermediate goods is above "normal". In this case, the supply of intermediate goods cannot meet its demand. Hence, $s_{2t} = 0$ and $\pi_t^m > 0$, implying that there is a stockout in intermediate goods. Hence, the optimal volume of orders for intermediate goods in equilibrium is simply

$$y_{1t} = y_{2t} + s_{2t-1}.$$

In this case, the tightness in the intermediate goods market pushes goods price upwards, so that $\lambda_t^m = \beta a + \pi_t^m > \beta a$. Clearly, whether the economy is in case A_2 or case B_2 depends not only on expected demand shock in period $t-1$ ($E_{t-1}\theta_t$), but also on the availability (production) of intermediate goods ($y_{2t} + s_{2t-1}$), which is determined earlier in history (period $t-2$).

To determine the optimal supply of intermediate goods (y_2) and the optimal orders for raw materials (n) in period $t-2$ based on expected demand for intermediate goods, the planner needs to compute the total demand for intermediate goods (= input demand as factors of production by upstream firms plus inventory investment demand for intermediate goods). Equation (5) shows that y_{2t} should be chosen such that the marginal cost of production, a, equals the expected next period value of marginal product (the shadow price of intermediate goods), $E_{t-2}\lambda_t^m$. Since λ^m depends on the tightness of the intermediate goods market (as discussed above under case A_2 and case B_2), there are two possibilities to consider. In the case A_2, $\lambda^m = \beta a$. In the case B_2, $\lambda^m = E_{t-1}\lambda^c - w = E_{t-1}(\theta_t - c_t) - w$ (by equations 3 and 1).

Denote

$$\begin{aligned} \zeta_t &\equiv E_{t-2}s_{2t} \\ &= y_{2t} + s_{2t-1} - \underbrace{E_{t-2}(k - s_{1t-1})}_{0} - (E_{t-2}\theta_t - \bar{\lambda}) \\ &= y_{2t} + s_{2t-1} - (E_{t-2}\theta_t - \bar{\lambda}) \end{aligned}$$

as the optimal target level of inventory holdings for intermediate goods such that the firm stocks out if and only if $\varepsilon_{t-1} \geq \zeta$.[8] In equation (5), the expected price

[8]Note that in the above expression, the equilibrium concept, $E_{t-2}s_{1t-1} = E_{t-1}s_{1t} = k$, is

of intermediate goods based on information available in period $t-2$ can then be expanded into two terms,

$$a = E_{t-2}\lambda_t^m = \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left(E_{t-1}\theta_t - E_{t-1}c_t - w\right) f(\varepsilon) d\varepsilon, \qquad (10)$$

where the first term represents the probability of case A_2 and the second represents the probability of case B_2. Equation (10) implicitely determines ζ_t and consequently the optimal production policy for y_2. The following proposition shows that the optimal solution for ζ_t is a constant.

Proposition 3.1. *The optimal target for intermediate goods inventory stock is a constant,* $\zeta_t = \widetilde{k}$.

Proof. See Appendix 001.■

Hence by the definition of ζ we have

$$y_{2t} = \left(E_{t-2}\theta_t - \bar{\lambda}\right) + (\widetilde{k} - s_{2t-1}). \qquad (11)$$

Equation (11) says that in equilibrium the optimal production plan for intermediate goods (y_2) is to meet the expected consumption demand $(E_{t-2}\theta_t - \bar{\lambda})$ plus a target level for inventory investment in intermediate goods $(\widetilde{k} - s_{2t-1})$. Notice that this is very similar to the policy of choosing finished goods production (y_{1t}) in equation (9) except in this case the possibility of a stockout in the raw material market is not considered here in the model (extending the model to include such a possibility is straightforward).

