Endogenous Timing and Efficiency in Coordination Games with Incomplete Information*

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October 15, 2003

Abstract

By adding a small amount of noise to the information structure, the theory of global games is able to select a unique equilibrium in coordination games with a finite number of players and two actions, a safe action and a risky action. As the noise vanishes, however, it is often the case that positive amount of inefficiency remains in the selected equilibrium. This paper argues that this is partly due to the simultaneity of the moves. If the game is played sequentially with the order of moves determined endogenously, and if the risky action is irreversible and the safe action is reversible, then efficiency will be asymptotically restored as the noise vanishes. However, if the safe action is irreversible, then dynamics will not make much difference to the possible inefficiency of equilibria. Thus two coordination games may look very similar if they are treated as simultaneous move games, yet they can be very different if they are treated as sequential move games. For example, there has been much recent interest in the phenomenon of currency attacks and its similarity to the well-known model of bank runs. However, we show that these games are quite different in the dynamic setting and endogenous timing might help to resolve inefficiencies in the first but not the second.

1 Introduction

Many economic and political situations can be modelled as a coordination game with a finite number of players and two actions, a risky action and a safe action. One example is industrialization, where simultaneous industrialization

*I am grateful to Kalyan Chatterjee for his guidance. I thank Susanna Esteban, Nezih Guner, Vijay Krishna, Dmitry Kvassov, James Jordan and Tao Zhu for helpful discussions. Special thanks to Tomas Sjöström for his insightful comments and editorial suggestions. All errors remain my own.

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of many sectors can be profitable even if no single sector can make money by industrializing alone. In this example the risky action is to commit to industrializing a sector, and the safe action is not to commit to industrializing. Such coordination games usually have multiple equilibria, but the theory of "global games" has provided methods for selecting a unique equilibrium. This equilibrium selection theory is developed by modelling the coordination situation as a simultaneous move game with incomplete information. In this paper, we argue that if it is more appropriate to model the situation as a sequential move game, then the prediction of the outcome may be very different from the prediction made by the theory of global games. If the risky action is irreversible, and the safe action is reversible, then the static model will exaggerate the likelihood of a coordination failure, compared to the dynamic model. However, if the risky action is reversible, and the safe action is irreversible, then dynamics will not make much difference to the possible inefficiency of equilibria. Hence two coordination games may look very similar if they are treated as simultaneous move games, yet they can be very different if they are treated as sequential move games.

For ease of exposition, we model the coordination situation as a stag hunt game. In the stag hunt game, a group of hunters need to decide whether to hunt for a stag or not. The risky action is to hunt for the stag, and the safe action is not to hunt for the stag. The stag will be caught if and only if all of them choose to hunt. Each hunter has private information about his cost of participating in the hunt. Each hunter must decide whether and when to commit to hunting. Commitment is a one time decision and irreversible in that once a hunter makes a commitment, his cost is sunk even if the stag is not caught. Each hunter in each period observes accurately how many hunters have committed to hunting so far. If there is no discounting, and if there are no dominant strategy types who strictly prefer not to hunt, then it is not hard to see that in any equilibrium the stag is caught for sure. I show that this result is robust to the introduction of dominant strategy types and discounting. It also holds if there is a common shock to the payoffs. This is in contrast to the static game in which there is always a "no hunting" equilibrium, and this equilibrium is unique in the presence of dominant strategy types under certain conditions on the distribution function.

2 A Simple Example

We illustrate the ideas using the following simple example.

	H	NH
H	$1-c_1, 1-c_2$	$-c_1, 0$
NH	$0, -c_2$	$0, 0$

Two hunters must decide whether to hunt for a stag or not. The risky action is to hunt for the stag, and the safe action is not to hunt for the stag. The stag will be caught if and only if both players choose to hunt. Once caught, the stag is worth 1 to each player. Let c_i denote player i's cost of hunting for the stag. If player i chooses to hunt, she loses c_i no matter what the other player plays. Let us first assume that $c_i \in [0, 1]$, $i = 1, 2$, and that the values of c_1 and c_2 are common knowledge between the two players. The complete information game has two pure strategy Nash equilibria, (H, H) and (NH, NH). Now we assume that it takes two periods for the players to play this game, and there is no discounting. The payoff of each player is read off from the simultaneous move game, according to the final actions of the two players. In the two period game, players choose whether and when to commit to hunting. Commitment is a one time decision and irreversible in that once a player makes a commitment, her cost is sunk even if the stag is not caught. We also assume that at the beginning of the second period, each hunter is able to observe whether the other hunter has committed to hunting or not in the first period. To be sure, this two period game also has multiple subgame perfect equilibria, but it is easy to see that the stag will be caught for sure in any subgame perfect equilibrium.

Next we come back to the static game, but let us assume that c_1 and c_2 are independently and uniformly distributed over $[0, 1]$. Now we have an incomplete information game, but it is still common knowledge between the two players that it is in the interest of both to hunt for the stag. There are multiple Bayesian Nash equilibria in this game. In particular, "always hunt no matter what the type is" is one of them, and "never hunt no matter what the type is" is another. Again, if we add one more period and allow the players to choose the timing of moves, the stag will be caught in any perfect Bayesian equilibrium.

Now let us assume that c_1 and c_2 are independently and uniformly distributed over $[0, 1 + \epsilon]$, where $\epsilon > 0$. The introduction of ϵ represents a small amount of doubt that the players have about each other's willingness to hunt. Now for some type c less than 1 but very close to 1, it is not profitable to hunt because the potential profit is so small that it is not worth taking the risk that the other player's type is above 1. Anticipating this, it is not profitable for some type c' less than c but very close to c to hunt either, and so on. With uniform distribution, the argument keeps going to the left until it reaches 0. Hence the unique Bayesian Nash equilibrium in the static game is for nobody to hunt no matter what the type is. What happens to the two period game with endogenous timing? Suppose that whenever a player is indifferent between hunting and no hunting, she always chooses hunting. Then it is easy to show that there is a unique perfect Bayesian equilibrium in the two period game, in which player i hunts in period 1 if and only if $c_i \leq \frac{2+\epsilon-\sqrt{\epsilon^2+4\epsilon}}{2}$. Hence for small ϵ, the probability that the stag is caught in the dynamic game is still near 1, even if the probability that the stag is caught in the static game is 0.

These simple examples suggest that if players have no doubt about other players' willingness to play the risky action, then the static game still has multiple equilibria, but the dynamic game refines away the inefficient ones. On the other hand, if players slightly doubt about other players' willingness to play the risky action, then the static game has a unique equilibrium, which is the most inefficient equilibrium. Dynamics gives a qualitatively different set of equilibria, in which the probability of the inefficient outcome is small in every equilibrium. Dynamics will not make such a difference, however, unless the risky action is irreversible, and the safe action is reversible. We will illustrate this by comparing currency attacks and bank runs in Section 6, and by studying an arms race game in Section 7. The main message of the paper is thus the following. Endogenous timing, combined with an irreversible risky action and a reversible safe action, will overcome coordination failure even in the presence of incomplete information. We will confirm this intuition in a generalized version of the above two player game.

The rest of the paper is organized as follows. Section 3 discusses some related literature. Section 4 presents the model and the asymptotic result. Section 5 establishes the existence of PBE in both the no discounting case and the discounting case. A common shock model is studied in Section 6, and the issue of irrversibility and the arms race game will be discussed in Section 7. Section 8 concludes.

3 Related Literature

To resolve the multiplicity problem in games described in the introduction, Carlsson and van Damme (1993) introduce a small amount of uncertainty about the payoff from the safe action. They create dominant strategy types of players who always choose the risky action or the safe action in the Bayesian game. These extreme (remote) types exert an influence on other types' reasoning, so that a process of iterative elimination of strictly dominated strategies will generate a unique Bayesian Nash Equilibrium. This idea is further developed by Morris and Shin in a series of papers. Morris and Shin find that introducing small heterogeneity in the information structure of games with strategic complementarity is likely to generate a unique equilibrium. Again, the games they consider are simultaneous move games. In a simultaneous move game, players eliminate strictly dominated strategies iteratively by introspection, which selects a unique strategy. An important difference between our model and the global games model is that we have independent types, while the global games model usually considers correlated types. The correlation is created by the players' noisy observations of a common fundamental variable. We will investigate a common shock model in Section 6.

While the coordination games literature is largely concerned with the static setting, a different set of papers has focused on how rational behavior could lead to individuals' rationally following the actions of others who moved earlier. This is the the so called informational cascade literature, initiated by Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992). In these papers players might ignore their own information about the environment and blindly yet rationally follow their predecessors' choices. One player's action has no direct impact on other players' payoff, and the order of moves is exogenously given. Chamley and Gale (1994) endogenize the order of actions. In their model, a random fraction of people have an investment opportunity. The payoff to investment depends on the number of people who have the opportunity, but not on the number of people who actually execute it. Another endogenous timing model with pure informational externalities is studied in Zhang (1997), where different players receive signals about the fundamentals with different precision. In equilibrium, the player with the highest precision waits the shortest, and her choice will be imitated immediately by everyone else. Like the informational cascade literature, herding might occur in our paper, too, but for a different reason. People follow other people because they rationally expect themselves to be followed, and this is due to the strategic complementarity in our model.

Our model also has some similarity with the war of attrition game. In fact, if only one hunter is required to catch the stag, and if the stag is a public good so that once it is caught everybody has a share, then it is exactly a war of attrition game (Bliss and Nalebuff (1984)). In the war of attrition game, it can be strictly optimal for a player to wait for a while and contribute to the public good only if there is no contribution during the waiting period. In our model this could also be the case if a player finds it worthwhile to postpone her action in order to synchronize her action with the "bottleneck" players. As a consequence, it might be that in equilibrium a player is not willing to move when she is more optimistic about other players' types, but she is willing to move then she becomes more pessimistic about their types. We will discuss this in Section 5.

Dynamic games with strategic complementarity and incomplete information are also studied in Choi (1997) and Dasgupta (2001). Choi (1997) studies a sequential technology adoption game in which agents refrain from experimenting with a new technology unless the expected value of it is sufficiently higher than the realized value of the established technology. The agents tend to herd on an established technology for fear of being stranded. Choi concludes that sequential adoption may be worse than simultaneous adoption in terms of ex-ante welfare. The major difference between Choi's model and our model is that in his model (i) both actions are risky and irreversible, (ii) players are symmetric and (iii) uncertainty about a technology is resolved immediately after the technology is tried by someone. All the three factors discourage the ex-ante efficient outcome from being realized.

Contrary to Choi's conclusion, Dasgupta (2001) argues that dynamics may be good when network effects and uncertainty are both present. Dasgupta considers a continuum of agents, who make private observations of the underlying state of the world. Agents choose whether and when to switch from a safe project to a risky one. Network externalities are present in the risky project. In particular, the fraction of investors it takes to succeed in the risky project is inversely related to the state of the world. Dasgupta shows that when heterogeneity is small enough, i.e. when agents' observations are close enough, there is a unique monotone equilibrium in which agents switch early if and only if their observations exceed some threshold value. Moreover, endogenous timing produces higher welfare than exogenous timing and the static counterpart of the switching game. Dasgupta (2001) is in spirit the closest paper to ours, yet there remain several major differences: (a) He has a continuum of agents, so that an individual player need not worry about her impact on the rest of the players, while we have an arbitrary but finite number of players, and an individual player's impact is not negligible ; (b) There are only two periods in his model, while there is a finite yet arbitrary number of periods in our model; (c) In his model agents have private observations of a common fundamental of the economy, and their observations are close to each other, in our model agents have private information about their own types, and they don't have to be close to each other and (d) Agents' observations are assumed to follow a normal distribution in his model, but we put no specific restrictions on the distribution of types except some regularity conditions.

The common shock section is based on an investment game in Morris and Shin (2000). Morris and Shin (2000) find that in the simultaneous move game, inefficiency does not disappear even if players observe the fundamentals of the economy with more and more accuracy. We show that inefficiency will disappear as the amount of noise vanishes, if the game is played out sequentially. The arms race game is studied in Baliga and Sjöström (2002), in which they use a cheap talk approach to restore efficiency in the "security dilemma". Dynamics does not change much in their model because the risky action (not to build weapons) is a reversible decision. It is hard to destroy weapons that have already been built, so the safe action (to build weapons) is irreversible.

If it is hard to renege on the risky choice and easy to switch from the safe one, then there is a role for leaders to play, and coordination with incomplete information may be achieved by sequential moving. Thus a crucial aspect of our model, missing in the static model, is whether or not the risky action is reversible. One example is political revolution. A successful revolution requires a critical mass of participants. However, people are uncertain about other people's willingness to participate. Usually it is hard for someone who already takes part to renege, but it's easier for someone who is silent earlier to change position and follow suit. Kuran (1989) gives a theory of revolution that relies on "preference falsification". Formally Kuran's model is static, but it implies a dynamic process in which people who are privately more pro-revolution revolt early, followed by people who are slightly less pro-revolution, and so on. Eventually a bandwagon effect takes on and a majority of people switch position publicly. We formalize this reasoning explicitly in a dynamic model. A significant difference is that we assume people differ in their cost, not their "political ideal point". If we take people's "political ideal point" seriously, then after a revolution only those who join the revolution late should be repressed, since they are most sympathetic for the old regime. What we observe, however, is that those who join the revolution at the very beginning sometimes get wiped out first. Our model implies that people with smaller cost to participate a revolution will take part earlier, and it is these people who are most capable of launching another revolution, hence they are likely to be repressed immediately after the success of the revolution.

