

# Ambiguous Beliefs and Mechanism Design

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## Abstract

This paper develops a payoff equivalence theorem for mechanisms with ambiguity averse participants with preferences of the Maxmin Expected Utility (MEU) form (Gilboa and Schmeidler [8]). We use our payoff equivalence result to explicitly characterize the revenue maximizing private value auction mechanism for agents with arbitrary forms of ambiguous beliefs. We also show that the revenue ranking between first and second price auctions is sensitive to the form of ambiguity aversion. Our payoff equivalence techniques allow us to study the constrained efficient, budget balanced bilateral trade mechanism and show that increased ambiguity improves the efficiency of the mechanism. In addition, we characterize the revenue maximizing, efficient bilateral trade mechanism and show that heightened ambiguity lowers ex ante budget deficits.

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## 1. Introduction

The study of mechanism design problems with agents that have subjective expected utility (SEU) preferences has generated an array of powerful theoretical models including auctions to privatize government assets and analyses of the possibility for efficient, budget balanced bargaining outcomes. However, the study of mechanisms with non-classical preference relations remains in its infancy. We develop a payoff equivalence theorem in order to analyze mechanisms with agents possessing ambiguity averse preferences representable in Minimum Expected Utility (MEU) form (Gilboa and Schmeidler 1989). Our payoff

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equivalence techniques allow us to explicitly characterize the revenue maximizing auction for arbitrary ambiguous beliefs;<sup>2</sup> analyze the revenue ranking of first and second price auctions; and derive the form of revenue maximizing, efficient bilateral trade mechanisms and constrained efficient, ex ante budget balanced bilateral trade mechanisms.

The theory of ambiguity aversion is based on a formal differentiation between risky and ambiguous lotteries. A risky lottery is a random variable with a known probability measure governing its behavior. An ambiguous lottery is a random variable with a probabilistic structure that is not known by the decision maker. When decision makers face ambiguous lotteries, the preferences over lotteries cannot be rationalized by any combination of a felicity function and a (unique) subjective prior belief. Attempts to provide decision theoretic foundations for preferences over ambiguous lotteries have led to a number of representations of ambiguity averse preferences with the MEU representation a classic example in this literature. The interested reader is directed to Gilboa and Marinacci (2011) for a survey of the motivation and history of ambiguity aversion and its relation to Bayesian decision theory.

Ellsberg (1961) presents a thought experiment that clarifies the difference between risk and ambiguity. The subject in the experiment is presented with an urn filled with 30 red balls and 60 balls that are a mixture of black and yellow colors in an unknown ratio. The subjects have unambiguous beliefs about the probability of drawing a red ball and ambiguous beliefs about the probability of drawing a black or yellow ball. The subject is first offered a choice between the lotteries A and B and then between C and D.

	Red	Black	Yellow
A	\$100	\$0	\$0
B	\$0	\$100	\$0
C	\$100	\$0	\$100
D	\$0	\$100	\$100

Most decision makers express a strict preference for A over B and D over C. However, the sure-thing principle (Savage 1954) implies that a preference for A over B should entail a preference for C over D. The aversion to ambiguous lotteries is the primitive choice data on which ambiguity averse preferences are founded.

We assume that the mechanism designer knows the distribution of agent types.<sup>3</sup> The agents, while knowing perfectly their own valuation, possess ambiguous beliefs about the distribution of types of the other agents.<sup>4</sup> The agents' ambiguous beliefs have behavioral consequences since each agent's choice

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<sup>2</sup>The term "ambiguous beliefs" refers to the convex set of priors in the MEU representation of ambiguity averse preferences.

<sup>3</sup>Ambiguity aversion on the part of the mechanism designer would not alter the comparative statics conclusions, which describe the effect of ambiguity aversion on the behavior and welfare of mechanism participants.

<sup>4</sup>We assume the strategies are common knowledge in equilibrium.

of action is a response to the valuation dependent actions of the other agents. Therefore, the mechanism designer's choices (i.e. auction format) will be driven by the designer's unambiguous beliefs about the distribution of agent actions that the agents' ambiguity averse preferences generate.

Introspection suggests that ambiguous beliefs will be most prevalent in situations with agents that have little information on which to formulate a prior belief about the fundamentals of the market and in economies with nonstationary or unlearnable environments that prevent the agents from learning the fundamentals with experience. Many of the important decisions in a person's life, such as bidding on a home or bargaining over a car purchase, are rare events that present little opportunity to practice. Important business decisions, such as bargaining with a potential merger partner or participating in an auction for government contracts, are often faced by firm leaders having little, if any, prior experience. In addition, agent beliefs about the economic primitives may become ambiguous following major economic events (e.g. financial crises) or institutional changes (e.g. FCC spectrum auctions).

Ambiguity averse preferences have not been examined in mechanism design studies with the exception of tractable examples. The analytical power of our payoff equivalence framework allows us to derive the form of the fully optimal auction under arbitrary ambiguous beliefs, which we find has connections with the reservation value formats studied by Myerson 1981 and Riley and Samuelson 1981. Although the first price auction generates greater revenues than the second price auction when bidders are risk averse (Maskin and Riley 1984), we provide examples where the relative revenues of the first and second price auction depend on the details of the ambiguous beliefs of the agents. These examples show a sharp differentiation between the bidding behavior of risk averse and ambiguity averse agents. We are also able to discuss the effect of ex ante entry costs on the number of bidders in the auction and the expected revenues. Given that high stakes auctions (e.g. FCC spectrum auction) attempt to recruit novice bidders (e.g. minority owned companies), it is important to understand how ambiguity affects the entry decisions of these novices.

We also provide applications of our framework to bilateral trade mechanisms. We show that with a well chosen mechanism design, increased ambiguity can move efficient bilateral trade protocols toward balanced budgets and render budget balanced bilateral trade mechanisms more efficient. These results suggest that ambiguity aversion can be turned into a force generating ex post Pareto superior outcomes.

### *1.1. Literature Review*

In the closest work to ours, Bose et al. 2006 derive the revenue maximizing transfer function under general MEU preferences for a fixed outcome function, but are unable to explicitly characterize the optimal allocation function except for the  $\varepsilon$ -contamination model defined in their paper. Lo 1998 discusses first and second price auctions under conditions of identical, ambiguous beliefs on the part of bidders and auctioneer. Under a restricted contamination model of ambiguity, Lo finds that sellers prefer to utilize a first price auction while buyers

(weakly) favor a second price format. Ozdenoren 2001 provides an analysis of auction and bargaining problems in the context of restricted forms of MEU preferences. Salo and Weber 1995 use a special form of the rank dependent utility model to show that revenues in a first price auction are greater than in a second price auction. Lopomo et al. 2006 study auctions within the Bewley 1986 model of Knightian uncertainty as incompleteness of preferences. Bose and Renou 2011 and de Castro and Yannelis 2010 study the issue of implementability in contexts with ambiguity aversion.

### 1.2. Outline

Section 2 presents the economic primitives we employ for analyzing agent preferences and the mechanisms within which they interact. Section 3 develops the payoff equivalence result for Bayesian-Nash implementations of a mechanism and shows that outcome function monotonicity and the incentive compatibility first order conditions are sufficient for the mechanism to be globally incentive compatible. The fourth section applies this payoff equivalence result to derive the revenue maximizing auction format, study revenue rankings between the first and second price auction, and study the effect of entry costs on our results. The fifth section discusses the effect of ambiguity on bilateral trade mechanisms, and the sixth section concludes. All proofs are relegated to appendices.

## 2. Model Primitives

In this section we discuss the MEU preference representation we use throughout the paper. In addition, we lay out the mathematical formalism for the mechanisms we consider.

### 2.1. Terminology: Aversion to Ambiguity

We employ the MEU model, which considers preferences representable by the utility function:

$$U(f) = \min_{\pi \in \Delta} E^\pi u(f(\omega))$$

where  $E^\pi$  is the expectation operator with respect to the additive probability measure  $\pi$ ,  $\Delta$  is a convex set of probability measures,  $u(\circ)$  is a felicity function, and  $f(\circ)$  is a gamble over the states of the world  $\omega \in \Omega$ . When referring to subsets  $E \subseteq \Omega$  we assume that  $E$  is measurable.