Proposition 3.2. *The optimal target for intermediate goods inventories is higher than that for finished goods inventories,*

$$\widetilde{k} > k.$$

Proof. See Appendix 002.■

The intuition behind proposition 2 can be understood as follows. The probability of a stockout in period t in the final consumption goods market is affected

applied because the optimal supply of intermediate goods (y_2) in any period should always be chosen such that not only does intermediate goods inventory level meet its target but also does the finished goods inventory level (whose expected target is k in any period as shown previously).

by the supply of finished goods ($y_{1t} = f(m,l)$) determined in period $t-1$, which in turn is affected by the demand and supply of intermediate goods as well as by the demand and supply of raw materials ($y_{2t} = g(n)$) in the primary goods market determined in period $t-2$. Hence, in a multi-stage production economy equilibrium consumption depends ultimately on the production decisions made by the most remote upstream firms at the start of the production chain which supplies intermediate goods to downstream firms. However, because stocking out and holding inventories are both costly, in order to determine optimally how much to produce at the upstream, firms must form correct expectations on final demand facing the most remote downstream. The longer the chain of production (i.e., the larger the number of production stages), the harder it is to forecast final demand at downstream, hence the more precautionary inventory investment it is needed in order to avoid possible stockouts.

Proposition 3.3. *The equilibrium decision rules for consumption (c), finished goods inventory holding (s_1), finished goods production (y_1) - which also equals equilibrium usage of intermediate goods (m), intermediate goods production (y_2) – which also equals equilibrium demand of raw materials (n), and intermediate goods inventory holding (s_2) are given respectively by:*

$$s_{1t} = \begin{cases} k - \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} \leq \widetilde{k} \\ \widetilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \widetilde{k} \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases}$$

$$c_t = \begin{cases} E_{t-1}\theta_t - \bar{\lambda} + \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \\ E_{t-1}\theta_t - \bar{\lambda} + k & ; \text{ if } \varepsilon_t > k \ \& \ \varepsilon_{t-1} \leq \widetilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \widetilde{k} & ; \text{ if } \varepsilon_t > k \ \& \ \varepsilon_{t-1} > \widetilde{k} \end{cases}$$

$$y_{1t} = \begin{cases} E_{t-1}\theta_t - \bar{\lambda} + \varepsilon_{t-1} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} \leq \widetilde{k} \\ E_{t-1}\theta_t - \bar{\lambda} + k - \widetilde{k} + \varepsilon_{t-1} + \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} > \widetilde{k} \\ E_{t-1}\theta_t - \bar{\lambda} + k & ; \text{ if } k < \varepsilon_{t-1} \leq \widetilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \widetilde{k} & ; \text{ if } \varepsilon_{t-1} > \widetilde{k} \end{cases}$$

$$s_{2t} = \begin{cases} \widetilde{k} - \varepsilon_{t-1} - \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} \leq \widetilde{k} \\ 2\widetilde{k} - k - (1+\rho)\varepsilon_{t-1} - \rho\varepsilon_{t-2} & ; \text{ if } \varepsilon_{t-1} \leq k \ \& \ \varepsilon_{t-2} > \widetilde{k} \\ \widetilde{k} - k - \rho\varepsilon_{t-1} & ; \text{ if } k < \varepsilon_{t-1} \leq \widetilde{k} \\ 0 & ; \text{ if } \varepsilon_{t-1} > k \end{cases}$$

$$
y_{2t} = \begin{cases} E_{t-2}\theta_t - \bar{\lambda} + \varepsilon_{t-2} + \rho\varepsilon_{t-3} & ; \textit{ if } \varepsilon_{t-2} \leq k \ \& \ \varepsilon_{t-3} \leq \widetilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + k - \widetilde{k} + (1+\rho)\varepsilon_{t-2} + \rho\varepsilon_{t-3} & ; \textit{ if } \varepsilon_{t-2} \leq k \ \& \ \varepsilon_{t-3} > \widetilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + k + \rho\varepsilon_{t-2} & ; \textit{ if } k < \varepsilon_{t-2} \leq \widetilde{k} \\ E_{t-2}\theta_t - \bar{\lambda} + \widetilde{k} & ; \textit{ if } \varepsilon_{t-2} > \widetilde{k} \end{cases}
$$

where $\bar{\lambda} = \beta(w + \beta a)$.