Murphy, Shleifer and Vishny (1989) consider a model where industrialization of many sectors can be profitable even if no single sector can break even by industrializing alone. They propose several scenarios in which a good equilibrium where all sectors industrialize and a bad equilibrium where no sector does can coexist. Thus industrialization requires a certain level of coordination, which in turn, seems to require the intervention of some conscious planning. In fact, Murphy, Shleifer and Vishny (1989) point out the role of government intervention: "..., the government can use investment subsidies as long as they are widely enough spread to bring about a critical mass of investment needed to sustain a big push." My paper suggests that even without conscious planning, coordination among many sectors might still be achieved spontaneously through a herding mechanism. Indeed, in the industrialization scenario the risky action is likely to be irreversible.

4 The Model and The Asymptotic Result

The simultaneous move stag hunt game is as follows. There are n players who simultaneously choose whether to hunt or not. The stag is caught if and only if all players hunt. Each player gets a benefit of 1 if the stag is caught. Players' costs of hunting are *i.i.d.* with distribution function F over support $[0,\ 1+\epsilon]$, where $\epsilon \geq 0$ and $f(\cdot)$ is bounded. A player loses her cost if she chooses to hunt, no matter what other players do. A player receives 0 if she chooses not to hunt. The only private information is cost.

The following proposition is essentially Theorem 1 in Baliga and Sjöström (2002).

Proposition 1 *In the static game, there always exists a Bayesian Nash Equilibrium(BNE) in which there is no hunting with probability* 1. *If* $\epsilon > 0$, *and* $F(c) < c$, $\forall c \in (0, 1+\epsilon]$, *then the no hunting equilibrium is unique, for any* n.

Proof: It is clear that nobody hunting whatever her type may be is always a Nash equilibrium of the Bayesian game. It is also clear that any BNE has the cutoff property: if no hunting is a best response for type c_i of player i, then it is also a best response for type $c_i' > c_i$.

Now fix any BNE. Let $\widehat{c}_i$ denote player i's cutoff type. If no hunting is not the unique equilibrium, then there must exist another equilibrium in which $1 > \widehat{c}_k > 0$, $\forall k$. Without loss of generality assume $\widehat{c}_i = \max_k \{\widehat{c}_k\}$, then we have

$$\widehat{c}_i = \prod_{k \neq i} F(\widehat{c}_k) < \prod_{k \neq i} \widehat{c}_k \leq \widehat{c}_i^{n-1},$$

which is a contradiction. ■

Notice that Proposition 1 holds for any $\epsilon > 0$, so long as the condition on the distribution function holds. Hence in the limit when $\epsilon = 0$, as long as the condition on the distribution function still holds, the equilibrium in which nobody hunts no matter what the type is can be selected as the unique prediction of the noise free ($\epsilon = 0$) game. As we will see next, this strong prediction is completely a consequence of the simultaneity of the moves.

From now on, we will consider a dynamic version of the stag hunt game. Assume that it takes the n players T periods to finish the game, where $n \leq T < \infty$. Players choose whether and when to hunt. Hunting is a one time, irreversible decision, i.e. once a player chooses to hunt, her cost is not refundable, no matter whether the stag is caught or not. No hunting is a reversible decision. In each period, each player observes accurately how many players have committed to hunting so far. Players discount both cost expenditures and potential rewards by the same discount factor δ, $0 < \delta \leq 1$. At the end of the Tth period, a player's payoff is determined by the *final* decisions of everybody, i.e. the payoff is read off from the simultaneous move game, according to the final decisions of each player, upto certain adjustments of discounting. Formally, let h^T denote an arbitrary terminal history, let $\pi_j\left(h^T, c_j\right)$ denote type c_j of player j's discounted payoff attached to h^T. If j never hunts, then $\pi_j\left(h^T, c_j\right) = 0$; if j hunts in period t, but at least one player never hunts, then $\pi_j\left(h^T, c_j\right) = \delta^{t-1}\left(-c_j\right)$; if j hunts in period t, and there is no player who never hunts, and the last hunt occurs in period t', then $\pi_j\left(h^T, c_j\right) = \delta^{t-1}\left(-c_j\right) + \delta^{t'-1} \cdot 1$.

Our solution concept is perfect Bayesian equilibrium (PBE). For the case where $\delta = 1$, it is convenient to restrict attention to the set of PBE that satisfies the following assumption.

(A) After any history, for any player j, for any two types c_j and c_j' of player j, if both types are indifferent between hunting and waiting, then they choose the same pure action.

As we will see in the following lemma, assumption (A) is not needed if $\delta < 1$. If $\delta = 1$, then there might exist a PBE in which a player plays a strategy that is not monotonic in her type. This is ruled out by assumption (A) and the following lemma.

Lemma 1 *In any PBE, if either $\delta < 1$, or $\delta = 1$ and (A) holds, then the following is true. After any history, if a player has not hunted so far, and if he is willing to hunt when his cost is c, then he is also willing to hunt when his cost is below c, i.e. any PBE has the cutoff property.*[1]

Proof: Fix a PBE, a player i, and a history h. Let c_1 and c_2 be two types of player i. Let $c_1 < c_2$. Let $H(c_j|h)$ denote the expected equilibrium payoff of type c_j, $j = 1, 2$, if type c_j chooses to hunt, conditional on history h; let $W(c_j|h)$ denote the expected equilibrium payoff of type c_j, $j = 1, 2$, if type c_j chooses to wait, conditional on h.

Case 1. After choosing to wait at history h, type c_1 and type c_2's equilibrium decisions in the continuation game are identical. In this case, we can write

$$W(c_1|h) = \delta(\alpha(h)(-c_1) + D(h)),$$

and

$$W(c_2|h) = \delta(\alpha(h)(-c_2) + D(h)),$$

where $\alpha(h)$ and $D(h)$ only depend on other players' equilibrium strategies, and $\alpha(h) \leq 1$. Since $\delta < 1$, if $H(c_2|h) \geq W(c_2|h)$, then $H(c_1|h) > W(c_1|h)$.

Case 2. After choosing to wait at history h, type c_1 and type c_2's equilibrium decisions in the continuation game are different. In this case, let c_2 mimic c_1's decision in each and every contingency in the continuation game. Let $\widetilde{W}(c_2|h)$ denote the resulting expected payoff of c_2. Then it must be that $\widetilde{W}(c_2|h) \leq W(c_2|h)$, by the incentive compatibility of perfect Bayesian equilibrium. Hence if $H(c_2|h) \geq W(c_2|h)$, then $H(c_2|h) \geq \widetilde{W}(c_2|h)$, which in turn, implies that $H(c_1|h) > W(c_1|h)$, by the argument in case 1. ■

Since every PBE has the cutoff property, the belief system of the equilibrium can be easily determined from the equilibrium strategy, using Bayes' rule. In particular, after any history, the belief about any remaining player's types must be some truncated distribution above some cutoff value. The belief about the players who have already moved is irrelevant. Having said this, from now on

[1] Usually in the literature, equilibrium with the cutoff property is called "monotone equilibrium", in that a player's action is monotonic in her own types. In this paper we use the term "monotone equilibrium" in a different sense. An equilibrium is monotone if a player's action is monotone in her belief about *other* players' types. That is, if she moves when she is more pessimistic about other players' types, then she also moves when she is more optimistic. We will refer to equilibrium with the cutoff property as "cutoff equilibrium".

we identify a PBE with a PBE strategy, omitting the supporting belief system, which can be derived from the strategy straightforwardly. Our proofs will use the following notation. Let $\overrightarrow{x}$ denote an n dimensional vector of lower bounds on the n players' types. The upper bound on each player's cost is $1+\epsilon$. Let $\Gamma\left(n, \epsilon, \delta, \overrightarrow{x}, T\right)$ denote the n player, T period game in which the discount factor is δ, the lower bounds on the players' cost types are $\overrightarrow{x}$, and the distribution over j's costs is given by F, truncated to the interval $[x_j, 1+\epsilon]$, where x_j is the jth coordinate of $\overrightarrow{x}$.

If there are no dominant strategy types and no discounting, then the results of Proposition 1 are completely reversed in the dynamic game.

Proposition 2 *If $\epsilon = 0$ and $\delta = 1$, then in any PBE that satisifies (A) the stag is caught with probability* 1*.*

Proof: We show this by induction on the number of players. This is obvious if $n = 1$. Now assume that this is true for $\Gamma\left(k, 0, 1, \overrightarrow{x}, T\right)$, $\forall k \leq n$, $\forall \overrightarrow{x} < (1, 1, .., 1)$, $\forall T \geq k$. We need to show that this is true for $\Gamma\left(n+1, 0, 1, \overrightarrow{x}, T\right)$, $\forall \overrightarrow{x} < (1, 1, .., 1)$, $\forall T \geq n+1$. Suppose otherwise that there exists a PBE of $\Gamma\left(n+1, 0, 1, \overrightarrow{x}, T\right)$, such that the stag is caught with probability less than 1. Then there exists a player j, a type $c_j < 1$, such that the equilibrium payoff of c_j is less than $1 - c_j$. However, if c_j hunts in period 1, then by the induction hypothesis, c_j obtains a payoff of $1 - c_j$, a contradiction. ■

Next we show that Proposition 2 is robust to introducing a small amount of dominant strategy types and discounting. That is, for small $\epsilon > 0$ and large $\delta < 1$, the probability that the stag is caught in any PBE is close to 1.

Proposition 3 *For all sequences $\left(\epsilon_k, \delta_k\right)_k$ such that $\left(\epsilon_k, \delta_k\right)_k \to (0, 1)$, and $\delta_k < 1$, $\forall k$, for all sequences $\left(E_k\right)_k$ such that E_k is a PBE of $\Gamma\left(n, \epsilon_k, \delta_k, \overrightarrow{0}, T\right)$, $p_k \to 1$, where p_k is the probability that the stag is caught in E_k.*

Proof: Fix a sequence $\left(\epsilon_k, \delta_k\right)_k \to (0, 1)$. Let $\overrightarrow{x}_k$ denote the n dimensional vector of lower bounds on the n players' types. Let S be a subset of the n players, let $\overrightarrow{x}_k^S$ be the restriction of $\overrightarrow{x}_k$ to S. Let $\Gamma\left(S, \epsilon_k, \delta_k, \overrightarrow{x}_k^S, T'\right)$ denote the game in which players in S play the corresponding game for T' periods, where $|S| \leq T' \leq T$. Let E_k denote an arbitrary equilibrium of $\Gamma\left(n, \epsilon_k, \delta_k, \overrightarrow{x}_k, T\right)$, let E_k^S denote an arbitrary equilibrium of $\Gamma\left(S, \epsilon_k, \delta_k, \overrightarrow{x}_k^S, T'\right)$.

We prove the proposition by induction on the number of players. First we state the induction hypothesis(IH).

(IH): $\forall x < 1$, $\forall \left(\overrightarrow{x}_k\right)_k$ such that $x_{kj} \leq x$, $\forall k$, $\forall j = 1,..,n$, $\forall S$, $\forall \left(E_k^S\right)_k$, $P\left(\text{ everybody in } S \text{ hunts in } E_k^S\right) \to 1$.

The IH obviously holds when $n = 1$. Suppose it holds for $n \geq 1$. Now suppose there are $n+1$ players. Let E_k denote an equilibrium of $\Gamma_k := \Gamma\left(n+1, \epsilon_k, \delta_k, \overrightarrow{x}_k, T\right)$, where $T \geq n+1$. We need to show that $\forall x < 1$, $\forall \left(\overrightarrow{x}_k\right)_k$ such that $x_{kj} \leq x$, $\forall k$, $\forall j = 1,..,n+1$, $\forall \left(E_k\right)_k$, $P\left(\text{ everybody hunts in } E_k\right) \to 1$.

Proof by way of contradiction. The contradiction hypothesis is

(CH): $\exists \left(\epsilon_k, \delta_k\right)_k \to (0,1)$, $\exists x < 1$, $\exists \left(\overrightarrow{x}_k\right)_k$ such that $x_{kj} \leq x$, $\forall k$, $\forall j = 1,..,n+1$. $\exists \left(E_k\right)_k$, $p_k := P\left(\text{ everybody hunts in } E_k\right) \to p < 1$.

The first implication of CH: By Proposition 2(i), E_k is characterized by a collection of cutoff points. Let c_k^j denote the first period cutoff type of player j in E_k, $j = 1,..,n+1$. Let $c_k := \max_j \left\{c_k^j\right\}$, then $c_k \nrightarrow 1$, since otherwise $p_k \to 1$ by IH. Takeing a subsequence if necessary let $c_k \to c < 1$. The first implication of CH is, $\forall$ type c' of any player j, if j hunts in period 1 in Γ_k, then her expected payoff is $-c' + \alpha_k$, where $\alpha_k \to 1$, by IH.