The set  $\Delta$  is interpreted as the set of measures that the agent entertains as possible probability rules. Confidence in the probability rule,  $\Delta = \{\pi\}$ , causes the MEU representation to collapse to the SEU form. When multiple measures are considered plausible by the agent, she makes her decision pessimistically by acting as if the measure generating the lowest possible expected value for a choice is the true measure.

**Definition 1.** *The set of priors  $\Delta$  is more ambiguous<sup>5</sup> than  $\tilde{\Delta}$  if  $\tilde{\Delta} \subseteq \Delta$*

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<sup>5</sup>This notion of comparative ambiguity is justified by Ghirardato and Marinacci (2002).

It is straightforward to see that if  $\Delta \subseteq \tilde{\Delta}$ , we have  $\min_{\pi \in \Delta} E^\pi f(x) \geq \min_{\pi \in \tilde{\Delta}} E^\pi f(x)$ . Therefore, increased ambiguity is weakly worse for the agent.<sup>6</sup>

**Definition 2.** *Agent beliefs are **correct** if the mechanism designer's prior,  $\pi^*$ , is contained within every type's set of priors*

Throughout our paper we assume that the mechanism designer has SEU preferences with a prior equal to  $\pi^*$ . Our definition of *correct but ambiguous* entails only that the agents agree that the mechanism designer's prior is a plausible distribution over the states of the world. Our analysis uses the designer's beliefs to evaluate the optimal auction format and the revenue ranking between the first and second price auction formats and assuming correct but ambiguous beliefs provides structure for our analysis. For example, if we did not assume that  $\pi^*$  is in  $\Delta$ , we would be able to generate arbitrary revenue rankings by creatively choosing a prior for the bidders ( $\Delta = \{\pi\}$ ).

## 2.2. Terminology: Mechanisms

We study static, private values direct revelation mechanisms of the form  $(\Theta, x, p)$ . The payoff type of agent  $i$  is an element  $\theta_i \in [\underline{\theta}, \bar{\theta}] = \Theta_i$ , and we denote agent type declarations as  $\hat{\theta}_i \in \Theta_i$ . The type space for all of  $N$  agents is then  $\Theta = \Theta_1 \times \dots \times \Theta_N$ .<sup>7</sup> Let  $x: \Theta \rightarrow \mathcal{X}$  denote the outcome function for the mechanism and  $p: \Theta \rightarrow \mathbb{R}^N$  be the mechanism transfer function. We assume that  $\mathcal{X}$  is a partially ordered space denoting the possible outcomes for each of the  $N$  agents participating in the mechanism. In the allocation mechanisms we consider,  $x_i(\hat{\theta})$  is the probability that agent  $i$  is allocated the object given the players declare type profile  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_N) \in \Theta$ .

The states of the world reflect the potential agent valuations ( $\Omega = \Theta$ ). Each measure  $\pi \in \Delta$  is a probability measure over  $[\underline{\theta}, \bar{\theta}]$  that is the marginal distribution of an independent and identically distributed product measure over  $\Theta$  equal to  $\times_{i=1}^N \pi$ . Let  $E_X^\pi$  denote an expectation taken with regard to the random variable  $X$  (usually the types of the other agents, denoted  $\theta_{-i}$ ) with respect to the measure  $\pi$ . If  $X$  is a random vector, we take the dimensions to be i.i.d. with distribution in each dimension equal to  $\pi$ .

We focus on mechanisms in which the agents possess felicity functions that are quasi-linear in outcome and payment function. Therefore, the utility representations at the interim and ex post stages have the forms

$$\min_{\pi \in \Delta(\theta_i)} E^\pi [v(x_i; \theta_i) - p_i(\hat{\theta})] \quad (\text{Interim})$$

$$v(x_i; \theta_i) - p_i(\hat{\theta}) \quad (\text{Ex Post})$$

<sup>6</sup>Measuring the welfare effects of changes in the set of priors,  $\Delta$ , can be conducted using standard comparative statics tools. Welfare effects of changes in  $\Delta$  are relatively easy to measure in our quasilinear setting by determining (welfare) equivalent deterministic transfers of the numeraire good.

<sup>7</sup>We reserve the notation  $\theta_i$  for the type of agent  $i$  and  $\theta$  to refer to a vector of agent types.

where  $v(x_i, \theta_i)$  is the utility an agent of type  $\theta_i$  reaps from mechanism outcome  $x_i$  and  $-p_i(\hat{\theta})$  is the agent's utility from paying transfer  $p_i(\hat{\theta})$ . An agent of type  $\theta_i$  has type dependent ambiguous beliefs equal to  $\Delta(\theta_i)$  where  $\Delta(\circ)$  is a convex valued correspondence from  $\Theta_i$  to the set of probability measures over  $\Theta_i$ . At the interim stage, each agent knows his or her own valuation but has ambiguous beliefs about the distribution of valuations of the other agents.

Although every prior entertained by every agent type assumes that the types of the agents are stochastically independent, it is true that agents of types  $\theta, \theta'$  such that  $\Delta(\theta) \neq \Delta(\theta')$  may evaluate actions using different priors.<sup>8,9</sup> In our application to the revenue differences between first and second price auctions, we do not emphasize the role that type dependent ambiguity might play. One potential application would be settings where agent valuations correlate with experience in the mechanism (e.g. low valuation agents are new to the mechanism) and those agents with little experience are represented as entertaining a larger set of probability rules governing the type distribution of the other participants. An interesting goal for future work would be to extend our envelope theorem to a multidimensional type space wherein agents are differentiated both by private valuations and privately know sets of priors. The mechanism designer would then be faced with the possible need to elicit both valuations and sets of priors from the agents.

Our analysis is concerned with two types of mechanisms differentiated by the timing of the incentive constraints. In a dominant strategy mechanism (DSM), the equilibrium type revelations at the interim stage are also an equilibrium at the ex post stage when all of the agents' types are common knowledge.

**Definition 3.** A *truthful dominant strategy mechanism* is one in which  $\forall \theta_{-i} \in \Theta_{-i}$  and  $\forall \theta_i, \hat{\theta}_i \in \Theta_i$  we have

$$v(x_i(\theta_i, \theta_{-i}); \theta_i) - p_i(\theta_i, \theta_{-i}) \geq v(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})$$

Bayesian-Nash mechanisms (BNMs) require that truthful revelation be a Bayesian-Nash equilibrium at the interim stage when each agent knows his own valuation but does not know the valuations of any other participants.

**Definition 4.** A *truthful Bayesian-Nash mechanism* is one in which  $\forall \theta_i, \hat{\theta}_i \in \Theta_i$

$$\min_{\pi \in \Delta(\theta_i)} E_{\hat{\theta}_{-i}}^{\pi} [v(x_i(\theta_i, \theta_{-i}); \theta_i) - p_i(\theta_i, \theta_{-i})] \geq \min_{\pi \in \Delta(\theta_i)} E_{\hat{\theta}_{-i}}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})]$$

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<sup>8</sup>The author thanks an anonymous referee for encouraging the elaboration of this point of interest.

<sup>9</sup>Some readers have suggested that if agents (in effect) act as if they had different, type dependent beliefs that a full extraction result might be possible. Theorem 3 below proves that full extraction is not (in general) possible in our setting. Theorem 3 uses lotteries with 0 expected value to the bidder to extract additional surplus, but the auctioneer is constrained in the amount of surplus that can be extracted with such lotteries since the agents evaluate each possible bid using the most pessimistic evaluation in his set of priors.

We require several regularity assumptions for our analysis.