Proof. See Appendix 003.■

The interpretation for $\bar{\lambda} = \beta(w + \beta a)$ is reasonably straightforward. Given that the slackness constraint for finished goods does not bind in the current period (hence the slackness constraint for intermediate goods will not bind in the next period according to proposition 3.2), the expected value of carrying one extra unit of finished goods inventory into the next period is determined by two terms: first, it can save on the labor cost of production next period (w); second, it can save on the intermediate goods used in production whose expected value next period equals the discounted marginal cost of raw materials (βa) given that the slackness constraint on intermediate goods does not bind next period. The present value of this compounded marginal cost is then $\beta(w + \beta a)$. Hence $\bar{\lambda}$ is simply the competitive price of finished goods when the economy has inventories.

4. Understanding Inventory Fluctuations

The following propositions characterize some important implications of the model for inventory fluctuations. To fix concepts, note that for finished goods sector, the production is y_{1t} and the sales are c_t; for intermediate goods sector, the usage is $m_t (= y_{1t})$, and the orders are y_{2t} (note that the one period production time for transforming n into y_2 can also be interpreted as a delivery time during which n units of goods are delivered from an upstream firm to an intermediate-goods producing firm which can produce finished goods instantaneously). Since we do not consider inventories of raw materials (n) in this model, the usage in n equals the orders of n. Extending the model to a 3-stage-fabrication economy to allow for raw material inventories is straightforward. To facilitate the proofs, we denote $P \equiv \Pr[\varepsilon_t \leq k]$ and $\widetilde{P} \equiv \Pr[\varepsilon_t \leq \widetilde{k}]$. Clearly, $\widetilde{P} > P$ since $\widetilde{k} > k$.

Proposition 4.1. *The variance of output exceeds the variance of sales for finished goods, and the variance of orders exceeds the variance of usages for inter-*

mediate goods, provided that demand shocks are positively autocorrelated.

Proof. See Appendix 004.■

The intuition for this proposition is reasonably straightforward. If we can understand why production (y_1) is more variable than sales (c), then we can also understand why orders (y_2) are more variable than usages (y_1), because usages are like demand (sales) and orders are equivalent to supply (production).

Since optimal production under demand uncertainty is to meet expected sales plus the target inventory investment, output will respond to an innovation in sales by more than one-for-one. This is because a positive demand shock has two effects: it reduces inventories one-for-one, and it increases expected demand. The first effect alone would make production exactly as volatile as sales in order to replenish the inventory stock; the second effect further increases production if demand shocks are serially correlated ($\rho > 0$) so that the expected next period shock is strictly positive. This important insight has already been provided by Kahn (1987) using a single-stage production partial equilibrium model of output inventories. Here it is shown that it continues to hold in a general equilibrium multi-stage production model.

Since production of finished goods (y_1) requires intermediate goods (m) as input ($y_1 = f(m, l)$), the volatility in sales at the very bottom of the production chain is translated into volatility in production ($\sigma^2_{y_1}$), which in turn is translated into volatility in sales (σ^2_m) from the upstream firms which supply intermediate goods to downstream as inputs. The transmission and amplification of uncertainty continues until the most remote upstream firms are reached. Hence the variance of production/orders exceeds the variance of sales/usages at all stages of production and the variances keep increasing along the chain of production towards upstream firms.

This chain-multiplier effect thus explains why in the US production of raw materials at upstream firms is usually about 50 percent more volatile than production of finished goods at downstream firms.

Proposition 4.2. *Productions (as well as sales) are positively correlated across all stages of fabrication.*

Proof. See Appendix 005.■

This proposition simply explains a well known phenomenon of sectorial co-movements observed in the US economy. In the existing RBC literature, such comovements are explained by aggregate technology shocks (see Hornstein, 2000, for detailed discussions on why sectorial technology shocks cannot explain such comovements). Here is it is shown that such comovements may be explained also by demand shocks to the downstream production sector.