The second implication of CH: Let A_k denote the event that the stag is not caught in E_k. Then $A_k \subseteq [0, 1+\epsilon]^{n+1}$. Moreover, by the cutoff property of E_k, A_k is a finite union of mutually disjoint product sets. That is $A_k = \cup_{i=1}^{I(k)} A_k^i$, where $I(k) \leq B < \infty$, and B only depends on the number of players and the number of cutoff points. A_k^i is the ith product set such that if the players' types fall into this set, then the stag will not be caught in E_k. A_k^i can be written as $A_k^i = \Pi_{j=1}^{n+1} A_{kj}^i$, where A_{kj}^i is the jth component of A_k^i, $j = 1,..,n+1$. Let $A_k^{i(k)}$ denote the event that receives the highest probability among all the A_k^i's, $i = 1,..,I(k)$. By CH, $P\left(A_k^{i(k)}\right) \to q > 0$. But $P\left(A_k^{i(k)}\right) = \Pi_{j=1}^{n+1} P(A_{kj}^{i(k)})$, hence $\forall j$, $\left(P\left(A_{kj}^{i(k)}\right)\right)_k \to q' > 0$. Hence $\forall j$, $\exists c < 1$, $\exists K_0 > 0$, such that $\forall k \geq K_0$, $\exists c(k) \leq c$, and $c(k) \in A_{kj}^{i(k)}$. Notice that once $c(k) \in A_{kj}^{i(k)}$, the equilibrium payoff to type $c(k)$ of player j is bounded away from below $1 - c(k)$, because so long as $c_l \in A_{kl}^{i(k)}$, $\forall l \neq j$, which happens with non-negligible probability $\Pi_{l \neq j} P(A_{kl}^{i(k)})$, type $c(k)$ of player j gets at most 0 in equilibrium. More precisely, $\exists z > 0$, $\exists K_1 > 0$, such that $\forall k \geq K_1$, $\Pi_{l \neq j} P(A_{kl}^{i(k)}) \geq z$. Therefore, $\forall k \geq K := \max\{K_0, K_1\}$, type $c(k)$ of player j gets at most $(1-z)(1-c(k))$ in E_k. The second implication of CH is, therefore, $\forall j$, $\exists c < 1$, $\exists z > 0$, $\exists K > 0$, such that $\forall k \geq K$, $\exists c(k) \leq c$, and type $c(k)$ of player j's equilibrium payoff is at most $(1-z)(1-c(k))$.

By the first implication of CH, for k sufficiently large, type $c(k)$ of player j can guarantee herself an expected payoff arbitrarily close to $1 - c(k)$. Hence if k is large enough, the two implications contradict each other. ■

Even if there is a significant discounting and the proportion of dominant strategy types is high, complete coordination failure is impossible in the dynamic game, as shown in the following proposition.

Proposition 4 *If $\epsilon > 0$ and $\delta < 1$, or $\delta = 1$ and (A) holds, then in any PBE that satisfies (A) the stag is caught with positive probability.*

Proof: We show this by induction on the number of players. This is obvious if $n = 1$. Now assume that this is true for $\Gamma\left(k, \epsilon, \delta, \overrightarrow{0}, T\right)$, $\forall T \geq k$, $\forall k \leq n$, where $\overrightarrow{0}$ is the k dimensional vector $(0, 0, .., 0)$. We need to show that this is true for $\Gamma\left(n+1, \epsilon, \delta, \overrightarrow{0}, T\right)$, $\forall T \geq n+1$, where $\overrightarrow{0}$ is the $n+1$ dimensional vector $(0, 0, .., 0)$. Suppose otherwise that there exists a PBE of $\Gamma\left(n+1, \epsilon, \delta, \overrightarrow{0}, T\right)$ in which the stag is caught with probability 0. Then it must be that in the first period, at least one player's cutoff type is 0, since otherwise there is a positive probability that everybody moves in the first period. Let S denote the set of players whose cutoff types in period 1 are 0. If $|S| < n+1$, then there is a positive probability that everybody in S^c moves in period 1, but then by the induction hypothesis, there is a positive probability that everybody in S follows up in the continuation game, which is a contradiction. If $|S| = n+1$, then for any player j, if c_j is small enough, then by the induction hypothesis, c_j should deviate by moving in period 1 and obtain a positive expected payoff. On the other hand, the equilibrium payoff to type c_j is 0, contradiction. ■

5 Existence of Equilibrium

The robustness result in the last section only applies when the set of PBE is non-empty. In this section, we establish the existence of equilibrium. For $\delta = 1$, we show existence by constructing an equilibrium that satisfies assumption (A). For $\delta < 1$, we offer a general existence proof, and explain why direct construction is difficult.

5.1 No Discounting Case

In this case we construct an equilibrium that not only satisfies assumption (A), but also meets the following three requirements.

(a) Symmetry.

(b) Whenever a player is indifferent between hunting and waiting, she always chooses to hunt.

(c) If a player does not hunt when she is more optimistic about other players' types, then she doesn't hunt when she is less optimistic.[2]

We do it in two steps. First we claim that any equilibrium in the *infinite* horizon game ($T = \infty$) that satisfies (a), (b) and (c) corresponds to an equilibrium in the finite horizon game ($n \leq T < \infty$) that also satisfies (a), (b) and (c). Then we show that there exists a unique equilibrium in the infinite horizon game that satisfies (a), (b) and (c).

Claim 1 *Let Γ denote the infinite horizon game, and Γ^T the T period game, $T \geq n$. Let E be an equilibrium of Γ that satisfies (a), (b) and (c). Then the restriction of the equilibrium path of E to the first T periods can also be supported as an equilibrium path in Γ^T.*

Proof: Let E_T be the strategy such that whenever the number of players left is less than or equal to the number of periods left, play according to E, otherwise don't hunt. Notice that E_T trivially satisfies (a), (b) and (c). We show that E_T is a PBE in Γ^T. We prove this in steps.

Step 1. After each history, if the number of players left is less than or equal to the number of periods left, then the expected payoff a player gets by following E_T is the same as she could get in Γ. Before we prove this, we state it formally as follows. Let $\pi^T(h^t, c_j)$ be the expected payoff to type c_j of player j in Γ^T after history h^t, if j follows E_T. Let $\pi(h^t, c_j)$ be the expected payoff to type c_j of player j in Γ after history h^t, if j follows E. Let $n(h^t)$ be the number of players left after h^t, and $T(h^t)$ the number of periods left after h^t. Step 1 claims that $\forall h^t$, if $n(h^t) \leq T(h^t)$, then $\pi^T(h^t, c_j) = \pi(h^t, c_j)$, $\forall c_j$, $\forall j$.

Now we prove Step 1.

Suppose after h^t, E_T prescibes "hunt" for type c_j of player j. Need to show $\pi^T(h^t, c_j) = \pi(h^t, c_j)$. Let $v(n, x, T)$ denote the expected discounted value of the stag in Γ^T when everybody follows E_T,[3] where x is the lower bound on the n players' cost. Let $v(n, x)$ denote the expected discounted value of the stag in Γ when everybody follows E. First we prove that $\forall x < 1$, $\forall n$, $\forall T \geq n$, $v(n, x, T) = v(n, x)$.

[2] Being "more optimistic" means putting a smaller lower bound on other players' types.

[3] Suppose there is an "outside" player whose cost is 0, whose participation is not essential, and whose payoff function is the same as the rest of the players. Then $v(n, x, T)$ is this player's payoff in Γ^T, if everybody else follows E_T.

Proof by induction on the number of players. First it is obvious that $v(1,x,T) = v(1,x)$. The induction hypothesis is that $\forall k \leq n$, $\forall x$, $\forall T \geq k$, $v(k,x,T) = v(k,x)$. We need to show $\forall T \geq n+1$, $v(n+1,x,T) = v(n+1,x)$.

For $i = 0,..,n+1$, let p_i denote the probability that in the first period i players hunt in Γ^T where everybody plays E_T. Let x' denote the first period cutoff prescribed by E_T. If everybody's type is above x', then nobody hunts in period 1. By (c), nobody hunts thereafter, the expected discounted value of the stag must be 0. Therefore,

$$v(n+1,x,T) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n,x',T-1) + .. + p_{n+1} \cdot 1,$$

and

$$v(n+1,x) = p_0 \cdot 0 + p_1 \cdot \delta \cdot v(n,x') + .. + p_{n+1} \cdot 1.$$

Hence by the induction hypothesis, $v(n+1,x,T) = v(n+1,x)$. Now if after h^t, E_T prescribes "hunt" for type c_j of j, then

$$\begin{aligned} & \pi^T\left(h^t, c_j\right) \\ = \; & p_0 \cdot \delta \cdot v\left(n\left(h^t\right)-1, x', T\left(h^t\right)-1\right) + p_1 \cdot \delta \cdot v\left(n\left(h^t\right)-2, x', T\left(h^t\right)-1\right) \\ & +.. + p_{n(h^t)-1} \cdot 1 - c_j \\ = \; & p_0 \cdot \delta \cdot v\left(n\left(h^t\right)-1, x'\right) + p_1 \cdot \delta \cdot v\left(n\left(h^t\right)-2, x'\right) + .. + p_{n(h^t)-1} \cdot 1 - c_j \\ = \; & \pi\left(h^t, c_j\right), \end{aligned}$$

where x' is the current period cutoff prescribed by E_T, and p_k, $k = 0,..,n(h^t)-1$, is the probability that k out of $n(h^t)-1$ players will move in the current period.

Suppose after h^t, E_T prescribes "wait" for type c_j of player j. In this case proof by induction on $n(h^t)$. The proof is trivial if $n(h^t) = 1$. Now suppose step 1 is true for $n(h^t) \leq n$. We need to show that step 1 is also true for $n(h^t) = n+1$. Let p_k, $k = 0,..,n(h^t)-1$, be the probability that k out of $n(h^t)-1$ players will move in the current period. Let h_i^{t+1}, $i = 0,1,..,n(h^t)-1$, be the history in which right after h^t, i players hunt in period t. Then,

$$\begin{aligned} & \pi^T\left(h^t, c_j\right) \\ = \; & p_0 \cdot \delta \cdot \pi^{T-1}\left(h_0^{t+1}, c_j\right) + p_1 \cdot \delta \cdot \pi^{T-1}\left(h_1^{t+1}, c_j\right) + .. + p_{n(h^t)-1} \cdot \delta \cdot \pi^{T-1}\left(h_{n(h^t)-1}^{t+1}, c_j\right) \\ = \; & p_0 \cdot \delta \cdot \pi\left(h_0^{t+1}, c_j\right) + p_1 \cdot \delta \cdot \pi\left(h_1^{t+1}, c_j\right) + .. + p_{n(h^t)-1} \cdot \delta \cdot \pi\left(h_{n(h^t)-1}^{t+1}, c_j\right) \\ = \; & \pi\left(h^t, c_j\right), \end{aligned}$$

where the second equality is because of the induction hypothesis and the fact that $\pi^{T-1}\left(h_0^{t+1}, c_j\right) = \pi\left(h_0^{t+1}, c_j\right) = 0$, due to (c).

Step 2. If the number of players left is more than the number of periods left, it is obviously optimal not to hunt given that nobody else does.

Step 3. The payoff a player gets by deviating after a given history h^t, however, can be no higher than the payoff she gets by deviating in the infinite horizon game. To see this, notice that there are two types of deviations, (1) A player should hunt but does not hunt. First of all, it must be the case that before the player makes the decision, $n\left(h^t\right) \leq T\left(h^t\right)$. If $n\left(h^t\right) < T\left(h^t\right)$, then the payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by step 1, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game. If $n\left(h^t\right) = T\left(h^t\right)$, then the continuation payoff after h_0^{t+1} of such deviation is 0 in Γ^T, and the continuation payoff after h_0^{t+1} of the same deviation is non-negative in Γ. At the same time, by step 1, the continuation payoff after h_i^{t+1} of the deviation in Γ^T is the same as the continuation payoff after h_i^{t+1} of the deviation in Γ, for any $i = 1, .., n\left(h^t\right) - 1$. Therefore, the expected payoff of the deviation in Γ^T can be no higher than the expected payoff of the deviation in Γ. (2) A player should not hunt but hunts. If right before such deviation the number of players left is more than the number of periods left, then the deviation is clearly not profitable, otherwise the payoff from such deviation is the same as the payoff from the same deviation in the infinite horizon game, because by step 1, the payoff from such deviation is the same convex combination of the same expected continuation payoffs as the payoff from such deviation in the infinite horizon game. ■

By Claim 1, in order to establish existence for the no discounting, finite horizon case, it suffices to show existence for the no discounting, infinite horizon case.

Proposition 5 *In the n player stag hunt game with infinite horizon and no discounting, there is a unique PBE that satisfies (a), (b) and (c). The equilibrium is characterized by a sequence of cutoff values*

$$1 = g(1) > g(2) > ... > g(n) > 0,$$

where $g(k)$ is such that in the continuation game with k players, any type $c \leq g\left(k\right)$ is indifferent between hunting and waiting in the current period, and any type $c > g\left(k\right)$ strictly prefers to wait in the current period, $1 \leq k \leq n$. Moreover, $g\left(n\right) \longrightarrow 0$, as $n \longrightarrow \infty$.