**A1.** Monotonicity (**M**):<sup>10</sup>  $x_i(\hat{\theta}_i, \theta_{-i})$  is monotone increasing in  $\hat{\theta}_i$  for all  $\hat{\theta}_{-i}$

**A2.**  $v_\theta(x_i; \circ)$  is a bounded, integrable, and continuous function.<sup>11</sup>  $v(x_i; \circ)$  is increasing and equidifferentiable, i.e.

$$\frac{v(x_i, \theta) - v(x_i, \theta')}{\theta - \theta'} \text{ converges uniformly as } \theta \rightarrow \theta'$$

**A3.**  $v(x_i; \theta_i)$  is supermodular in  $(x_i, \theta_i)$

**A4.**  $v(x_i; \theta_i)$  is increasing in  $x_i$

Assumptions 1 through 4 are standard within the mechanism design literature and will not be discussed.

**A5.** The set  $\Delta(\theta_i)$  is compact under the weak\* topology

**A6.** For all  $\theta_i$ ,  $\Delta(\theta_i)$  is correct but ambiguous

Assumption 5 is required to assure the existence of solutions to the optimization problems. Assumption 6 implies that the agents agree that the mechanism designer's beliefs,  $\pi^*$ , represent a possible distribution over the states of the world.

**A7.** We can represent each agent's set of priors by  $\Delta(\theta_i) = \{\pi : g(\pi, \theta_i) \geq 0\}$  for some  $g : \Sigma \times \Theta \rightarrow \mathbb{R}$  where  $\Sigma$  is the set of (Borel) probability measures over  $\Theta$ . We assume  $g(\pi, \theta_i)$  is equidifferentiable in  $\theta_i$ , bounded and quasiconcave in  $\pi$ . We also assume  $g_\theta(\pi, \circ)$  is continuous and integrable.

Assumption 7 provides structure for employing the envelope theorem in cases where  $\Delta(\theta_i)$  is not constant in  $\theta_i$ . Since  $g(\pi, \theta_i)$  is used only to define the constraint set, the requirement that  $g$  be bounded is trivial. The theoretical content is that  $g(\pi, \theta_i)$  is equidifferentiable in  $\theta_i$  and quasiconcave in  $\pi$ . Quasiconcavity in  $\pi$  insures that the set  $\Delta(\theta_i)$  is convex as required by the MEU representation. A sufficient condition for equidifferentiability is to assume that  $\{g_\theta(\pi, \circ)\}_{\pi \in \Sigma}$  is equicontinuous. Importantly, equidifferentiability implies that  $\Delta(\theta_i)$  does not change discontinuously with  $\theta_i$ . Prior applications have assumed that ambiguity was type independent, which entails that  $g_\theta(\pi, \theta_i) = 0$  and thus trivially satisfies A7.

**A8.**  $E_{\hat{\theta}_{-i}}^\pi [v_\theta(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) + \lambda g_\theta(\pi, \theta_i)]$  is continuous in  $\hat{\theta}_i$  and  $\pi \in \Delta(\theta_i)$ <sup>12</sup>

Assumption 8 is a technical condition required for the application of the envelope theorem to saddle point problems.

<sup>10</sup>This condition is not necessary. For example, with linear payoffs we have  $v_\theta(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i) = x_i(\hat{\theta}_i, \hat{\theta}_{-i})$ . If we let  $\pi(\theta_i) \in \arg \min_{\pi \in \Delta(\theta)} E_{\hat{\theta}_{-i}}^\pi [v(x_i(\hat{\theta}_i, \hat{\theta}_{-i}); \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})]$ ,

then we require  $E_{\hat{\theta}_{-i}}^{\pi(\theta_i)} x_i(\hat{\theta}_i, \hat{\theta}_{-i})$  be increasing in  $\hat{\theta}_i$  for all  $i$ .

<sup>11</sup>Let  $v_\theta(x_i; \circ)$  denote  $\frac{\partial}{\partial \theta_i} v(x_i, \circ)$

<sup>12</sup>This is trivially true if  $x$  is continuous. Typically even when  $x$  is discontinuous, continuity can be attained by requiring that  $\Delta(\theta_i)$  contain only measures that are absolutely continuous with respect to the Lebesgue measure.

### 3. The Ambiguous Mechanism Problem

At the interim stage each agent faces the following optimization problem.

$$U(\theta_i) = \max_{\hat{\theta}_i \in \Theta_i} \min_{\pi \in \Delta(\theta_i)} E_{\hat{\theta}_i}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})] \quad (\text{P1})$$

The inner minimization problem is generated by the agent's ambiguity averse preferences. The outer maximization problem is the decision problem generated by the revelation mechanism. We denote the solution to this problem as  $(\hat{\theta}, \hat{\pi})$ . If the mechanism is truthful, then we have  $(\hat{\theta}_i, \hat{\pi}) = (\theta_i, \pi(\theta_i))$  where

$$\pi(\theta_i) \in \min_{\pi \in \Delta(\theta_i)} E_{\theta_{-i}}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})]$$

We refer to  $\pi(\theta_i)$  as a *minimal measure* for type  $\theta_i$ .

We now employ the Generalized Envelope Theorem of Milgrom and Segal (2002, Theorem 4) to prove our payoff equivalence result. (ICFOC) is a necessary condition for truthful revelation to be an equilibrium of our revelation mechanism.

**Theorem 1.** (*Payoff Equivalence*) *Assume (A2), (A5), (A7), and (A8) hold. Then the utility function,  $U(\theta_i)$ , is differentiable almost everywhere with a derivative equal to*

$$\frac{\partial}{\partial \theta_i} U(\theta_i) = U'(\theta_i) = E_{\theta_{-i}}^{\pi(\theta_i)} v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i) \quad (\text{ICFOC})$$

for some multiplier<sup>13</sup>  $\lambda : \Theta_i \rightarrow \mathbb{R}_+$

In the case of SEU preferences we recover the usual payoff equivalence result. In the context of participants with MEU preferences, the agent payoffs depend not only on the outcome function of the mechanism,  $x$ , but on the minimizing measure of each type  $\pi(\theta_i)$  and the multiplier  $\lambda$ . In effect the set of priors for each agent  $\Delta(\theta_i)$  becomes part of the mechanism  $(\Theta, x, p)$ . While the outcome function is an object the designer is free to choose, the minimal measures of the agents are endogenous to the designer's choice of outcome and transfer function.

Our analysis strategy is to identify a class of mechanisms and a set of priors such that the outcome function and minimal measure are the same across the class of mechanisms, thereby implying payoff equivalence across the mechanisms. We then use Theorem 1 to compare the expected revenue generated by the transfer schemes of the mechanisms within this class where the expectation is with respect to  $\pi^*$ , the mechanism designer's unambiguous beliefs. The

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<sup>13</sup>The multiplier  $\lambda(\theta_i)$  can be interpreted as the **shadow cost of ambiguity**, which allows us to study how interim utility changes as a function of changing ambiguity levels. The shadow cost can be interpreted empirically with respect to unambiguous acts such as monetary transfers.

payoff equivalence theorem for risk and ambiguity neutral agents asserts an independence of agent payoff from transfer function. In our case the transfer function can impact the choice of minimal measure by the agent, which makes it potentially relevant for payoff equivalence. In many problems of interest the minimal measure for an agent can be ascertained immediately by inspection (see lemma 3 in the appendix).

Theorem 2 proves that any monotone increasing outcome function paired with any transfer function that provides the agent with the ICFOC mandated utilities forms a truthful mechanism. Note that if  $\Delta(\theta_i) = \{\pi^*\}$ , we recover the usual theorem on interim incentive compatibility in BNMs.

**Theorem 2.** *Assume assumptions (A1), (A3), and (A4) hold. Given (IC-FOC) and (M) hold, the mechanism is truthful and globally incentive compatible.*

We assume that (A1) through (A8) hold for the remainder of our analysis.

#### 4. Optimal Auctions, Revenue (Non-) Equivalence, and Entry Costs

Section 4.1 derives the optimal auction outcome and transfer functions for MEU preferences. Section 4.2 demonstrates that there does not exist a general revenue ranking between the first price auction (FPA) and second price auction (SPA) formats with MEU preferences. Section 4.3 discusses the effect of entry costs on our results.