Proposition 4.3. *Input inventories are uncorrelated with but more volatile than output inventories.*

Proof. See Appendix 006.■

The puzzle that inventories of materials and good-in-progress are significantly more volatile than but essentially uncorrelated with inventories of finished goods was first documented by Feldstein and Auerbach (1976), but it has never been explained in the literature. The multi-stage production model offers a clear-cut explanation for this puzzle. The reason for the volatility difference between the two types of inventories is that uncertainty in final demand gets transmitted and amplified along the chain of production towards upstream. Hence inventories held by upstream firms (i.e., intermediate-goods inventories) are more volatile than inventories held by downstream firms (i.e., finished-goods inventories) due to the chain-multiplier effect. The reason for the weak correlation between the two types of inventories is that systematic changes in final demand (e.g., $E(\theta_t)$) are picked up by production while only noise components in demand (i.e., the innovations, ε) are picked up by inventories because any systematic (forecastable) changes in inventories would have been immediately replenished by production/orders according to the optimal inventory targeting policies. Since the noise contained in the two types of inventories reflect demand innovations in different time periods, they are thus uncorrelated.

5. Conclusion

This paper offers a general equilibrium multi-stage production model to explain the co-existence and co-movement of output and input inventories, and provides a neoclassical perspective on the propagation mechanism of demand uncertainty. It reveals that uncertainty in demand at downstream can be transmitted and amplified towards upstream by inventory investment at all stages of production via input-output linkages, leading to a chain-multiplier effect on aggregate output and employment. The model is capable of explaining several long-standing puzzles of the business cycle associated with inventories. Since the chain-multiplier reflects nothing more than optimal responses of a perfectly functioning market economy towards demand uncertainty, interventionary policies in an attempt to reduce aggregate output volatility via demand management may prove counter-productive, unless they can target reducing demand uncertainty and the lags involved in firms' production decisions.

The model may be extended to explain durable goods inventories and seasonal cycles in production and inventories. It is well know that the durable goods sector is far more volatile than the nondurable goods sector and that seasonal movements in production and inventories are much larger than business-cycle frequency movements. These stylized facts have not been well explained by theories. Clearly, durability of consumption goods and seasonal demand at Christmas can only be modeled through preferences. The general equilibrium inventory model with preference shifts outlined in this paper provides a suitable framework for addressing these issues in the future.

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Appendix 001 (proof for proposition 3.1):
Note that

$$E_{t-1}(\theta_t - c_t) = \int_{\underline{\alpha}}^{k} \bar{\lambda} f(\varepsilon) d\varepsilon + \int_{k}^{\bar{\alpha}} (\theta_t - y_{1t} - s_{1t-1}) f(\varepsilon) d\varepsilon.$$

We can rewrite (10) as

$$\begin{aligned}
a &= \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left(\int_{\underline{\alpha}}^{k} \bar{\lambda} f(\varepsilon) d\varepsilon + \int_{k}^{\bar{\alpha}} (\theta_t - \underbrace{y_{1t}}_{y_{2t}+s_{2t-1}} - s_{1t-1}) f(\varepsilon) d\varepsilon - w \right) f(\varepsilon) d\varepsilon \\
&= \int_{\underline{\alpha}}^{\zeta} \beta a f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \bar{\lambda} f(\varepsilon) d\varepsilon \\
&\quad + \int_{\zeta}^{\bar{\alpha}} \left(\int_{k}^{\bar{\alpha}} (\theta_t - y_{2t} - s_{2t-1} - s_{1t-1} - \bar{\lambda}) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon - w \int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon
\end{aligned}$$

where $y_{1t} = y_{2t} + s_{2t-1}$ since $s_{2t} = 0$ if $\varepsilon_{t-1} > \zeta$. Assuming that $\zeta > k$ (see proposition 3.2 for a confirmation of this assumption), hence $\varepsilon_{t-1} > \zeta$ implies $\varepsilon_{t-1} > k$, which in turn implies $s_{1t-1} = \dot{0}$. Then conditioned on $\varepsilon_{t-1} > \zeta$, the third term in the above equation becomes,