Proof: See the Appendix. ■

We will sketch the intuition for the two player game here. First we introduce some notations for the n player game in general. Let $p(k, x)$ denote the probability of a failure in a k-player game, where $1 \leq k \leq n$, and x is the lower bound on the players' type. Let $p(F, k, x)$ denote the probability of a failure (the stag is not caught) in the k player game when one player hunts for sure and the other player's cost is at least x, where $2 \leq k \leq n$. Let $p^{k\ m}(c|x)$ denote the probability that k players out of m players have types no higher than c, conditional on that the lower bound on everybody's cost is x, where $0 \leq k \leq m \leq n$. Let $v(k, c)$ denote the equilibrium continuation payoff to the cutoff type in the k player game where the opponents' types are above c, where $2 \leq k \leq n$. The two player indifference equation can thus be written as

$$-c + (1 - p(F, 2, x)) \cdot 1 = p^{1\ 1}(c|x)(1 - c) + p^{0\ 1}(c|x)\, v(2, c). \tag{1}$$

By Lemma 1, if $\delta = 1$, then any PBE that satisfies (A) has the cutoff property. Since condition (b) implies (A), any PBE that satisisfies (a), (b) and (c) must have the cutoff property. This implies that we can write out $v(2, c)$ as follows.

$$v(2, c) = \max\{0, -c + (1 - p(F, 2, c))\}.$$

We rewrite equation (1) as

$$LHS_2(x, c, c) = RHS_2(x, c, c),$$

where the first argument is the lower bound on cost, the second argument is the reference player's type, and the third argument is the cutoff that the opponent uses.

Now any PBE that satisfies (a) and (b) can be characterized by a cutoff type $\widehat{c_2}$, such that $LHS_2(x, c, \widehat{c_2}) \geq RHS_2(x, c, \widehat{c_2})$, $\forall c \leq \widehat{c_2}$, and $LHS_2(x, c, \widehat{c_2}) < RHS_2(x, c, \widehat{c_2})$, $\forall c > \widehat{c_2}$.

Such $\widehat{c_2}$ must solve (1), but not every solution to (1) can be $\widehat{c_2}$. I show that such $\widehat{c_2}$ exists, it is unique, and it doesn't depend on x. As proved in the Appendix, (1) has a continuum of solutions, and $\widehat{c_2}$ is the largest one. In fact, a solution to (1) can become $\widehat{c_2}$ if and only if it solves

$$-c + (1 - p(F, 2, c)) = 0.$$

It is easy to show that $p(F, 2, c) = p(1, c)$. Hence $\widehat{c_2}$ solves

$$-c + (1 - p(1, c)) = 0.$$

Figure 1 summarizes the discussion.

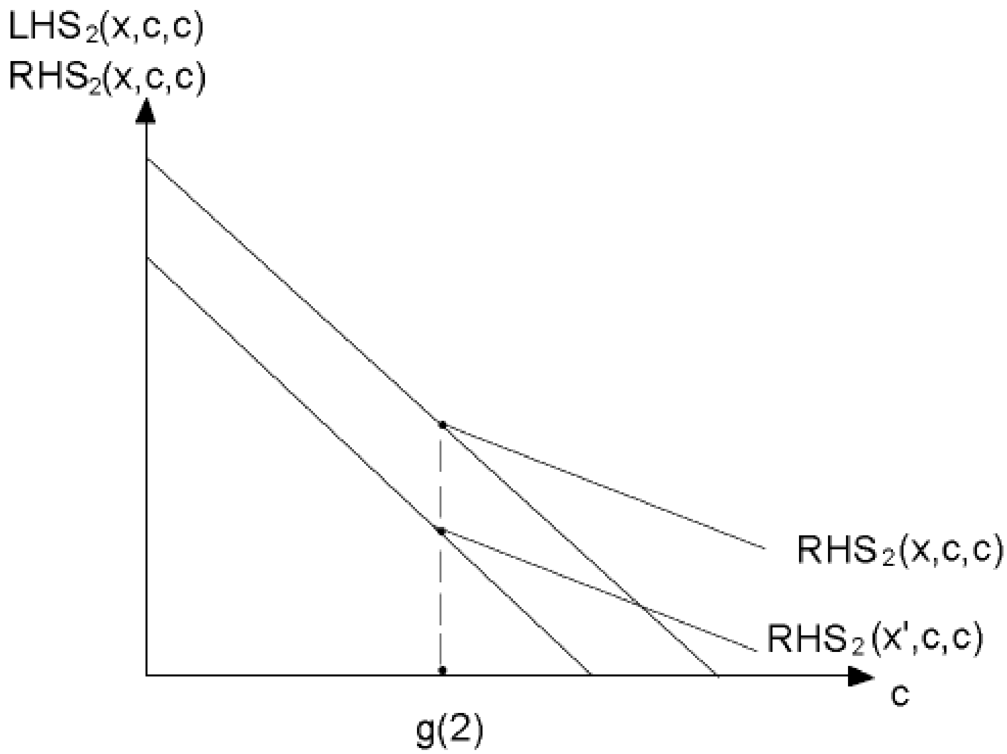

Figure 1: Two player no discounting

In Figure 1, the l.h.s. and r.h.s. of (1) coincide up to $g(2)$, where $g(2) := \widehat{c_2}$, then the r.h.s. kinks upward. As we change x into $x' > x$, both sides will shift downward, but the kink point remains the same.

In general, the sequence of the cutoffs can be found inductively as follows.

$$
\begin{aligned}
g(1) &= 1, \\
p(1,x) &= p^{0\ 1}(g(1)|x), \\
1-g(2) &= p(1,g(2)), \\
p(2,x) &= p^{0\ 2}(g(2)|x)\cdot 1 + p^{1\ 2}(g(2)|x)\, p(1,g(2)), \\
1-g(3) &= p(2,g(3)), \\
&\quad . \\
&\quad . \\
p(n-1,x) &= p^{0\ n-1}(g(n-1)|x)\cdot 1 + .. + p^{n-2\ n-1}(g(n-1)|x)\, p(1,g(n-1)), \\
1-g(n) &= p(n-1,g(n)).
\end{aligned}
$$

As we can see from these equations, the cutoff values and the conditional probabilities of failure are like two strands that feed on each other.

5.2 Discounting Case

Now we focus on the case where players discount both cost and reward at the same rate $\delta < 1$. In this case, each player prefers the stag to be caught as early as possible, but also prefers his cost to be incurred as late as possible. The tradeoff between hunting and waiting is the following. Hunting may accelerate the process to success, it may even make the difference between success and failure; waiting may avoid the cost when it turns out that too few people hunt, it may also synchronize the expenditure with the reward, i.e., without affecting the time at which the stag is caught, a player always wants to be the last one who hunts.

We keep the three restrictions (a), (b) and (c) on equilibrium as before. If we can construct an equilibrium in the infinite horizon game that satisfies (a), (b) and (c), then by Claim 1 (which does not depend on δ), we are done. Hence in the following discussion, up to but not including the existence proposition, we consider the infinite horizon game. As we will see next, the three restrictions can be satisfied in the two player game. Once there are more than two players, there may not exist any equilibrium that satisfies these restrictions.

Let $c = g(n,x)$ denote the first period cutoff in an n-player game with lower bound on cost being x. Let $w(k,c)$ denote the expected gross reward to the cutoff player when there are k players left in the game (the cutoff player has already moved) whose types are above c, $k = 1, ..., n-1$. Let $v(k,c)$ denote the expected net reward to the cutoff player when there are k players left, including the cutoff player himself, whose type is c, and the other $k-1$ players costs are above c. Note that $g(n,x)$ cound be empty-valued or multi-valued, but it must satisfy the following indifference equation.

$$\begin{aligned}
& -c+p^{n-1\ n-1}(c|x)\cdot 1 \\
& +p^{n-2\ n-1}(c|x)\delta w(1,c) \\
& +... \\
& +p^{0\ n-1}(c|x)\delta w(n-1,c) \\
= \quad & \delta[p^{n-1\ n-1}(c|x)(1-c) \\
& +p^{n-2\ n-1}(c|x)v(2,c) \\
& +... \\
& +p^{0\ n-1}(c|x)v(n,c)].
\end{aligned} \tag{2}$$

The value fuctions in (2) need to be specified to make sure that it is a well defined equation. Let us first solve the two player game.

The two player equation can be written as

$$\begin{aligned}
& -c+p^{1\ 1}(c|x)\cdot 1+p^{0\ 1}(c|x)\delta w(1,c) \\
= \quad & \delta[p^{1\ 1}(c|x)(1-c)+p^{0\ 1}(c|x)v(2,c)],
\end{aligned} \tag{3}$$

where $w(1,c)=p^{1\ 1}(1|c)$, and $v(2,c)=\max\{0,-c+\delta w(1,c)\}$.

I can write out $v(2,c)$ in this way because (i) in this situation, type c is the smallest type, so if equilibrium doesn't allow type c to move, then by the cutoff property and symmetry of PBE, nobody else is allowed to move; (ii) if the opponent doesn't move when he is more optimistic, then he doesn't move when he is less optimistic, by condition (c).

Let c_2^* solve

$$-c+\delta w(1,c)=0.$$

Since $w(1,c)$ is non-increasing in c, c_2^* is unique. Moreover, $c_2^*(\delta)$ is increasing in δ, and $c_2^*(\delta)\longrightarrow g(2)$ as $\delta\longrightarrow 1$.

Lemma 2 $\forall x\in[0,c_2^*]$, $\exists g(2,x)\in[x,1]$, such that $g(2,x)$ solves (3).

Proof: Rewrite (3) as $LHS_2(x,c)=RHS_2(x,c)$.

Since $-c+\delta w(1,c)$ is decreasing in c, we have $LHS_2(x,x) > RHS_2(x,x)$ and $LHS_2(x,1) < RHS_2(x,1)$, $\forall x \in [0,c_2^*]$. By the Intermediate Value Theorem, $\exists c \in [x,1]$ that solves (2). ■

Remark If $x > c_2^*$, define $g(2,x) = 0$[4], i.e., if both players are above c_2^*, then neither hunts in the current period. It is equilibrium behavior since $-c+\delta w(1,x) < 0$ for $c \geq x$ and $x > c_2^*$.

Let $\underline{g}(2,x)$ denote the smallest solution to (3) for $x \in [0,c_2^*]$. By Lemma 1 and continuity of both sides of (3), $\underline{g}(2,x)$ is well defined.

Lemma 3 $\underline{g}(2,x)$ is strictly decreasing on $[0,c_2^*]$, and $\underline{g}(2,c_2^*) = c_2^*$.

Proof: We first prove monotonicity. Let $p^{k\ m}([x,x']\,|x)$ denote the probabilty that k players out of m players have types falling into the interval $[x,x']$, conditional on that their types being no less than x, where $0 \leq k \leq m \leq 1$. Let $I_{k\ m}$ denote the event that k players out of m players fall into the interval $[x,x']$.

Decompose the conditional probabilities in (3) in the following way:

$$\begin{aligned} & p(\bullet|x) \\ = \ & p^{1\ 1}([x,x']\,|x)p(\bullet|I_{1\ 1}) + p^{0\ 1}([x,x'])p(\bullet|I_{0\ 1}) \\ = \ & p^{1\ 1}([x,x']\,|x)p(\bullet|I_{1\ 1}) + p^{0\ 1}([x,x'])p(\bullet|x'). \end{aligned}$$

Decompose c into $c[p^{1\ 1}([x,x']\,|x) + p^{0\ 1}([x,x']\,|x)]$.

Plug the decompositions into $LHS_2(x,c)$ and $RHS_2(x,c)$, rearrange, we have

$$\begin{aligned} & LHS_2(x,c) - RHS_2(x,c) \\ = \ & p^{1\ 1}([x,x']\,|x)\left((1-c) - \delta(1-c)\right) \\ & + p^{0\ 1}([x,x']\,|x)\left(LHS_2(x',c) - RHS_2(x',c)\right). \end{aligned} \tag{4}$$

Fix $x \in [0,c_2^*)$, let $c = \underline{g}(2,x)$, fix $x' \in (x,c)$, then by (4)

$$\begin{aligned} & LHS_2(x,c) - RHS_2(x,c) \\ = \ & p^{1\ 1}([x,x']\,|x)\left((1-c) - \delta(1-c)\right) + p^{0\ 1}([x,x']|x)\left(LHS_2(x',c) - RHS_2(x',c)\right) \\ = \ & 0. \end{aligned}$$

[4] In fact, for $x > c_2^*$, if δ is large, then no solution exists for equation (3), hence $g(2,x) = 0$ is not only an equilibrium, but also the unique equilibrium. I leave the proof of this to the Appendix.

This implies that $LHS_2(x', c) - RHS_2(x', c) < 0$. Hence at x' there is a solution below c, hence $\underline{g}(2, x') < c$.