##### 4.1. Optimal Auctions

In this section we use our payoff equivalence result to explicitly characterize the revenue maximizing auction for agents with ex post utility

$$v(\theta_i) * x_i(\hat{\theta}_i, \hat{\theta}_{-i}) - p_i(\hat{\theta}_i, \hat{\theta}_{-i})$$

and arbitrary compact, convex valued ambiguity belief correspondence  $\Delta(\theta_i)$ . Bose et al. (2006) find that for any incentive compatible allocation mechanism  $(\Theta, x, p)$  where  $U(\theta_i)$  is the interim expected payoff of an agent with type  $\theta_i$ , the revenues for the auctioneer can be increased by using a transfer scheme of the form:

$$p_i^{INS}(\theta_i, \theta_{-i}) = x_i(\theta)v(\theta_i) - U(\theta_i)$$

These transfers insure the agents at the interim stage against ex-post variation in the other agents' types. In effect, the agents evaluate the insurance as an actuarially fair side-bet (given the minimal measure used in the bet-free mechanism), while the auctioneer assesses the insurance as a side-bet with a positive expected value. The transfer function,  $p_i^{INS}(\theta)$ , depends on  $x : \Theta^N \rightarrow [0, 1]^N$  directly (the term  $x_i(\theta)$  appears in the definition of  $p_i^{INS}(\theta)$ ) and indirectly (the choice of  $x(\theta)$  influences the interim expected utilities  $U(\theta_i)$ ).

The payoff equivalence theorem holds in this context, and so  $U(\theta_i)$  must obey (ICFOC). If we assume that the outcome function,  $x_i(\theta)$ , is monotone in

$\theta_i$ , we immediately have from Theorem 2 that the mechanism  $(\Theta, x, p^{INS})$  is truthful and globally incentive compatible. First we show that the insurance transfer scheme is optimal among the class of payment functions resulting from BNM implementations of the outcome function.

**Theorem 3.** *Given a mechanism  $(\Theta, x(\theta), p(\theta))$  such that the (M) and (IC-FOC) hold, the full-insurance mechanism  $(\Theta, x(\theta), p^{INS}(\theta))$  is globally incentive compatible and generates weakly greater revenues.*

Our next corollary shows that auctioneers, when they have control over the ambiguity of bidder beliefs, have an incentive to increase that ambiguity when employing an insurance transfer scheme.<sup>14</sup> Since the bidders face a greater degree of ambiguity, the auctioneer can increase the magnitude of the side bets, which increases the auctioneer's expected revenues.

**Corollary 1.** *Suppose  $\tilde{\Delta}(\theta) \subseteq \Delta(\theta)$ . Then the expected revenues from the insurance transfer scheme are higher under  $\Delta(\theta)$ .*

Given the optimal transfer function, we now derive the revenue maximizing allocation function. The problem of the mechanism designer is:

$$\max_{\{x_i: \Theta^N \rightarrow [0,1]\}_{i=1}^N} \sum_{i=1}^N \int_{\Theta^N} p_i^{INS}(\theta) * F^*(d\theta) \text{ such that } \sum_{i=1}^N x_i(\theta) \in [0,1] \text{ and (M)} \tag{P}$$

where  $F^*(\circ)$  is the cumulative distribution function (CDF) of the designer's beliefs,  $\pi^*$ . The mechanism designer's problem is comprised of two steps. First, the mechanism designer must find the mechanism  $(\Theta, x, p)$  that minimizes the interim expected utility of the agents. In an SEU optimal auction problem, this is only a matter of determining the allocation function  $x : \Theta \rightarrow [0, 1]^N$  as the transfer function is irrelevant for determining interim utilities of incentive compatible mechanisms. However, in the MEU setting our problem is complicated as the transfer function can affect the minimal measure of agent, which in turn can affect the interim utilities. The second step for deriving the optimal auction is to use the insurance transfer scheme associated with the allocation function of our interim utility minimizing allocation mechanism.

**Theorem 4.** *The revenue maximizing auction uses the insurance transfer scheme and allocation function associated with the mechanism  $(\Theta, x, p)$  that maximizes*

$$\sum_{i=1}^N \int_{\underline{\theta}}^{\bar{\theta}} \left( v(\theta_i) - [1 - F^*(\theta_i)] * \frac{v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i)}{E_{\theta_{-i}}^{\pi^*} [x_i(\theta)]} \right) * E_{\theta_{-i}}^{\pi^*} [x_i(\theta)] * d\theta_i$$

*subject to the constraints*

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<sup>14</sup>This contrasts with our analysis of the FPA and SPA where we show that increased ambiguity can either raise or lower auction revenues. See Examples 1 and 2 of section 4.2.

1.  $\sum_{i=1}^N x_i(\theta) \in [0, 1]$
2.  $x_i(\theta)$  is increasing in  $\theta_i$
3.  $\pi(\theta_i) \in \arg \min_{\pi \in \Delta(\theta_i)} v_\theta(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_\theta(\pi(\theta_i), \theta_i)$

In the type independent ambiguity case the interim utility minimizing allocation function and minimal measure are simple to characterize. We are able to reduce condition (3) from Theorem 4 to the simpler condition 3' below. Note that  $\pi(\theta_i)$  is the minimal measure from the first price or all pay auction.<sup>15</sup>

**Theorem 5.** Assume that  $\Delta(\theta_i) = \Delta$ . The revenue maximizing auction sets  $x : \Theta^N \rightarrow [0, 1]^N$  to maximize

$$\sum_{i=1}^N \int_{\underline{\theta}}^{\bar{\theta}} \left( v(\theta_i) - [1 - F^*(\theta_i)] * v_\theta(\theta_i) * \frac{E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta)}{E_{\theta_{-i}}^{\pi^*} [x_i(\theta)]} \right) * E_{\theta_{-i}}^{\pi^*} [x_i(\theta)] * d\theta_i$$

subject to the constraints

1.  $\sum_{i=1}^N x_i(\theta) \in [0, 1]$
2.  $x_i(\theta)$  is increasing in  $\theta_i$
- 3'.  $\pi(\theta_i) \in \arg \min_{\pi \in \Delta} E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta)$

In the type independent ambiguous beliefs case, we find the ambiguity averse virtual valuation function

$$V(\theta_i) = v(\theta_i) - [1 - F^*(\theta_i)] * \frac{v_\theta(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} [x_i(\theta)]}{E_{\theta_{-i}}^{\pi^*} [x_i(\theta)]}, \quad \pi(\theta_i) \in \arg \min_{\pi \in \Delta(\theta_i)} E_{\theta_{-i}}^\pi [x_i(\theta)]$$

assumes a form similar to that found for SEU optimal auctions with an adjustment to account for the ambiguous beliefs of the bidders.<sup>16</sup> If we assume that  $V(\theta_i)$  is increasing, the optimal outcome function  $x : \Theta^N \rightarrow [0, 1]^N$  takes the form of an auction with a reservation price defined by the value  $\theta_0$  at which the ambiguity averse virtual valuation becomes 0.<sup>17</sup>

Finally, we show that more ambiguity aversion implies a lower reservation value. We cannot provide a comparative static for type dependent ambiguity due to the difficulty of signing differences in  $g_\theta$  corresponding to different prior correspondences  $\Delta(\theta_i)$  and  $\hat{\Delta}(\theta_i)$ .

<sup>15</sup>The intuition underlying this analysis is that if  $p_i(\theta)$  varies with  $\theta_{-i}$  then the variables  $v(\theta_i) * x_i(\theta)$  and  $p_i(\theta)$  provide a hedge for each other. The MEU preference relation can be characterized as a strict preference for the hedging of ambiguous lotteries. By choosing  $p_i(\theta)$  to be independent of  $\theta_{-i}$ , we remove the possibility of such a hedge and minimize the interim expected utility. Using these insights, it is easy to show that the all pay and first price auctions are the interim bidder utility minimizing efficient auctions in the type independent ambiguity case.