$$\begin{aligned}
& \int_{\zeta}^{\bar{\alpha}} \left(\int_{k}^{\bar{\alpha}} (\underbrace{\theta_t}_{E_{t-2}\theta_t + \varepsilon_t + \rho\varepsilon_{t-1}} - y_{2t} - s_{2t-1} - \underbrace{s_{1t-1}}_{0} - \bar{\lambda}) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon \\
&= \int_{\zeta}^{\bar{\alpha}} \left[\underbrace{(E_{t-2}\theta_t - y_{2t} - s_{2t-1} - \bar{\lambda})}_{-\zeta} \int_{k}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right] f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left[\int_{k}^{\bar{\alpha}} (\varepsilon_t + \rho\varepsilon_{t-1}) f(\varepsilon) d\varepsilon \right] f(\varepsilon) d\varepsilon \\
&= -\zeta \int_{\zeta}^{\bar{\alpha}} \left(\int_{k}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon + \int_{\zeta}^{\bar{\alpha}} \left(\int_{k}^{\bar{\alpha}} (\varepsilon_t + \rho\varepsilon_{t-1}) f(\varepsilon) d\varepsilon \right) f(\varepsilon) d\varepsilon \\
&\equiv G(\zeta_t, \Omega_1),
\end{aligned}$$

where Ω_1 is a set of fixed parameters. Hence we have

$$\begin{aligned}
a &= \beta a \int_{\underline{\alpha}}^{\zeta} f(\varepsilon) d\varepsilon + \bar{\lambda} \left(\int_{\underline{\alpha}}^{k} f(\varepsilon) d\varepsilon \right) \left(\int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \right) + G(\zeta_t, \Omega_1) - w \int_{\zeta}^{\bar{\alpha}} f(\varepsilon) d\varepsilon \\
&\equiv \widetilde{\Gamma}(\zeta_t, \Omega_2),
\end{aligned}$$

where Ω_2 is a set of fixed parameters. This equation implies that the optimal inventory target (ζ_t) for intermediate goods is a constant that solves the identity $\widetilde{\Gamma}(\zeta_t, \Omega_2) = a$.■

Appendix 002 (proof for proposition 3.2):

Finished goods inventory stock satisfies

$$s_{1t} = \begin{cases} y_{1t} + s_{1t-1} - (\theta_t - \bar{\lambda}) & ;\ \text{if } \varepsilon_t \leq k \\ 0 & ;\ \text{if } \varepsilon_t > k \end{cases}.$$

Since the optimal production for y_{1t} satisfies

$$y_{1t} = \begin{cases} (E_{t-1}\theta_t - \bar{\lambda}) + (k - s_{1t-1}) & ;\ \text{if } \varepsilon_{t-1} \leq \widetilde{k} \\ y_{2t} + s_{2t-1} & ;\ \text{if } \varepsilon_{t-1} > \widetilde{k} \end{cases}$$

and the optimal production for y_{2t} satisfies

$$y_{2t} = (E_{t-2}\theta_t - \bar{\lambda}) + (\widetilde{k} - s_{2t-1}),$$

we have

$$\begin{aligned} s_{1t} &= \begin{cases} (E_{t-1}\theta_t - \bar{\lambda}) + k - (\theta_t - \bar{\lambda}) & ;\ \text{if } \varepsilon_t \leq k \ \&\ \varepsilon_{t-1} \leq \widetilde{k} \\ (E_{t-2}\theta_t - \bar{\lambda}) + \widetilde{k} + s_{1t-1} - (\theta_t - \bar{\lambda}) & ;\ \text{if } \varepsilon_t \leq k \ \&\ \varepsilon_{t-1} > \widetilde{k} \\ 0 & ;\ \text{if } \varepsilon_t > k \end{cases} \\ &= \begin{cases} k - \varepsilon_t & ;\ \text{if } \varepsilon_t \leq k \ \&\ \varepsilon_{t-1} \leq \widetilde{k} \\ s_{1t-1} + \widetilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ;\ \text{if } \varepsilon_t \leq k \ \&\ \varepsilon_{t-1} > \widetilde{k} \\ 0 & ;\ \text{if } \varepsilon_t > k \end{cases} \end{aligned}$$