Finally $\underline{g}(2, c_2^*) = c_2^*$ because $LHS_2(c_2^*, c_2^*) = RHS_2(c_2^*, c_2^*)$. It's important to notice that even if $\underline{g}(2, x)$ is not left continuous at c_2^*, there can not be an upper jump at c_2^*, therefore, $\underline{g}(2, x) \geq c_2^*$, $\forall x \in [0, c_2^*]$. ■

Lemma 4 $\exists \underline{\delta} < 1$, such that $\forall \delta > \underline{\delta}$, (3) has a unique solution for each $x \in [0, c_2^*(\delta)]$.

Proof: We rewrite (3) as

$$LHS_2(x, c, \delta) = RHS_2(x, c, \delta).$$

Let $\Delta_2(x, c, \delta) := LHS_2(x, c, \delta) - RHS_2(x, c, \delta)$. Notice that $\Delta_2(x, c, \delta)$ kinks at $c_2^*(\delta)$.

Recall from the no discounting case that $p(1, x)$ is the probability of a failure in a one player game, x being the lower bound on the player's type, and $p(1, x)$ reaches its minimum at $x = 0$, and $p(1, 0) < 1$. Let $\widetilde{c} := 1 - p(1, 0)$.

First we show that

$$LHS_2(x, c, \delta) < 0, \forall x \in [0, c_2^*], \forall c > \widetilde{c}, \forall \delta \in (0, 1].$$

This is because

$$\begin{aligned} LHS_2(x, c, 1) &= -c + 1 - p(1, x) \leq -c + 1 - p(1, 0) \\ &\implies \forall c > \widetilde{c}, LHS_2(x, c, 1) < 0. \end{aligned}$$

But $LHS_2(x, c, \delta) \leq LHS_2(x, c, 1)$, hence $LHS_2(x, c, \delta) < 0$.

By Lemma 3, there is no solution in $[0, c_2^*(\delta)]$ to (3) at any $x \in [0, c_2^*(\delta)]$. Moreover, there is no solution in $(\widetilde{c}, 1]$, either, because we show above that on that range $LHS_2(x, c, \delta) < 0$, and $RHS_2(x, c, \delta)$ is always nonnegative . Hence to prove uniqueness, it suffices to show that $\Delta_2(x, c, \delta)$ is strictly decreasing over $c \in [c_2^*(\delta), \widetilde{c}]$, for sufficiently large δ, that is independent of x.

Notice that over this range of c, (3) becomes

$$\begin{aligned} & -c + p^{1\ 1}(c|x) \cdot 1 + p^{0\ 1}(c|x)\delta w(1,c) \\ = \quad & \delta p^{1\ 1}(c|x)(1-c). \end{aligned}$$

Pick c and c' in this range such that $c < c'$. It suffices to show that $\Delta_2(x,c,\delta) - \Delta_2(x,c',\delta) > 0$.

We can write $w(1,c) = p^{1\ 1}([c,c']|c) \cdot 1 + p^{0\ 1}([c,c']|c)\, w(1,c')$. Substituting the decomposition into $\Delta_2(x,c,\delta)$, we find that

$$\begin{aligned} & \Delta_2(x,c,\delta) - \Delta_2(x,c',\delta) \\ = \quad & (c'-c)\left(1 - \delta p^{1\ 1}(c'|x)\right) + p^{1\ 1}([c,c']|x)(2\delta - \delta c - 1). \end{aligned}$$

Hence $\exists \underline{\delta} < 1$, such that $\forall \delta > \underline{\delta}$, $\Delta_2(x,c,\delta) - \Delta_2(x,c',\delta) > 0$, regardless of x, c, and c'. ■

Lemma 5 If δ is large enough, $g(2,x)$ is continuous in x on $[0, c_2^*]$.

Proof: By Lemma 4, for sufficiently large δ, $g(2,x) = \overline{g}(2,x) = \underline{g}(2,x)$, where $\overline{g}(2,x)$ is the largest solution to (3). We show that $\overline{g}(2,x)$ is *u.s.c.* and $\underline{g}(2,x)$ is *l.s.c.*.

It suffices to prove the following general result.

Let $F(x,y)$ be continuous in (x,y). Suppose $\forall x \in [a,b]$, $\exists y \in [0,1]$, such that $F(x,y) = 0$. Let $\overline{f}(x) := \max\{y \in [0,1] \,|F(x,y) = 0\}$, then $\overline{f}$ is *u.s.c.* in $[a,b]$.

Proof by way of contradiction. Suppose $\exists x_0 \in [a,b]$, such that $\overline{f}$ is not *u.s.c.* at x_0. Then $\exists \epsilon_0 > 0$, such that $\forall \delta > 0$, $\exists x \in B(x_0,\delta)$, such that $\overline{f}(x) \geq \overline{f}(x_0) + \epsilon_0$. Then we can construct a sequence $\{x_n\}$, such that $x_n \longrightarrow x_0$, and $\overline{f}(x_n) \geq \overline{f}(x_0) + \epsilon_0$, $\forall n$. Choosing a subsequence if necessary, let $\overline{f}(x_n) \longrightarrow y_0$. By continuity of F, $F(x_0,y_0) = 0$, contradicting to $\overline{f}(x_0)$ being the largest solution.

That $\underline{g}(2,x)$ is *l.s.c.*is proved analogously. ■

We summarize the above results in the following proposition.

Proposition 6 *In the 2 player game with discounting, if δ is sufficiently large, then there exists a unique PBE that satisfies (a), (b) and (c). Moreover, the equilibrium can be characterized by a continuous and strictly decreasing function $g(2,x)$, such that $g(2,x)$ is the cutoff type in the 2 player game with lower bound x.*

Now consider the 3-player equation. When I write down the three player equation below, I take $g(2,x)$ to be the continuation policy function in the two player continuation game.

Let

$$w(1,c) = p^{1\ 1}(1|c),$$

$$w(2,c) = \begin{cases} p^{2\ 2}(g(2,c)|c) \cdot 1 + p^{1\ 2}(g(2,c)|c)\delta w(1,g(2,c)) & \text{if } c \leq c_2^* \\ 0 & \text{if } c > c_2^* \end{cases},$$

$$\widehat{v}(2,c) = \begin{cases} -c + p^{1\ 1}(g(2,c)|c) \cdot 1 + p^{0\ 1}(g(2,c)|c)\delta w(1,g(2,c)) & \text{if } c \leq c_2^* \\ 0 & \text{if } c > c_2^* \end{cases},$$

and

$$v(3,c) = \max\left\{0, -c + \delta w(2,c)\right\}.$$

The three player indifference equation can thus be written as

$$\begin{aligned} & -c + p^{2\ 2}(c|x) \cdot 1 + p^{1\ 2}(c|x)\delta w(1,c) + p^{0\ 2}(c|x)\delta w(2,c) \\ = \ & \delta\left[p^{2\ 2}(c|x)(1-c) + p^{1\ 2}(c|x)\widehat{v}(2,c) + p^{0\ 2}(c|x)v(3,c)\right]. \end{aligned} \qquad (5)$$

Next we show that the solution to (5) may not satisfy condition (c). Hence an equilibrium that satisfies (a), (b) and (c) in the three player game may not exist. Why is it possible that some player is willing to hunt in period 2 when she is more pessimistic about other players, but she is not willing to hunt in period 1 when she is more optimistic? There are three effects going on here: Delay effect, if you anticipate that others are going to move early, you want to move early, too; Leading effect, if you anticipate that others are going to move late, you want to move early to encourage them to follow you; Synchronization effect, if you anticipate that someone is going to move early, and someone else is going to move late, then you want to move late to synchronize your action with the "bottleneck" player. It is the third effect that might frustrate condition (c). Before we construct a counter-example, we make the following preparations.

Lemma 6 $w(2,c) < w(1,c)$, $\forall c \in [0, c_2^*]$.

Proof : Let i be the player in a one player game. Imagine there is another player j, who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let i and j denote i's type and j's type. Then the set of i's types for which i will succeed in the one player game with lower bound c can be written as the union of $\{(i,j) \geq (c,c) \,|\, i \in [c, g(2,c)],\ j \text{ is anywhere}\}$ and $\{(i,j) \geq (c,c) \,|\, i \in (g(2,c), 1],\ j \text{ is anywhere}\}$. The union in turn, contains the union of $\{(i,j) \geq (c,c) \,|\, i \in [c, g(2,c)],\ j \in [c,1]\ \}$ and $\{(i,j) \geq (c,c) \,|\, i \in (g(2,c), 1],\ j \in [c, g(2,c)]\}$, which is equal to the set of i and j's types for which i and j will succeed in the two player game with lower bound c.

Now there are two reasons $w(1,c)$ must be larger than $w(2,c)$. One is that two players succeed only if one player succeed, the other is two players can never succeed earlier than one player does. ■

Lemma 7 $w(2,c)$ is strictly decreasing in c over $[0, c_2^*]$.

Proof: $\forall c'$ *s.t.* $c < c' < c_2^*$. From Lemma 2 we know that $c' < \overline{g}(2,c)$. We can decompose $w(2,c)$ as follows.

$$w(2,c) = p^{2\ 2}([c,c']\,|c) \cdot 1 + p^{1\ 2}([c,c']\,|c)\, w(1,c') + p^{0\ 2}([c,c']\,|c)\, w(2,c').$$

Hence to show $w(2,c) > w(2,c')$, it suffices to show that $w(1,c') \geq w(2,c')$, which follows from Lemma 6. ■

Let c_3^* solve

$$-c + \delta w(2,c) = 0.$$

Then c_3^* is unique, and $c_3^* < c_2^*$, by Lemma 6 and Lemma 7. Moreover, $c_3^*(\delta)$ is increasing in δ, and $c_3^*(\delta) \longrightarrow g(3)$ as $\delta \longrightarrow 1$.

Let $g(3,x)$ denote a solution to (5). If we can find $\epsilon > 0$, $\delta < 1$ and a distribution function F, such that $g(3,x)$ is the unique solution to (5) and $g(3,x) < c_3^*$, then we have a counter-example. To see why, for type $c \in (g(3,x), c_3^*)$, this type does not hunt in period 1 when she is more optimistic about other players' types. In any monotone equilibria, she should not hunt in the second period upon seeing inaction in the first period, and nobody should for the same reason. Hence her continuation payoff in equilibrium after seeing inaction in period 1 is 0. But if she deviates, her expected payoff is $-c+\delta w(2,c) > -c_3^*+\delta w(2,c_3^*) = 0$.

Synchronization effect is likely to make a difference if the distribution function F is not skewed, so that the probability that the other two players are located on two sides of the cutoff is relatively high. At the same time, for large δ, c_3^* is close to $g(3)$, and for small $\epsilon > 0$, $g(3)$ is close to 1. So for small ϵ and large δ, c_3^* is large, hence makes it easier for $g(3,x)$ to fall below it. It is therefore no surprise that a counter-example occurs at a combination of small ϵ, large δ and the least skewed distribution, uniform distribution. Numerical computation shows that at the combination where $\epsilon = 0.0001, \delta = .999, x = 0$, and F =uniform distribution, $g(3,0) \approx 0.52$, but $c_3^* \approx 0.96$.

This example shows that in general, it is difficult to explicitly construct an equilibrium in the discounting case. Nevertheless, existence of equilibrium is still established by the following proposition.

Proposition 7 *$\forall \epsilon \geq 0$, $\forall \delta < 1$, $\forall 1 \leq n < \infty$, $\forall \overrightarrow{x} \in [0, 1+\epsilon]^n$,$\forall 1 \leq T < \infty$, there exists a PBE in $\Gamma(n, \epsilon, \delta, \overrightarrow{x}, T)$.*

Proof: For simplicity, we only prove the case where $\overrightarrow{x} = \overrightarrow{0}$. For other $\overrightarrow{x}$'s, the proof is essentially the same, except that more notations are needed. The basic idea of the proof is to approximate the original game by a sequence of games with finite type spaces. Existence of PBE in a game with finite type space is guaranteed, we then show that as the type space becomes arbitrarily finer, the limiting strategy profile exists, and it constitutes a PBE of the continuous type game. We establish this in steps.

Step 1 We first discretize the type space $\Theta := [0, 1+\epsilon]$ in the following way. Let $\Theta_k := \left\{\theta_i = \frac{i(1+\epsilon)}{2^k}, i = 0, .., 2^k\right\}$. Let $P_k(\theta_i) := F(\theta_i) - F(\theta_{i-1})$ for $i \geq 1$ and $P_k(\theta_0) := 0$ be the probability distribution over Θ_k. Let F_k denote the *c.d.f.* induced by P_k. Since F is continuous over the closed interval $[0, 1+\epsilon]$, F is uniformly continuous, which implies that F_k converges to F uniformly. Now let Γ_k denote the game that is the same as the original game except we replace Θ and F by Θ_k and F_k.

Step 2 By Theorem 4.6 in Myerson (1991), there exists a sequential equilibrium in Γ_k, hence there exists a PBE in Γ_k. We choose an arbitrary equilibrium of Γ_k, denote it by E_k.