<sup>16</sup>In the case of ambiguity neutral bidders,  $\pi(\theta_i) = \pi^*$ , the above formula reduces to the SEU optimal auction formula (Myerson 1981, Riley and Samuelson 1981).

<sup>17</sup>Since we assume monotonicity of the outcome function to insure global incentive compatibility, if  $V(\theta)$  is not monotone we may require ironing procedures to be employed (Bulow and Roberts 1989).

**Corollary 2.** *Assume the ambiguity averse virtual valuation,  $V(\theta)$ , is increasing under the type independent sets of priors  $\Delta \subset \tilde{\Delta}$ . Then the optimal reservation value is lower for  $\tilde{\Delta}$*

Increases in bidder ambiguity raise the auctioneer's revenues through two channels. First, holding the outcome function fixed, the insurance scheme transfers are increased for each type when the agents exhibit a higher degree of ambiguity aversion. Second, the optimal reservation value drops as the agent's degree of ambiguity aversion increases, which decreases the probability that the auctioneer will retain the item following the auction.

#### 4.2. First Price Auction versus Second Price Auction

Although the first and second price auction formats generate equal revenues under the assumption of quasi-linear private values with risk and ambiguity neutral agents, we use the payoff-equivalence theorem above to examine whether any ranking can be established in the MEU model. We examine several special cases with the conclusion that no general revenue ordering exists.

Our analysis framework has the following four steps:

1. Define the space of measures  $\Delta$
2. Show for all types  $\theta_i$  that the minimal measure is equivalent across BNM and DSM implementations of the outcome function
3. Utilize the payoff equivalence formula to compare expected payments
4. Compute expected revenues for each format under the designer's beliefs  $\pi^*$

Consider the felicity function:

$$v(x_i(\theta_i, \theta_{-i}), \theta_i) = v(\theta_i) * x_i(\theta_i, \theta_{-i})$$

Both the FPA and SPA are efficient allocation mechanisms, so

$$x_i(\theta_i, \theta_{-i}) = \begin{cases} 1 & \text{if } \theta_i = \theta^{(1)} \\ 0 & \text{otherwise} \end{cases}$$

where  $\theta^{(k)}$  is the  $k^{th}$ -highest valuation declared amongst the  $N$  agents. In the cases of ties, we assume the object is allocated with uniform probability to the agents who submitted the highest bids. The transfer functions for the FPA and the SPA, denoted  $p^{FPA}$  and  $p^{SPA}$  respectively, have the form

$$\begin{aligned} p_i^{FPA}(\theta_i, \theta_{-i}) &= p^{FPA}(\theta_i) * x_i(\theta_i, \theta_{-i}) \\ p_i^{SPA}(\theta_i, \theta_{-i}) &= \theta^{(2)} * x_i(\theta_i, \theta_{-i}) \end{aligned}$$

Clearly  $v(x_i(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i})$  is weakly decreasing in  $\theta_{-i}$  for all weakly undominated declarations  $\hat{\theta}_i$ . Assuming there exists a measure in  $\Delta(\theta_i)$  that is first order stochastically dominated by the other measures in the set of priors,

we know from Lemma 3 that the dominated measure is a minimal measure for all agent types.

Assume that the type space for each bidder is  $[0, 1]$  and that the agents' set of priors can be bounded by additive measures  $L$  and  $U$ .<sup>18</sup>

$$\Delta = \{\pi : \text{For all } E \subseteq [0, 1], L(E) \leq \pi(E) \leq U(E), \pi(\Theta) = 1\}$$

We refer to this as a **general contamination model of ambiguity**. This structure bounds the ambiguity of the agent's beliefs about the probability of event  $E \subset [0, 1]$  between  $L(E) \leq U(E)$ .<sup>19</sup> If  $L(E) = U(E)$ , then the agent has no ambiguity about the probability of event  $E$ . If  $L(E) = 0$  and  $U(E) = 1$ , then the agent's beliefs are maximally ambiguous about the probability of event  $E$ . We show that within this simple structure, we can choose  $L$  and  $U$  so that the FPA can provide either strictly more or strictly less revenue than the SPA. Therefore, revenue ranking results are not robust to changes in the set of priors entertained by the agents.

For notational simplicity, we assume that both  $L, U$  are absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}$  and denote the respective Radon-Nikodym derivatives as  $f^L, f^U$  and the Riemann-Stieltjes integrals as  $F^L, F^U$ . We can write

$$\Delta = \{\pi : f^L(\theta) \leq f^\pi(\theta) \leq f^U(\theta), F^\pi(1) = 1\}$$

Finally, assume that  $\pi^*$  has probability density function (PDF)  $f^*$  that is positive for all  $\theta \in [0, 1]$ .

The set  $\Delta$  contains a lower bound in the strong stochastic order that we denote  $\pi^{MIN}$ , and Lemma 3 proves that  $\pi^{MIN}$  is the minimal measure for all types of bidders.  $\pi^{MIN}$  has CDF

$$F^{MIN}(\theta) = \min_{\pi \in \Delta} F^\pi(\theta) = \begin{cases} F^L(\theta) & \text{for all } \theta \leq \theta^+ \\ F^U(\theta) - F^U(\theta^+) + F^L(\theta^+) & \text{for all } \theta > \theta^+ \end{cases}$$

where  $F^U(\theta) - F^U(\theta^+) = 1 - F^L(\theta^+)$ . The associated PDF is

$$f^{MIN}(\theta) = \begin{cases} f^L(\theta) & \text{for all } \theta \leq \theta^+ \\ f^U(\theta) & \text{for all } \theta > \theta^+ \end{cases}$$

Note that  $\pi^{MIN} \in \Delta$  by construction.

Interim utilities in each auction format are

$$\min_{\pi \in \Delta(\theta_i)} E_{\theta_i}^\pi [v(\theta_i)x_i(\theta) - p_i^{FPA}(\theta)] = F^{MIN}(\theta_i) * [v(\theta_i) - p^{FPA}(\theta_i)] \quad (\text{FPA})$$

<sup>18</sup>Note that  $L$  and  $U$  are not probability measures, so it can be the case that  $L(\Theta) < 1 < U(\Theta)$ .

<sup>19</sup>Our notion of contamination is a generalization of Lo (1998), where he uses a constant upper and lower restriction on the Radon-Nikodym derivatives of  $\pi \in \Delta$  with respect to  $\pi^*$ . Lo finds a sharp revenue ranking between the SPA and FPA, and our analysis highlights the special nature of the example developed by Lo.

$$\min_{\pi \in \Delta(\theta)} E_{\theta_{-i}}^{\pi} [v(\theta_i)x_i(\theta) - p_i^{SPA}(\theta)] = \quad (\text{SPA})$$

$$F^{MIN}(\theta_i) * [v(\theta_i) - E_{\theta_{-i}}^{MIN}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)]$$

where  $E_{\theta_{-i}}^{MIN}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)$  denotes the conditional expectation with respect to the measure  $\pi^{MIN}$ . Theorem 1 yields  $p_i^{FPA}(\theta_i) = E_{\theta_{-i}}^{MIN}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)$ . The revenue ranking of the FPA and the SPA hinges on whether the expectation of  $p_i^{SPA}(\theta)$  taken under  $\pi^{MIN}$  contingent on bidder  $i$  winning is greater or less than the conditional expectation taken under the true distribution  $\pi^*$ .

**Example 1. Example: (FPA Revenues Greater than SPA Revenues)**

Suppose that  $f^L(\theta) = \alpha(\theta) * f^*(\theta)$  where  $\alpha(\theta) : \Theta \rightarrow (0, 1)$  is strictly increasing and  $f^U(\theta) = \beta(\theta) * f^*(\theta)$  where  $\beta(\theta) : \Theta \rightarrow (1, +\infty)$  is bounded and strictly increasing. Let  $F^{MIN}(\theta|\theta^{(1)})$  denote the CDF of the minimal measure and  $F^*(\theta|\theta^{(1)})$  denote the CDF of the auctioneer's beliefs where both distributions are conditional on the event that  $\theta \leq \theta^{(1)}$  where  $\theta^{(1)}$  is the highest type announced amongst the participants.