Notice that the lagged variable (s_{1t-1}) can be further iterated backwards using the above dynamic equation:

$$s_{1t-1} = \begin{cases} k - \varepsilon_{t-1} & ;\ \text{if } \varepsilon_{t-1} \leq k \ \&\ \varepsilon_{t-2} \leq \widetilde{k} \\ s_{1t-2} + \widetilde{k} - \varepsilon_{t-1} - \rho\varepsilon_{t-2} & ;\ \text{if } \varepsilon_{t-1} \leq k \ \&\ \varepsilon_{t-2} > \widetilde{k} \\ 0 & ;\ \text{if } \varepsilon_{t-1} > k \end{cases},$$

which can lead to infinite regression, resulting in the series $\{s_{1t}\}$ being nonstationary. This cannot be an equilibrium unless

$$s_{1t-1} = 0$$

always holds. But this requires that the condition, $\varepsilon_{t-1} > k$, always hold so that $s_{1t-1} = 0$ with probability one (the third line in the above equation for s_{1t-1}). However, given the second line in the equation for s_{1t},

$$s_{1t} = \ s_{1t-1} + \widetilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} \quad ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \widetilde{k} \ ,$$

if the condition, $\widetilde{k} > k$, holds, then $\varepsilon_{t-1} > \widetilde{k}$ automatically implies $\varepsilon_{t-1} > k$, which in turn implies (by the third line in the equation for s_{1t-1})

$$s_{1t-1} = 0 \quad ; \text{ if } \varepsilon_{t-1} > \widetilde{k},$$

hence we have

$$s_{1t} = \begin{cases} k - \varepsilon_t & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} \leq \widetilde{k} \\ \widetilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} & ; \text{ if } \varepsilon_t \leq k \ \& \ \varepsilon_{t-1} > \widetilde{k} \\ 0 & ; \text{ if } \varepsilon_t > k \end{cases} ,$$

which is stationary and is therefore an equilibrium. Hence, $\widetilde{k} > k$ must be true.■

Appendix 003 (proof for proposition 3.3):

The decision rule for s_{1t} is proved in the proof for proposition 3.2. The rest can be obtained by following the discussions in section 3 above using straightforward substitutions. To show that $\bar{\lambda} = \beta(w + \beta a)$, note $\lambda_t^c = \bar{\lambda}$ if $\varepsilon_t \leq k$ (i.e., if $\pi_t^c = 0$). On the other hand, since the current period shadow price of intermediate goods is known based on last period information set, as all variables in the resource constraint (7) are known in period $t-1$, we then have $E_{t-1}\lambda_t^m = \lambda_t^m$ and $E_t\lambda_{t+1}^m = \lambda_{t+1}^m$. Since $\lambda_t^m = \beta a$ if $\varepsilon_{t-1} \leq \widetilde{k}$ according equation (4) and equation (5),[9] we have $\lambda_{t+1}^m = \beta a$ if $\varepsilon_t \leq \widetilde{k}$. According to proposition 3.2, $k < \widetilde{k}$, hence $\lambda_t^c = \bar{\lambda}$ implies $\lambda_{t+1}^m = \beta a$ (since $\varepsilon_t \leq k$ implies $\varepsilon_t \leq \widetilde{k}$). According to equation (3), $E_t\lambda_{t+1}^c = w + \lambda_{t+1}^m$, substituting this into equation (2) gives

$$\lambda_t^c = \beta(w + \lambda_{t+1}^m) = \beta(w + \beta a), \quad \text{if } \varepsilon_t \leq k.$$

■

Appendix 004 (proof for proposition 4.1):

We need to show $\sigma_{y_2} > \sigma_{y_1} > \sigma_c$. Using the law of motion for preference shocks, the variance of c is given by

$$\sigma_c^2 = P(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2) + (1-P)\widetilde{P}\rho^2\sigma_\theta^2 + (1-P)(1-\widetilde{P})\rho^4\sigma_\theta^2.$$

[9] Equation (5) implies $E_{t-1}\lambda_{t+1}^m = a$.