Step 3 Suppose that there are H non-terminal histories in the original game. Here by a history we mean public history that is observed by everybody in the game. Since $n < \infty$, $T < \infty$, it must be that $H < \infty$. Then E_k is simply a collection of $n \times H$ functions, each mapping Θ_k to a probability distribution over $\{0, 1\}$, where 0 stands for no hunting, and 1 for hunting. $\forall k$, $\forall$ player j, $\forall$ history h, there is at most one type $\theta_i \in \Theta_k$ who is indifferent between 0 and 1, hence there is at most one type who mixes. To see this, notice that if the length of h is $T-1$, then if type θ is indifferent between 0 and 1, it must be that $\forall \theta' > \theta$, θ' strictly prefers 0, and $\forall \theta' < \theta$, θ' strictly prefers 1. If the length of h is less than $T-1$, the above claim also holds because $\delta < 1$(Refer to the proof of Proposition 2(i)). Therefore, E_k is a collection of $n \times H$ nonincreasing functions. Notice that E_k is undefined over $\theta \notin \Theta_k$. Before we go to the next step, define $E_{kj}(h)(\theta) := E_{kj}(h)(\theta_{+1})$, $\forall \theta \notin \Theta_k$, $\forall j$, $\forall h$, where θ_{+1} is the closest point in Θ_k to the right of θ, and $E_{kj}(h)(\cdot)$ is player j's action in E_k at h.

Step 4 By Helly's selection theorem (Kolmogorov and Fomin 1970), there exists a monotone strategy profile E, such that $E_k \to E$ pointwise, meaning $E_{kj}(h)(\cdot) \to E_j(h)(\cdot)$ pointwise, $\forall j$, $\forall h$.

Step 5 Fix j and h. The limiting function $E_j(h)$ has at most one point at which the value of the function is neither 0 nor 1. Consider Figure 2.

Suppose otherwise that there are two such points, θ_1 and θ_2. Let $p_1 = E_j(h)(\theta_1)$, $p_2 = E_j(h)(\theta_2)$, then by the monotonicity of $E_j(h)(\cdot)$ and the contradiction hypothesis, $0 < p_2 \leq p_1 < 1$. Since the grid can be made arbitrarily fine, $\exists \theta_{-1}, \theta, \theta_{+1} \in \Theta_k$ for some k such that $\theta_1 < \theta_{-1} < \theta < \theta_{+1} < \theta_2$. By the monotonicity of $E_j(h)(\cdot)$, $E_j(h)(\theta) \in [p_2, p_1]$. By the convergence result, $\exists K > 0$, such that $\forall k > K$, $E_{kj}(h)(\theta)$ is sufficiently close to $E_j(h)(\theta)$. By Step 3, $\forall k > K$, $E_{kj}(h)(\theta_{-1}) = 1$ and $E_{kj}(h)(\theta_{+1}) = 0$. But $E_j(h)(\theta_{-1}) \in [p_2, p_1]$ and $E_j(h)(\theta_{+1}) \in [p_2, p_1]$, a contradiction.

Step 6 Fix j and h. Let θ denote the cutoff point of the limiting function $E_j(h)(\cdot)$. $\forall k$, let θ_{+1} denote the right closest grid point in Θ_k to θ, similarly for θ_{-1}, θ_{+2}, and θ_{-2}. We claim that $\exists K > 0$, such that $\forall k' > K$, $E_{k'j}(h)(\theta) = E_j(h)(\theta)$, $\forall \theta \geq \theta_{+2}$, and $\forall \theta \leq \theta_{-2}$. Since k is chosen arbitrarily, this claim implies that the sequence of functions $(E_{k'j}(h)(\cdot))_{k'}$ coincides with $E_j(h)(\cdot)$ over an arbitrarily large set(relative to the type space). To prove the claim, consider Figure 3(if θ is an end point, the proof is analogous).

By Step 5, $E_j(h)(\theta_{-1}) = 1$, $E_j(h)(\theta_{+1}) = 0$. By an argument similar to Step 5, $\exists K > 0$, such that $\forall k > K$, $E_{kj}(h)(\theta_{+2}) = 0$ and $E_{kj}(h)(\theta_{-2}) = 1$. The rest of the step is finished by the monotonicity of $E_{kj}(h)(\cdot)$ and $E_j(h)(\cdot)$.

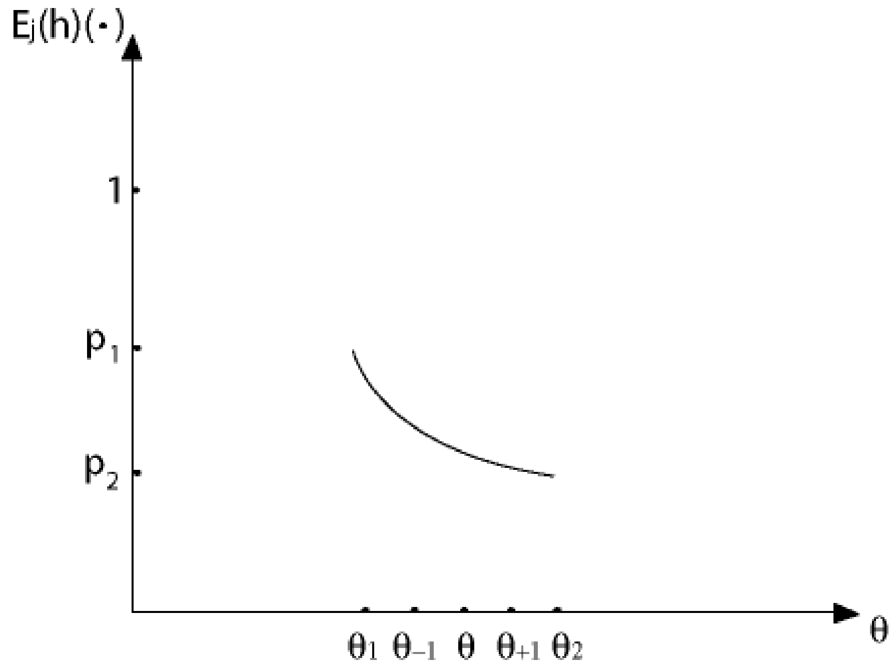

Figure 2: Step 5 of Proposition 7

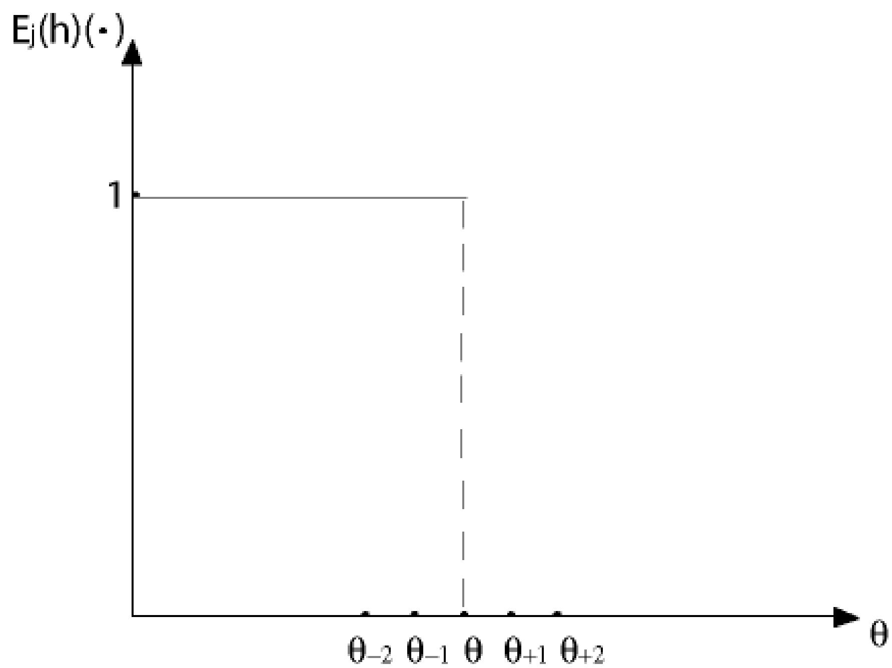

Figure 3: Step 6 of Proposition 7

Step 7 $\forall h$, $\forall j$, $\forall \theta_j \in \Theta$ such that $\theta_j \in \Theta_k$ for some k, type θ_j of player j does not want to deviate. To see this, let $P(\cdot|h, \theta_j, no\ dev)$ denote the lottery over the terminal histories of the game induced by E, conditional on history h, player j's type being θ_j, and player j following E_j throughout the continuation game. Let $P(\cdot|h, \theta_j,\ dev)$ denote the lottery over the terminal histories of the game induced by E, conditional on history h, player j's type being θ_j, and player j deviating right after h but following E_j for the rest of the continuation game. Let $P^{k'}(\cdot|h, \theta_j, no\ dev)$ and $P^{k'}(\cdot|h, \theta_j, dev)$ be defined similarly for $\Gamma_{k'}$, induced by $E_{k'}$. By Step 6 and the fact that $F_{k'}$ converges to F uniformly, we have $P^{k'}(\cdot|h, \theta_j, no\ dev) \to P(\cdot|h, \theta_j, no\ dev)$, and $P^{k'}(\cdot|h, \theta_j, dev) \to P(\cdot|h, \theta_j,\ dev)$.

Since a player's payoff is continuous in the lotteries over the terminal histories, if $P(\cdot|h, \theta_j, dev) \succ_{\theta_j} P(\cdot|h, \theta_j,\ no\ dev)$, then $P^{k'}(\cdot|h, \theta_j, dev) \succ_{\theta_j} P^{k'}(\cdot|h, \theta_j,\ no\ dev)$ for sufficiently large k', contradiction.

Step 8 $\forall h$, $\forall j$, $\forall \theta_j \in \Theta$ such that $\theta_j \notin \Theta_k$ for all k, if type θ_j wants to deviate, then $\exists \theta'_j \in \Theta_k$ for some k, such that θ'_j also wants to deviate.

There are two possibilities. 1. θ_j is never a cutoff point in E_j at any history. In this case, $\forall r > 0$, $\exists k$, $\exists \theta'_j \in \Theta_k$, such that (i) $\left|\theta'_j - \theta_j\right| < r$ and (ii) $E_j(h')(\theta_j) = E_j(h')\left(\theta'_j\right)$, $\forall h'$. Therefore, if we choose r sufficiently small, then a profitable deviation for type θ_j implies a profitable deviation for θ'_j, which is impossible by Step 7. 2. At some history h', θ_j is a cutoff point in E_j. First of all, it can not be that θ_j strictly prefers to hunt at h', since otherwise we can find a grid point θ''_j close enough to the right of θ_j who also strictly prefers to hunt at h', but $E_j(h')\left(\theta''_j\right) = 0$, which implies that θ''_j has a profitable deviation at h', impossible by Step 7. Hence in this case if necessary we can always redefine the value of $E_j(h')(\cdot)$ at θ_j to be equal to 0 without affecting any type of any player's payoff. But then we go back to the first possibility, that is $\forall r > 0$, $\exists k$, $\exists \theta'_j \in \Theta_k$, such that (i) $\left|\theta'_j - \theta_j\right| < r$ and (ii) $E_j(h')(\theta_j) = E_j(h')\left(\theta'_j\right)$, $\forall h'$. If two types are arbitrarily close and they behave the same way at any history, then if one type has a profitable deviation at some history, so does the other. ■

6 Common Shocks

In this section we study the investment game in Morris and Shin(2000), and show that the asymptotic results in Section 2 applies to common shocks models to some extent. The model is as follows.

	Invest	*Refrain*
Invest	θ, θ	$\theta - z, 0$
Refrain	$0, \theta - z$	$0, 0$

Two players must decide whether to invest or refrain from investing. If both invest, the payoff to each is θ, which follows standard normal distribution $N(0,1)$. If only one player invests, the investor receives $\theta - z$, where z is a positive constant. Player i observes θ with some noise ϵ_i that follows $N(0, 1/\beta)$. That is, player i's signal $x_i = \theta + \epsilon_i$. Assume that ϵ_1 and ϵ_2 are independent, and they are independent from θ. Morris and Shin (2000) show that if β is large enough, namely if the players' signals are precise enough, then there is a unique BNE of the game, which is characterized by a switching point $\widehat{x}(\beta)$, such that player i invests if and only if $x_i \geq \widehat{x}(\beta)$. Interestingly enough, $\widehat{x}(\beta) \to z/2 > 0$ as β goes to infinity, hence positive amount of inefficiency remains as precision of observation goes to infinity.

What we are going to show next is that if we can think of the game as being played sequentially, with "invest" being an irreversible action, and "refrain" being a reversible action, then efficiency can be asymptotically restored. For the ease of exposition, assume that there are only two periods, the result still holds for more than two periods. We focus on cutoff equilibria[5] in which after any history, if a player is willing to invest at some signal, then she is also willing to invest at any higher signals.

Proposition 8 *$\forall (\beta_n)_n \to \infty$, for any sequence of cutoff equilibria $(E_n)_n$, $P(\text{both invest in } E_n | \theta > 0) \to 1$.*

Proof: First we construct a symmetric equilibrium for fixed $\beta < \infty$, which can be characterized by a pair of numbers $(x^*(\beta), \widetilde{x}(\beta))$, such that a player invests in the first period if and only if $x \geq x^*(\beta)$; in the second period, if the opponent invests in the first period, follow him if and only if $x \geq \widetilde{x}(\beta)$, if the opponent refrains in the first period, refrain in the second period.