**Lemma 1.**  $F^{MIN}(\theta|\theta^{(1)})$  strictly first order stochastically dominates (FOSD)  $F^*(\theta|\theta^{(1)})$

**Proof.** By definition we have that for all  $x, y \in \Theta$  where  $\theta^{(1)} \geq x > y$  that  $\frac{f^{MIN}(x)}{f^{MIN}(y)} \geq \frac{f^*(x)}{f^*(y)}$ . Therefore we have that for  $\bar{x} \in (y, x]$

$$\begin{aligned} & \frac{\int_{\bar{x}}^{\theta^{(1)}} f^{MIN}(x)dx}{f^{MIN}(y)} > \frac{\int_{\bar{x}}^{\theta^{(1)}} f^*(x)dx}{f^*(y)} \\ \rightarrow & \frac{f^{MIN}(y)}{\int_{\bar{x}}^{\theta^{(1)}} f^{MIN}(x)dx} < \frac{f^*(y)}{\int_{\bar{x}}^{\theta^{(1)}} f^*(x)dx} \\ \rightarrow & \frac{\int_0^{\bar{x}} f^{MIN}(y)dx}{\int_{\bar{x}}^{\theta^{(1)}} f^{MIN}(x)dx} < \frac{\int_0^{\bar{x}} f^*(y)dx}{\int_{\bar{x}}^{\theta^{(1)}} f^*(x)dx} \\ \rightarrow & \frac{F^{MIN}(\bar{x}|\theta^{(1)})}{1 - F^{MIN}(\bar{x}|\theta^{(1)})} < \frac{F^*(\bar{x}|\theta^{(1)})}{1 - F^*(\bar{x}|\theta^{(1)})} \\ \rightarrow & F^{MIN}(\bar{x}|\theta^{(1)}) < F^*(\bar{x}|\theta^{(1)}) \end{aligned}$$

■  
From the strict FOSD property and the fact that  $p_i^{SPA}(\theta)$  is increasing in  $\theta_{-i}$  conditional on  $\theta_i = \theta^{(1)}$  for the weakly undominated strategy  $\hat{\theta}_i = \theta_i$ , we have that that  $E_{\theta_{-i}}^{MIN}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i) > E_{\theta_{-i}}^{\pi^*}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)$  (see Theorem 1.2.16 from Muller and Stoyan 2002). Therefore,  $p_i^{FPA}(\theta_i) > E_{\theta_{-i}}^{\pi^*}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)$  for each type  $\theta_i$ , implying that the FPA generates strictly higher revenue than the SPA.

In this example the minimal measure overweights the likelihood of high valuation competitors relative the likelihood of low valuation competitors. Importantly, this overweighting occurs for all subintervals of  $[0, 1]$ . Therefore, every bidder in the auction perceives competition to be intense and this encourages more aggressive bidding in the first price auction.

**Example 2. Example: (FPA Revenues Less than SPA Revenues)** Suppose that  $f^L(\theta) = \alpha(\theta) * f^*(\theta)$  where  $\alpha(\theta) : \Theta \rightarrow (0, 1)$  is strictly decreasing and  $f^U(\theta) = \beta(\theta) * f^*(\theta)$  where  $\beta(\theta) : \Theta \rightarrow (1, +\infty)$  is bounded, decreasing, and continuous.

Note the definition of  $\theta^+$  is  $F^U(1) - 1 = F^U(\theta^+) - F^L(\theta^+)$  and that we can increase  $F^U(1)$  by increasing  $\beta(\theta)$  within an arbitrarily small interval  $(\theta^*, 1]$  where  $\theta^* < 1$ . Therefore,  $\theta^+$  can be forced arbitrarily close to (but less than) 1 by selecting  $\beta$  appropriately.

For  $x, y, \theta^{(1)} \in [0, \theta^+]$  and  $\theta^{(1)} \geq x > y$  the distribution  $f^{MIN}(x|\theta^{(1)})$  exhibits the monotone decreasing likelihood ratio property. By methods as in Example 1, we see that for  $\theta_i \leq \theta^+$  we have that  $p_i^{FPA}(\theta_i) < E_{\theta_i}^{\pi^*}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)$ . Therefore the SPA generates strictly higher revenue for these types. Note that for all  $\theta \in [\theta^+, 1]$  we have that  $|p_i^{FPA}(\theta_i) - E_{\theta_i}^{\pi^*}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)| \leq 1$ . For any  $\varepsilon > 0$  we can choose  $\beta$  such that  $\theta^+$  is sufficiently close to 1 that the probability that  $\theta_i \in [\theta^+, 1]$  under  $\pi^*$  is less than  $\varepsilon$ . Combining these facts, we have that we can choose  $\beta(\circ)$  such that we have

$$E_{\theta_i}^{\pi^*}[E_{\theta_i}^{\pi^*}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i)] > E_{\theta_i}^{\pi^*}[p_i^{FPA}(\theta_i)]$$

In Example 2 the minimal measure pushes agents to bid as if they will rarely win, but the bidders believe conditional on winning the auction that the opponent valuations will be low and the item can be won with a relatively low bid. While the bids are determined by the shape of the minimal measure, the auctioneer's beliefs determine the expected revenues. Therefore the agents bid low and win more frequently than the bidders expect. We now provide a parameterized example to illustrate these effects.

**Example 3.** Let  $\Theta = [0, 1]$ ,  $f^*(\theta) = 1$ , and consider the FPA and SPA with 2 bidders. For reference, if we assume the bidders have SEU preferences and beliefs equal to  $f^*$ , the equilibrium bidding strategy is

$$b^*(\theta) = \frac{1}{2}\theta$$

First consider case 1,  $f^L(\theta) = \theta < 1 < \theta + 1 = f^U(\theta)$ . Then we have

$$f^{MIN}(\theta) = \begin{cases} \theta & \text{if } \theta < \frac{1}{2} \\ 1+\theta & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

Using our payoff equivalence formulation, we find equilibrium FPA bidding function

$$b_1(\theta) = \begin{cases} \frac{2}{3}\theta & \text{if } \theta < \frac{1}{2} \\ \frac{8*\theta^3+12*\theta^2-3}{12*\theta^2+24*\theta-12} & \text{if } \theta \geq \frac{1}{2} \end{cases}$$

Now consider case 2, wherein the set of priors defined by  $f^L(\theta) = 1 - \theta < 1 < 10^{10} = f^U(\theta)$ .<sup>20</sup> Then the equilibrium FPA bidding function is

$$b_2(\theta) = \frac{3 * \theta - 2 * \theta^2}{6 - 3 * \theta}$$

Since  $b_2(\theta) < b^*(\theta) < b_1(\theta)$ , the auctioneer prefers to use a FPA in case 1 and a SPA in case 2. The bidding functions are plotted in Figure 1.