The variance of y_1 is given by

$$\begin{aligned}\sigma_{y_1}^2 &= P\widetilde{P}(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2) \\ &+P(1-\widetilde{P})(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2) \\ &+(1-P)\widetilde{P}\rho^2\sigma_\theta^2 \\ &+(1-\widetilde{P})\rho^4\sigma_\theta^2.\end{aligned}$$

And the variance of y_2 is given by

$$\begin{aligned}\sigma_{y_2}^2 &= P\widetilde{P}(\rho^4\sigma_\theta^2 + \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho^2\sigma_\varepsilon^2 + 2\rho^4\sigma_\varepsilon^2) \\ &+P(1-\widetilde{P})(\rho^4\sigma_\theta^2 + (1+\rho)^2\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2(1+\rho)\rho^2\sigma_\varepsilon^2 + 2\rho^4\sigma_\varepsilon^2) \\ &+(1-P)\widetilde{P}(\rho^4\sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2) \\ &+(1-\widetilde{P})\rho^4\sigma_\theta^2.\end{aligned}$$

All terms in these expressions are strictly positive. Notice that the last two terms in σ_y^2 exceed the last two terms in σ_c^2. Hence, $\sigma_{y_1} > \sigma_c$ if and only if the first two terms in σ_y^2 exceed the first term in σ_c^2, namely,

$$P\widetilde{P}(\rho^2\sigma_\theta^2+\sigma_\varepsilon^2+2\rho\sigma_\varepsilon^2)+P(1-\widetilde{P})(\rho^2\sigma_\theta^2+\sigma_\varepsilon^2+\rho^2\sigma_\varepsilon^2+2\rho\sigma_\varepsilon^2+2\rho^3\sigma_\varepsilon^2) > P(\rho^2\sigma_\theta^2+\sigma_\varepsilon^2).$$

Since the left-hand side of this inequality can be expressed as

$$P(\rho^2\sigma_\theta^2 + \sigma_\varepsilon^2) + P\widetilde{P}(2\rho\sigma_\varepsilon^2) + P(1-\widetilde{P})(\rho^2\sigma_\varepsilon^2 + 2\rho\sigma_\varepsilon^2 + 2\rho^3\sigma_\varepsilon^2),$$

where the first term is the same as the right-hand side of the inequality, hence $\sigma_y^2 > \sigma_c^2$ as long as $\rho > 0$. In order to show $\sigma_{y_2} > \sigma_{y_1}$, first notice that both $\sigma_{y_2}^2$ and $\sigma_{y_1}^2$ are increasing functions of ρ. Secondly, notice that if $\rho = 0$, then $\sigma_{y_2}^2 = \sigma_{y_1}^2 = P\widetilde{P}\sigma_\varepsilon^2 + P(1-\widetilde{P})\sigma_\varepsilon^2$. And thirdly, notice that if $\rho = 1$, then

$$\begin{aligned}\sigma_{y_1}^2 &= P\widetilde{P}(\sigma_\theta^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\ &+P(1-\widetilde{P})(\sigma_\theta^2 + \sigma_\varepsilon^2 + \sigma_\varepsilon^2 + 2\sigma_\varepsilon^2 + 2\sigma_\varepsilon^2) \\ &+(1-P)\widetilde{P}\sigma_\theta^2 \\ &+(1-\widetilde{P})\sigma_\theta^2,\end{aligned}$$