Given $x^*(\beta)$, $\widetilde{x}(\beta)$ should satisfy

$$E(\theta | x_2 = \widetilde{x}(\beta), x_1 \geq x^*(\beta)) = 0. \tag{6}$$

On the other hand, $x^*(\beta)$ should make a player(say player 1) indifferent between investing and refraining in the first period. Notice that the payoff of investing in the first period is given by

$$\begin{aligned}
&P(x_2 \geq x^*(\beta) | x_1 = x^*(\beta)) \cdot E(\theta | x_1 = x^*(\beta), x_2 \geq x^*(\beta)) \\
&+P(x_2 < x^*(\beta) | x_1 = x^*(\beta)) \cdot \\
&(E(\theta | x_1 = x^*(\beta), x_2 < x^*(\beta)) - z \cdot P(x_2 < \widetilde{x}(\beta) | x_1 = x^*(\beta), x_2 < x^*(\beta))),
\end{aligned}$$

and the payoff of refraining in the first period is

[5] Refer to footnote 1.

$$P\left(x_2 \geq x^*\left(\beta\right)|x_1 = x^*\left(\beta\right)\right) \cdot E\left(\theta|x_1 = x^*\left(\beta\right), x_2 \geq x^*\left(\beta\right)\right)$$
$$+P\left(x_2 < x^*\left(\beta\right)|x_1 = x^*\left(\beta\right)\right) \cdot 0.$$

Hence $x^*(\beta)$ must satisfy

$$E[\theta|x_1 = x^*\left(\beta\right), x_2 < x^*\left(\beta\right)] - z \cdot P[x_2 < \widetilde{x}\left(\beta\right)|x_1 = x^*\left(\beta\right), x_2 < x^*\left(\beta\right)] = 0. \tag{7}$$

Next we show that (a) In the first period, $\forall x > x^*(\beta)$, type x will invest, and $\forall x < x^*(\beta)$, type x will refrain; (b) In the second period, upon seeing the opponent investing in the first period, type x will follow up if and only if $x > \widetilde{x}(\beta)$; (c) In the second period, upon seeing the opponent not investing in the first period, it is optimal not to invest in the second period.

First of all, (b) and (c) immediately follow from equations (1) and (2), respectively. Next we prove that $\forall x > x^*(\beta)$,

$$E\left(\theta|x_1 = x, x_2 < x^*\left(\beta\right)\right) - z \cdot P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x, x_2 < x^*\left(\beta\right)\right) > 0.$$

First we show that $P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x, x_2 < x^*\left(\beta\right)\right)$ is decreasing in x. Notice that $(x_2|x_1 = x) \sim N\left(\frac{\beta x}{1+\beta}, \frac{1}{1+\beta} + \frac{1}{\beta}\right)$, hence

$$\begin{aligned} & P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x, x_2 < x^*\left(\beta\right)\right) \\ = & \frac{P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x\right)}{P\left(x_2 < x^*\left(\beta\right)|x_1 = x\right)} \\ = & \frac{\Phi\left(a\left(\widetilde{x} - bx\right)\right)}{\Phi\left(a\left(x^* - bx\right)\right)}, \end{aligned}$$

where $a = \sqrt{\frac{\beta(1+\beta)}{1+2\beta}}$, $b = \frac{\beta}{1+\beta}$

Differentiating w.r.t. x, the numerator is

$$ab\left(\Phi\left(a\left(\widetilde{x} - bx\right)\right)\Phi'\left(a\left(x^* - bx\right)\right) - \Phi\left(a\left(x^* - bx\right)\right)\Phi'\left(a\left(\widetilde{x} - bx\right)\right)\right).$$

Since $\widetilde{x} < x^*$, it suffices to show that $\frac{\Phi(z)}{\Phi'(z)}$ is increasing in z over $\mathbb{R}$. Since $sign\left(\frac{\partial}{\partial z}\left(\frac{\Phi(z)}{\Phi'(z)}\right)\right) = sign\left(\Phi'(z) + z\Phi(z)\right)$, and $\frac{\partial}{\partial z}\left(\Phi'(z) + z\Phi(z)\right) > 0$, and $\lim_{z\to-\infty}\left(\Phi'(z) + z\Phi(z)\right) = 0$, we have $\Phi'(z) + z\Phi(z) > 0$, as was to be shown.

Now let $x < y$, let F denote the distribution of $(x_2|x_1 = x)$ conditional on $x_2 < x^*(\beta)$, let G denote the distribution of $(x_2|x_1 = y)$ conditional on $x_2 < x^*(\beta)$. Then by the above argument, $\forall z < x^*(\beta)$,

$$P(x_2 < z|x_1 = x, x_2 < x^*(\beta)) > P(x_2 < z|x_1 = y, x_2 < x^*(\beta)).$$

Hence G first order stochastic dominates F. Therefore,

$$\int_{-\infty}^{x^*} x_2 dG \geq \int_{-\infty}^{x^*} x_2 dF.$$

This implies that $E(x_2|x_1 = x, x_2 < x^*(\beta))$ is increasing in x, which in turn, implies that $E(\theta|x_1 = x, x_2 < x^*(\beta))$ is increasing in x.

Hence $\forall x > x^*(\beta)$

$$\begin{aligned} & E(\theta|x_1 = x, x_2 < x^*(\beta)) - z \cdot P(x_2 < \widetilde{x}(\beta)|x_1 = x, x_2 < x^*(\beta)) \\ > \ & E(\theta|x_1 = x^*(\beta), x_2 < x^*(\beta)) - z \cdot P(x_2 < \widetilde{x}(\beta)|x_1 = x^*(\beta), x_2 < x^*(\beta)) \\ = \ & 0. \end{aligned}$$

For fixed $\beta < \infty$, it is easy to see that there is a unique pair $(x^*(\beta), \widetilde{x}(\beta))$ that solves 6 and 7. Now fix any cutoff equilibrium of the two period game. Since it has the cutoff property, it can be characterized by six cutoff numbers, $\widehat{x}_{11}, \widehat{x}_{12}, \widehat{x}_{21}, \widehat{x}_{22}, \widetilde{x}_{12}, \widetilde{x}_{22}$, where $\widehat{x}_{jt}$ is the cutoff type of player j in period t when nobody has invested yet, and $\widetilde{x}_{jt}$ is the cutoff type of player j in period t when the other player has already invested. We show that in this equilibrium,

$$P(\text{both invest}|x_1 \geq x^*(\beta), x_2 \geq x^*(\beta)) = 1.$$

Suppose not. Then it must be that $\widehat{x}_{11} > x^*(\beta), \widehat{x}_{21} > x^*(\beta)$, and at least one of $\widehat{x}_{12}$ and $\widehat{x}_{22}$ is also greater than $x^*(\beta)$, say it is $\widehat{x}_{12}$. Consider type x_1 of player 1 such that

$$x^*(\beta) < x_1 < \min\{\widehat{x}_{11}, \widehat{x}_{12}\}.$$

If type x_1 follows her equilibrium strategy, her expected payoff is

$$P(x_2 \geq \widehat{x}_{21}|x_1) \cdot E(\theta|x_1, x_2 \geq \widehat{x}_{21}).$$

If she deviates by investing in the first period, her expected payoff is

$$
\begin{aligned}
&P\left(x_2 \geq \widehat{x}_{21}|x_1\right) \cdot E\left(\theta|x_1, x_2 \geq \widehat{x}_{21}\right) \\
&+P\left(x_2 < \widehat{x}_{21}|x_1\right) \cdot \left(E\left(\theta|x_1, x_2 < \widehat{x}_{21}\right) - z \cdot P\left(x_2 < \widetilde{x}_{22}|x_1, x_2 < \widehat{x}_{21}\right)\right).
\end{aligned}
$$

Since $x_1 > x^*\left(\beta\right)$, $\widehat{x}_{21} > x^*(\beta)$, and $\widetilde{x}_{22} < \widetilde{x}\left(\beta\right)$, it must be that

$$
\begin{aligned}
&\left(E\left(\theta|x_1, x_2 < \widehat{x}_{21}\right) - z \cdot P\left(x_2 < \widetilde{x}_{22}|x_1, x_2 < \widehat{x}_{21}\right)\right) \\
> \; & E\left(\theta|x_1 = x^*\left(\beta\right), x_2 < x^*\left(\beta\right)\right) - z \cdot P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x^*\left(\beta\right), x_2 < x^*\left(\beta\right)\right) \\
= \; & 0.
\end{aligned}
$$

Hence type x_1 has a profitable deviation, a contradiction.

Now fix $\beta > 0$, fix a cutoff equilibrium E. Since

$P\left(\text{both invest in } E \mid \theta > 0\right)$
$\geq P\left(x_1 \geq x^*\left(\beta\right), x_2 \geq x^*\left(\beta\right)|\theta > 0\right) \cdot P\left(\text{both invest in } E \mid x_1 \geq x^*\left(\beta\right), x_2 \geq x^*\left(\beta\right)\right)$

It suffices to show that $\lim_{\beta \longrightarrow \infty} x^*\left(\beta\right) = 0$. Suppose otherwise that $\exists \left(\beta_k\right)_k \longrightarrow \infty$, such that $\lim_k x^*\left(\beta\right) = b > 0$, then by (1) it must be that $\lim_k \widetilde{x}\left(\beta_k\right) = -b < 0$. But this implies that $P\left(x_2 < \widetilde{x}\left(\beta\right)|x_1 = x^*\left(\beta\right), x_2 < x^*\left(\beta\right)\right) \longrightarrow 0$, which by 7 implies that $x^*\left(\beta\right)$ converges to 0, a contradiction. ■

What if both "invest" and "refrain" are irreversible actions? Then the two period game with endogenous timing is the same as the simultaneous move game, hence Proposition 8 no longer holds, and we come back to the prediction of Morris and Shin (2000). The reversibility of the actions in a coordination game depends on the context of the game. In a bank run model, for example, if we replace "invest" by "withdraw late", and "refrain" by "withdraw early", then both actions are irreversible. In a currency attack model, however, if we replace "invest" by "attack", and "refrain" by "not attack", then "attack" is an irreversible action, and "not attack" is a reversible action. Bank runs and currency attacks are usually considered as very similar models in the literature, we argue that once we add endogenous timing so that reversibility becomes an issue, they fall into very different categories.

7 Irreversibility

In this section we follow up the discussion at the end of last section, by studying a dynamic version of the arms race game in Baliga and Sjöström(2002).

Two countries must decide whether and when to build new weapons. Building weapons is a one time and irreversible decision. Not building weapons is a reversible action. The cost to build weapons is a one time expense, and it is players' private information. Without loss of generality, assume there are only two periods, and there is no discounting. Each country's payoff is determined from the simultaneous move game according to the final decisions of the two countries. The payoffs of the one period simultaneous move game are given in the following matrix.

	B	N
B	$-c_1, -c_2$	$\mu - c_1, -d$
N	$-d, \mu - c_2$	$0, 0$

where c_i is player i's cost to build weapons. The c_i's follow $i.i.d.$ F over $[0, \overline{c}]$. $\mu > 0$ is the advantage of a better armed country over a less armed country; $d > \overline{c}$ is the disadvantage of a less armed country over a better armed country. Baliga and Sjöström show that if $F(c) \cdot d \geq c$, $\forall c \in [0, \overline{c}]$, then the only BNE is (B, B) for all types. The question is, under the same conditions of the distribution function(the multiplier conditions in Baliga and Sjöström (2002)), is there an equilibrium where (N, N) occurs with positive probability, if the two countries play the game sequentially with endogenous timing?

Proposition 9 *In the two-period arms race game with endogenous timing, if the multiplier condition holds, then in any PBE, there is probability* 0 *that* (N, N) *is the final outcome.*

Proof: Suppose by way of contradiction that there exists a PBE in which (N, N) is the final outcome with positive probability. Let P_1 denote the probability that country 1will build in period 1. Let π_1 denote the probability that country 1 will build in period 2, conditional on that (N, N) is the outcome in period 1. By the contradiction hypothesis, $P_1 < 1$, $\pi_1 < 1$. Now we show that for any country i, for any type c_i of country i, c_i does not build in period 1 in this equilibrium. Suppose otherwise that, say, type c_2 of country 2 prefers to build in period 1. Then it must be that

$$-c_2 \geq P_1(-c_2) + (1 - P_1) \max\{-c_2 + (1 - \pi_1)\mu, \ -\pi_1 d\}.$$

Hence $-c_2 \geq \max\{-c_2 + (1 - \pi_1)\mu, \ -\pi_1 d\}$, which is impossible since $\pi_1 < 1$ and $\mu > 0$.