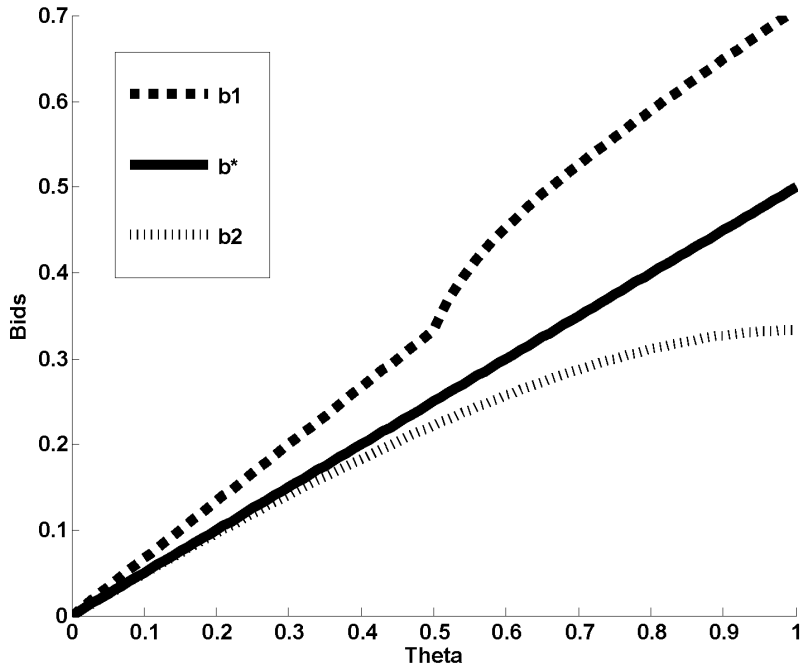


Figure 1: Equilibrium Bidding Strategies

Two significant conclusions should be drawn from this analysis. We have shown that risk and ambiguity aversion can have sharply different effects on auction revenues. While risk aversion causes the SPA to generate lower revenues than the FPA (Maskin and Riley 1984), ambiguity aversion can drive the revenue ranking in either direction. Therefore relying on familiar intuitions about mechanism design with risk averse participants can be misleading when applied to the case of risk neutral, MEU agents. The second conclusion is that analyses of the effect of ambiguity aversion on mechanism outcomes are sensitive

<sup>20</sup>The measure  $U$  effectively places an atom in the neighborhood of  $\theta = 1$ .

to the choice of ambiguity assumed. Without standards for a benchmark form of ambiguity to employ in a model, theoretical economists should be careful to test the robustness of their conclusions against a variety of assumptions. However, this non-robustness implies a valuable role for empirical and experimental economists to add to the applied theory by studying what forms of ambiguity decision makers exhibit.<sup>21</sup>

Finally, we provide a comparative static regarding the bids in the first price auction in the generalized contamination model of ambiguity. Define the sets of priors  $\Delta \subset \tilde{\Delta}$  as follows

$$\begin{aligned}\Delta &= \{\pi : f^L(\theta) \leq f^\pi(\theta) \leq f^U(\theta), F^\pi(1) = 1\} \\ \tilde{\Delta} &= \{\pi : \tilde{f}^L(\theta) \leq f^\pi(\theta) \leq \tilde{f}^U(\theta), F^\pi(1) = 1\}\end{aligned}$$

where  $\tilde{f}^L(\theta) \leq f^L(\theta)$  and  $f^U(\theta) \leq \tilde{f}^U(\theta)$ . It is immediate that for all  $\theta$

$$\min_{\pi \in \tilde{\Delta}} F^\pi(\theta) = \tilde{F}^{MIN}(\theta) \leq F^{MIN}(\theta) = \min_{\pi \in \Delta} F^\pi(\theta)$$

Let  $p_\Delta^{FPA} : \Theta_i \rightarrow \mathbb{R}_+$  and  $p_{\tilde{\Delta}}^{FPA} : \Theta_i \rightarrow \mathbb{R}_+$  represent the incentive compatible equilibrium bidding functions under  $\Delta$  and  $\tilde{\Delta}$  respectively.

**Theorem 6.** For all  $\theta_i \in \Theta_i$ ,  $p_\Delta^{FPA}(\theta_i) \leq p_{\tilde{\Delta}}^{FPA}(\theta_i)$

**Proof.** Lemma 3 implies  $F^{MIN}$  and  $\tilde{F}^{MIN}$  are the CDFs of the minimal measures under  $\Delta$  and  $\tilde{\Delta}$  respectively. As above we have payoff equivalence between the FPA and SPA under both  $\Delta$  and  $\tilde{\Delta}$ .  $\tilde{F}^{MIN}$  dominates  $F^{MIN}$  in the strong stochastic order, so from our payoff equivalence theorem and the definition of strong stochastic dominance we have

$$\begin{aligned}p_\Delta^{FPA}(\theta_i) &= E_{\theta_{-i}}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i, F^{MIN}) \\ &\leq E_{\theta_{-i}}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i, \tilde{F}^{MIN}) = p_{\tilde{\Delta}}^{FPA}(\theta_i)\end{aligned}$$

where  $E_{\theta_{-i}}(p_i^{SPA}(\theta)|\theta^{(1)} = \theta_i, F)$  is the expectation of  $p_i^{SPA}(\theta)$  conditional on  $\theta^{(1)} = \theta_i$  under the minimal measure with CDF  $F$ . ■

Increased ambiguity could be a side effect of other goals of the auctioneer. For example, the auctioneer might refuse to disclose information about the set of bidders in order to remove the possibility of collusion, explicit or otherwise. This has been cited as a concern in the FCC spectrum auctions. The FCC also pursues a mandate to provide licenses to small and minority owned businesses. To the extent that these groups are formed of auction novices, one should expect that the new bidders may be ambiguity averse.

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<sup>21</sup>There is a growing experimental literature calibrating models of ambiguity aversion (e.g. Ahn et al. 2009).

An identical comparative static can be provided for the all pay auction using a similar argument. The all pay auction is used as a model of wars of attrition, which includes such diverse activities as political lobbying and R&D races. A lobbyist may be very familiar with his own client's interest but have ambiguous beliefs regarding the valuation other groups place on a piece of legislation. Our comparative static suggests that laws designed to clamp down on bribery could increase the value of the bribes received by public officials if the laws have the effect of making disambiguating information transmission more difficult. Both of these results are due to the fact that competition in the model becomes fiercer as ambiguity increases, whether that competition is for political favors or R&D products.

#### 4.3. Ex Ante Entry Costs

Consider a two stage game amongst a pool of  $\bar{N}$  ex ante symmetric potential auction participants. Each participant has to decide whether to enter the auction by paying a cost  $c > 0$  to learn his valuation for the good.<sup>22</sup> Potential bidders who choose not to pay  $c$  cannot bid in the auction. We assume that at the ex ante stage, all of the potential bidders are ambiguity averse with preferences defined by the set of priors  $\Delta$ . The set of priors defines ex ante beliefs about the distribution of opponent valuations as well as beliefs about each agent's own interim valuation. Strategies consist of the decision as to whether or not to enter the auction and, contingent on entry, how to bid given each possible valuation. We consider only pure strategies.

Bidders choose to enter the auction until the ex ante expected benefit of entry is outweighed by the costs of entering. Formally, the number of entrants  $N$  solves

$$\begin{aligned} V(N) &= \min_{\pi \in \Delta} E_{\theta_i}^{\pi} U_N(\theta_i) \geq c \\ V(N+1) &= \min_{\pi \in \Delta} E_{\theta_i}^{\pi} U_{N+1}(\theta_i) < c \end{aligned} \quad (4.1)$$

where  $U_N(\theta_i)$  is the interim utility of type  $\theta_i$  given the presence of  $N$  participants in the auction.<sup>23</sup>

<sup>22</sup>Levin and Ozdenoren (2004) provide an analysis of auctions with an unambiguous distribution of agent valuations and an exogenous number of bidders that are unknown at the time bids are evaluated. In this work agents are assumed to have ambiguous beliefs about the number of other participants. In our model the bidders know the number of bidders at the interim stage when bids are submitted.

<sup>23</sup>At the ex ante stage, bidders evaluate their entry decision taking into account the utilities reaped at the interim stage of the auction. Unpacking condition 4.1, we have:

$$V(N) = \min_{\pi \in \Delta} E_{\theta_i}^{\pi} U_N(\theta_i) = \min_{\pi \in \Delta} E_{\theta_i}^{\pi} \left[ \max_{\tilde{\theta}_i \in \Theta_i} \min_{\tilde{\pi} \in \Delta(\theta_i)} E_{\theta_{-i}}^{\tilde{\pi}} [v(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - p_i(\tilde{\theta}_i, \theta_{-i})] \right]$$

Alternately, we could assume that at the ex ante stage the agent is naive about his future behavior and treats his interim self as having the same beliefs as his ex ante self. In this case

The interim utility is weakly declining in  $N$  and asymptotes to 0, implying that there is some  $N$  for which these conditions are satisfied.