and

$$\begin{aligned}\sigma_{y_2}^2 &= P\widetilde{P}(\sigma_\theta^2+\sigma_\varepsilon^2+\sigma_\varepsilon^2+2\sigma_\varepsilon^2+2\sigma_\varepsilon^2)\\&+P(1-\widetilde{P})(\sigma_\theta^2+4\sigma_\varepsilon^2+\sigma_\varepsilon^2+4\sigma_\varepsilon^2+2\sigma_\varepsilon^2)\\&+(1-P)\widetilde{P}(\sigma_\theta^2+\sigma_\varepsilon^2+2\sigma_\varepsilon^2)\\&+(1-\widetilde{P})\sigma_\theta^2,\end{aligned}$$

and clearly each term in $\sigma_{y_2}^2$ exceeds the corresponding term in $\sigma_{y_1}^2$. Hence, as long as $\rho > 0$, we have $\sigma_{y_2} > \sigma_{y_1} > \sigma_c$.■

Appendix 005 (proof for proposition 4.2):

Eye-ball inspection on the decision rules indicates that every possible case for c in the decision rule (sales of finished goods) is strictly positively correlated with every possible case for $m(= y_1$, sales of intermediate goods), and every possible case for y_1 in the decision rule (production of finished goods) is strictly positively correlated with every possible case for y_2 (production of intermediate goods). Hence productions as well as sales are unambiguously positively correlated across all stages of fabrication. Furthermore, these correlations increase with ρ since $E_{t-j}\theta_t = \rho^j\theta_{t-j}$.■

Appendix 006 (proof for proposition 4.3):

Note that the only term in s_{1t} that is correlated with terms in s_{2t} is ε_{t-1}. Hence we need only to consider the case where

$$s_{1t} = \widetilde{k} - \varepsilon_t - \rho\varepsilon_{t-1} \quad ; \text{ if } \varepsilon_t \le k \;\&\; \varepsilon_{t-1} > \widetilde{k} \ .$$

However, given $\widetilde{k} > k$, the only case in s_{2t} that meets the condition, $\varepsilon_{t-1} > \widetilde{k}$, which supports the above case for s_{1t} is

$$s_{2t} = 0 \quad ; \text{ if } \varepsilon_{t-1} > k \ .$$

Hence the two types of inventories are uncorrelated. The variances of the two types of inventories, on the other hand, are given respectively by

$$\sigma_{s_1}^2 = P\widetilde{P}\sigma_\varepsilon^2 + P(1-\widetilde{P})(\sigma_\varepsilon^2+\rho^2\sigma_\varepsilon^2)$$

$$\sigma_{s_2}^2 = P\widetilde{P}(\sigma_\varepsilon^2+\rho^2\sigma_\varepsilon^2) + P(1-\widetilde{P})\left((1+\rho)^2\sigma_\varepsilon^2+\rho^2\sigma_\varepsilon^2\right) + (1-P)\widetilde{P}(\rho^2\sigma_\varepsilon^2).$$

Term-by-term comparison shows that $\sigma_{s_2} > \sigma_{s_1}$ as long as $\rho > 0$ (the variances are equal if $\rho = 0$).■

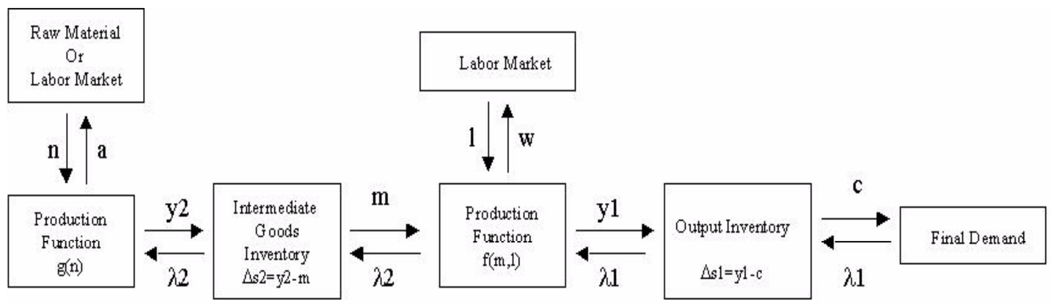

Fig. 1. Stage-of-Fabrication Model Diagram.