Now if (N, N) is the outcome for sure in period 1, it does not reveal any information. We essentially go back to the one shot game in which, under the multiplier condition, (B, B) is the unique outcome, which is a contradiction. ■

The arms race game is very similar to the stag hunt game in the paper. It is also a coordination game that may have multiple equilibria if information is complete. When introducing some amount of dominant strategy types, it also generates a unique BNE in which the inefficient action is chosen by all, and so on. So what makes the two models behave so differently in a dynamic setup, when they behave so similarly in a static one? In the arms race game, the safe action (B) is irreversible, while the risky action (N) is reversible. In our game, it is just the opposite. In both games, the risky action is the cooperative action. But if it is reversible, and if the safe (selfish) action is not, then leaders have no incentive to take the risk early. Only when the leaders anticipate that they will be followed by those who choose safe actions today are they willing to lead. In the arms race game, a leader will not be followed by countries which already build arms today, hence nobody wants to lead, and dynamics doesn't make much difference.

8 Conclusion

We build a simple model that combines strategic complementarity and incomplete information in a sequential setup with endogenous timing. We use the model to argue that it makes a difference which way we model a coordination situation. If it is more appropriate to model the situation as a dynamic game, then the predictions in the static model might be over pessimistic, depending on the nature of the actions of the game. The equilibrium analysis reveals a recursive structure behind a herding mechanism. Initially there is a large scale coordination problem. Then players with extreme types sort out themselves, which reduces the coordination problem to a smaller scale, and the process repeats. Herding may be completed (everybody participates), it may also die out prematurely, depending on the realization of players' types. People follow other people not because they are afraid of being left alone (Choi (1997)), or because they suppress their own information and free ride on predecessors' information (Bikhchandani, Hirshleifer and Welch (1992)), or because doing so gives them a reputational advantage (Scharfstein and Stein (1990)), but simply because they rationally expect that they themselves will be followed. If the types are distributed "correctly", which will be judged according to the cutoff points in equilibrium in a recursive way, then we may expect to observe a domino effect, which unfolds itself quickly.

Several extensions remain to be investigated. First, we assume that it takes everybody to catch the stag. I don't expect that the results will change qualitatively if it takes only a fraction of people to succeed, but it would still be interesting to check this out, especially when the rest of the players can free ride on the hunters' catch. Second, we assume that once everybody hunts, the stag is caught for sure, what happens if there is some uncertainty about this and different players have different perceptions about such uncertainty? That is, what if we combine private shocks on players' costs with common shocks on the fundamentals of the environment?

9 Appendix

Proof of Proposition 5: The idea of the proof is the following. Conditions (a), (b) and (c) allow us to write out the indifference equations that the cutoffs must satisfy. Once we have an explicit expression of the indifference equations, we can prove the existence and uniqueness of the solutions. Then it is trivial to check that conditions (a), (b) and (c) are indeed satisfied.

The notations in this proof can be found in Section 5 right after Proposition 5.

We first consider the case where $n = 2$, and the lower bound on the players' cost is $x \in [0, 1]$. Let $g(2, x)$ denote the first period cutoff type such that $\forall c_i \leq g(2, x)$, player i hunts in period 1; $\forall c_i > g(2, x)$, player i waits in period 1. We show that $g(2, x)$ is unique, and doesn't depend on x.

We organize the argument into small steps.

Step 1 No discounting implies that in any equilibrium that satisfies (a), (b) and (c), the payoff of hunting today is equal to the payoff of waiting today but hunting tomorrow no matter what happens today.

Step 2 We write down the indifference condition that $g(2, x)$ must satisify:

$$-g(2,x)+(1-p(F,2,x)) = p^{1\ 1}(g(2,x)\,|x)(1-g(2,x))+p^{0\ 1}(g(2,x)\,|x)\,v(2,g(2,x))\,.$$

Step 3 We can decompose $p(F, 2, x)$ as follows

$$p(F,2,x) = p^{1\ 1}(g(2,x)\,|x)\cdot 0 + p^{0\ 1}(g(2,x)\,|x)\,p(1,g(2,x)) \ = p(1,x)\,.$$

Step 4 Step 1, (a), (b) and (c) imply that

(i) On the equilibrium path, once there is inaction in some period, there will be inaction forever.

(ii) If type $g(2, x)$ doesn't hunt in period 1, then it hunts in period 2 no matter what.

(iii) $v(2, g(2,x)) = -g(2,x) + 1 - p(1, g(2,x)) = 0.$

It is easy to see that (i) directly follows from condition (c).

To see (ii), suppose otherwise that $g(2,x)$ doesn't hunt if he observes inaction in period 1, then the payoff to waiting in the first period is strictly higher than the payoff to waiting in period 1 and hunting in period 2 no matter what. By step 1, the latter is equal to the payoff to hunting in period 1. Hence type $g(2,x)$ is not indifferent between hunting and waiting in period 1, contradiction.

To see (iii), note that (ii) implies

$$v(2, g(2,x)) = -g(2,x) + 1 - p(F, 2, g(2,x)) = -g(2,x) + 1 - p(1, g(2,x)),$$

where the second equality comes from step 3.

The continuation value function $v(2, g(2,x))$ can not be less than 0, since otherwise (ii) is violated; it can't be greater than 0, since otherwise some type slightly above $g(2,x)$ should also hunt in period 2, even if nobody hunts in the first period. But this violates (i). Therefore $v(2, g(2,x)) = 0$.

Step 5 $-g(2,x) + 1 - p(1, g(2,x)) = 0$ has a unique solution in $(0,1)$, which doesn't depend on x.

It suffices to show that $-x + 1 - p(1,x) = 0$ has a unique solution in $(0,1)$. To that end, it suffices to show that $p(1,x)$ is increasing in x over the interval $[0,1]$. $\forall x'$ $s.t.$ $x < x' < 1$, We can decompose $p(1,x)$ in the following way

$$p(1,x) = p^{1\ 1}([x,x']\,|x) \cdot 0 + p^{0\ 1}([x,x']\,|x) \cdot p(1,x') < p(1,x'),$$

as was to be shown.

Now denote the solution to the equation $-x + 1 - p(1,x) = 0$ by $g(2)$. Note that if the lower bound x exceeds $g(2)$, then no hunting will ever occur in equilibrium because $\forall y \geq x$,

$$\begin{aligned}
& \text{Payoff of hunting} \\
= \; & -y + 1 - p(1,x) \\
\leq \; & -x + 1 - p(1,x) \\
< \; & -g(2) + 1 - p(1, g(2)) \\
= \; & 0.
\end{aligned}$$

Now we consider the case where $n = 3$, and the lower bound on the players' cost is $x \in [0, 1]$. Let $g(3, x)$ denote the first period cutoff type such that $\forall c_i \leq g(3, x)$, player i hunts in period 1; $\forall c_i > g(3, x)$, player i waits in period 1. We show that $g(3, x)$ is unique, and doesn't depend on x.

We still have step 1 as in the two player case.

Step 1 No discounting implies that in any equilibrium that satisfies (a), (b) and (c) , payoff of hunting today is equal to payoff of waiting today but hunting tomorrow no matter what.

Step 2 The indifference condition that $g(3, x)$ must satisfy is

$$\begin{aligned} & -g(3,x) + (1 - p(F,3,x)) \\ = \quad & p^{2\ 2}(g(3,x)|x)(1 - g(3,x)) + p^{1\ 2}(g(3,x)|x)\, v(2, g(3,x)) + p^{0\ 2}(g(3,x)|x)\, v(3, g(3,x)). \end{aligned}$$

Step 3 We can decompose $p(F, 3, x)$ as follows.

$$\begin{aligned} & p(F,3,x) \\ = \quad & p^{2\ 2}(g(3,x)|x) \cdot 0 + p^{1\ 2}(g(3,x)|x)\, p(1, g(3,x)) + p^{0\ 2}(g(3,x)|x)\, p(2, g(3,x)). \end{aligned}$$

Step 4 Step 1, (a), (b) and (c) imply that

(i) $g(3, x)$ always hunts in period 2 no matter what happens in period 1;

(ii) $g(3, x) \leq g(2)$, $\forall x$;

(iii) $p(F, 3, x) = p(2, x)$.

To see (i), suppose otherwise that $g(3, x)$ doesn't hunt after some observation in period 1, then the payoff to waiting in the first period is strictly higher than the payoff of waiting in period 1 and hunting in period 2 no matter what. By step 1, the latter is equal to the payoff of hunting in period 1, hence type $g(3, x)$ is not indifferent between hunting and waiting in period 1, a contradiction.

(ii) follows from (i) since if $\exists x, s.t.\ g(3, x) > g(2)$, then by the two player argument type $g(3, x)$ doesn't hunt in period 2 if only one player hunts in period 1, contradicting (i).

(iii) follows from (ii) and step 3.

Step 5 (i) and (iii) implies that

$$(iv)\ v(3, g(3, x)) = -g(3, x) + 1 - p(F, 3, g(3, x)) = -g(3, x) + 1 - p(2, g(3, x)).$$

Step 6 Condition (c) implies that

(v) On equilibrium path, if nobody hunts in period 1, then nobody hunts forever.

Step 7

$$\left.\begin{array}{c}(i)\\(iv)\\(v)\end{array}\right\} \Longrightarrow (vi)\quad v(3, g(3, x)) = -g(3, x) + 1 - p(2, g(3, x)) = 0.$$

It can't be less than 0 since otherwise (i) is violated; it can't be more than 0 since otherwise (v) is violated.

Step 8 $-g(3, x) + 1 - p(2, g(3, x)) = 0$ has a unique solution in $(0, 1)$, that doesn't depend on x, and is smaller than $g(2)$.

It suffices to show that $-x + 1 - p(2, x) = 0$ has a unique solution in $(0, 1)$. To that end, it suffices to show that $p(2, x)$ is increasing in x over the interval $[0, g(2)]$ ($p(2, x) = 1$ if $x > g(2)$). $\forall x'$ $s.t.$ $x < x' < g(2)$, we can decompose $p(2, x)$ as

$$p(2, x) = p(2/2 \in [x, x'] \,|x) \cdot 0 + p(1/2 \in [x, x'] \,|x)\, p(1, x') + p(0/2 \in [x, x'] \,|x)\, p(2, x').$$

To show $p(2, x) < p(2, x')$, it suffices to show that $p(2, x) \geq p(1, x)$, $\forall x$.

Let i be the player in a one player game. Imagine there is another player j, who is a dummy player in the one player game, but is a normal player in a two player game. To slightly abuse notation, also let i and j denote i's type and j's type.

Now

$$\begin{aligned} p(1,x) &= p\{(i,j) \geq (x,x) \,|i \in (1,1+\epsilon],\, j \text{ is anywhere}\} \\ &\leq p[\{(i,j) \geq (x,x) \,|i \in (1,1+\epsilon],\, j \text{ is anywhere}\} \\ &\quad \cup \{(i,j) \geq (x,x) \,|i \in [x, g(2)],\, j \in (1,1+\epsilon]\ \}] \\ &\leq p(2,x)\,. \end{aligned}$$

Finally, $g(3) < g(2)$ since $-g(2) + 1 - p(2, g(2)) = -g(2) < 0$.

In general when there are n players, we can follow the same steps as above to show that

(1) If type $g(n,x)$ doesn't hunt in period 1, then it will hunt in period 2 no matter what.

(2)

$$\begin{aligned} v(n, g(n,x)) &= -g(n,x) + 1 - p(F, n, g(n,x)) \\ &= -g(n,x) + 1 - p(n-1, g(n,x)) \\ &= 0. \end{aligned}$$

(3) $p(n,x)$ is increasing in n and x.

The sequence of the cutoffs can be found inductively as follows.

$$\begin{aligned} g(1) &= 1, \\ p(1,x) &= p^{0\ 1}(g(1)\,|x)\,, \\ 1 - g(2) &= p(1, g(2))\,, \\ p(2,x) &= p^{0\ 2}(g(2)\,|x) \cdot 1 + p^{1\ 2}(g(2)\,|x)\, p(1, g(2))\,, \\ 1 - g(3) &= p(2, g(3))\,, \\ &\cdot \\ &\cdot \\ p(n-1,x) &= p^{0\ n-1}(g(n-1)\,|x) \cdot 1 + .. + p^{n-2\ n-1}(g(n-1)\,|x)\, p(1, g(n-1))\,, \\ 1 - g(n) &= p(n-1, g(n))\,. \end{aligned}$$

Note that $p(n,x) \geq p(n,0)\,, \forall x > 0$, and $p(n,0) \longrightarrow 1$ as $n \longrightarrow \infty$, hence $p(n,x) \longrightarrow 1$ as $n \longrightarrow \infty$, uniformly with respect to x. Therefore, taking the limit of both sides of $1-g(n)=p(n-1,g(n))$ as n goes to infinity, it must be that $g(n)$ converges to 0. ■

Proof of footnote 4: Footnote 4 claims that if δ is large enough, then $\forall x>c_2^*$, no solution $g(2,x)\in[x,1]$ exists for equation (3). Following the proof of Lemma 4, if $x>\widetilde{c}$, then the claim is true; if $x\in(c_2^*,\widetilde{c}]$, then $\Delta_2(x,c,\delta)-\Delta_2(x,c',\delta)$ has the same expression as in the proof of Lemma 4. Moreover, $LHS_2(x,x)<0=RHS_2(x,x)$, hence if δ is large, then the r.h.s. is always larger than the l.h.s., for any $x\in(c_2^*,\widetilde{c}]$, hence no solution. ■

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