**Theorem 7.** *Consider  $\Delta(\theta) \subset \tilde{\Delta}(\theta)$  with corresponding equilibrium numbers of entrants  $N$  and  $\tilde{N}$ . Then we have  $N \geq \tilde{N}$*

**Proof.** It suffices to note that  $\min_{\pi \in \Delta(\theta)} E_{\theta_i}^\pi U_N(\theta_i) \geq \min_{\pi \in \tilde{\Delta}(\theta)} E_{\theta_i}^\pi U_N(\theta_i)$ . Therefore,  $\min_{\pi \in \tilde{\Delta}(\theta)} E_{\theta_i}^\pi U_N(\theta_i) \geq c$  implies  $\min_{\pi \in \Delta(\theta)} E_{\theta_i}^\pi U_N(\theta_i) \geq c$ , which in turn yields  $N \geq \tilde{N}$ . ■

This implies that entry is depressed when agents have ambiguous beliefs regarding the valuations of other bidders relative to the case where all bidders have the unique prior  $\pi^*$ .

We show that there is no tension between choosing an auction format that induces entry and one that maximizes revenue if the mechanisms are payoff equivalent for the bidders. This implies that the optimal auction and revenue rankings derived above are not sensitive to the presence of entry costs.

**Theorem 8.** *Consider a set of payoff equivalent auction mechanisms. Then the introduction of entry fees does not affect the revenue ranking amongst these mechanisms.*

**Proof.** Since the interim payoff equivalence result still holds, we have holding  $N$  fixed that the revenue ranking holds. For payoff equivalent auctions the interim utilities,  $U_N(\theta_i)$ , are identical, which implies that  $V(N)$  is the same across all of the mechanisms and the amount of entry is identical. Therefore, for mechanisms that are payoff equivalent, the revenue rankings derived above hold unconditional on  $N$ . ■

## 5. Optimal Bilateral Trade Mechanisms

In this section we analyze two bilateral trade mechanisms using our payoff equivalence theorem. First, we consider the problem of ex-ante budget deficits in efficient bargaining mechanisms. We show that higher levels of agent ambiguity are associated with smaller ex ante budget deficits when the revenue maximizing transfer scheme is used. Second, we consider the range of bargaining outcomes that can be implemented in an ex-ante budget balanced bilateral trade mechanism. We find that efficient trade increases with the ambiguity level when the revenue maximizing transfer scheme is employed.

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we have:

$$V^{EA}(N) = \min_{\pi \in \Delta} E_{\theta_i}^\pi \left[ \max_{\tilde{\theta}_i \in \Theta_i} E_{\theta_{-i}}^\pi [v(x_i(\tilde{\theta}_i, \theta_{-i}), \theta_i) - p(\tilde{\theta}_i, \theta_{-i})] \right]$$

### 5.1. Efficient Bargaining

Myerson and Satterthwaite (1983) show that any efficient bilateral trade mechanism implies an ex-ante budget deficit in an SEU setting. The revenue equivalence result for Bayesian-Nash mechanisms under SEU preferences implies that the budget deficit is equal across efficient, interim incentive compatible bargaining mechanisms. Payoff equivalence does not imply revenue equivalence when agents are ambiguity averse. Therefore, different incentive compatible transfer schemes can change the magnitude of the ex-ante budget deficit.

We assume that the buyer and the seller have identical ambiguous beliefs and quasi-linear felicity functions

$$v(x_i(\theta), \theta_i) = \theta_i * x_i(\theta) - p_i(\theta)$$

In an efficient bargaining protocol we have exchange if and only if the buyer values the good more than the seller.

$$\begin{aligned} x_B^G(\theta_S, \theta_B) &= 1\{\theta_B \geq \theta_S\} \\ x_S^G(\theta_S, \theta_B) &= 1\{\theta_B < \theta_S\} \end{aligned}$$

where  $B$  and  $S$  subscripts denote outcome and transfer functions for the buyer and seller. Denote the minimal measures by

$$\begin{aligned} \pi_B(\theta_B) &\in \arg \min_{\pi \in \Delta} E_S^\pi[\theta_B * x_B^G(\theta) - p_B(\theta)] \\ \pi_S(\theta_S) &\in \arg \min_{\pi \in \Delta} E_B^\pi[\theta_S * x_S^G(\theta) - p_S(\theta)] \end{aligned}$$

where the notation  $E_S^\pi f(\theta_S, \theta_B)$  denotes an expectation taken using measure  $\pi$  over values of  $\theta_S$ , and  $E_B^\pi h(\theta_S, \theta_B)$  denotes an expectation taken using measure  $\pi$  over values of  $\theta_B$ .

We will employ insurance transfer schemes of the form

$$\begin{aligned} p_B^{INS}(\theta_S, \theta_B) &= \theta_B * 1\{\theta_B \geq \theta_S\} - U_B(\theta_B) \\ p_S^{INS}(\theta_S, \theta_B) &= \theta_S * 1\{\theta_B < \theta_S\} - U_S(\theta_S) \end{aligned}$$

In order to minimize the budget deficit, our goal is to maximize the payments from each party, which is the equivalent of minimizing the interim expected utility. Using our payoff equivalence theorem, we have

$$\begin{aligned} U_S(\theta_i) &= U_S(\bar{\theta}) - \int_{\theta_i}^{\bar{\theta}} E_B^{\pi_S(s)} 1\{\theta_B < s\} * ds \\ U_B(\theta_i) &= U_B(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} E_S^{\pi_B(s)} 1\{s \geq \theta_S\} * ds \end{aligned}$$

where  $U_S(\bar{\theta}) = \bar{\theta}$  and  $U_B(\underline{\theta}) = 0$ . Recollecting that any measure is minimal under the insurance transfer schemes, to minimize the interim utility we set

$$\begin{aligned} \pi_S(\theta_S) &\in \arg \max_{\pi \in \Delta} E_B^\pi 1\{\theta_B < \theta_S\} = \arg \max_{\pi \in \Delta} F^\pi(\theta_S) \\ \pi_B(\theta_B) &\in \arg \min_{\pi \in \Delta} E_S^\pi 1\{\theta_B \geq \theta_S\} = \arg \min_{\pi \in \Delta} F^\pi(\theta_B) \end{aligned}$$

Denote the interim utility given a type  $\theta_i$  and a set of priors  $\Delta$  as  $U_i(\theta_i; \Delta)$ . Note that if  $\Delta \subset \tilde{\Delta}$ , we then have

$$U_i(\theta_i; \Delta) \geq U_i(\theta_i; \tilde{\Delta})$$

This in turn implies that the insurance transfer under  $\Delta$  and  $\tilde{\Delta}$ , denoted  $p_i^{INS}(\theta; \Delta)$  and  $p_i^{INS}(\theta; \tilde{\Delta})$  respectively, obey

$$\begin{aligned} p_B^{INS}(\theta_B; \Delta) + p_S^{INS}(\theta_S; \Delta) &= \max\{\theta_B, \theta_S\} - U_B(\theta_B; \Delta) - U_S(\theta_S; \Delta) \\ &\leq \max\{\theta_B, \theta_S\} - U_B(\theta_B; \tilde{\Delta}) - U_S(\theta_S; \tilde{\Delta}) \\ &= p_B^{INS}(\theta_B; \tilde{\Delta}) + p_S^{INS}(\theta_S; \tilde{\Delta}) \end{aligned}$$

Therefore payments to the auctioneer are higher under  $\tilde{\Delta}$ , implying a lower ex ante budget balance. Furthermore, when  $\Delta$  admits any probability measure over  $\Theta$  we have that  $\pi_S(\theta_i)$  is an atom at  $\underline{\theta}$  and  $\pi_B(\theta_i)$  is an atom at  $\bar{\theta}$ , which implies

$$\begin{aligned} U_B(\theta_B) &= 0 \\ U_S(\theta_S) &= \theta_S \end{aligned}$$

In effect, the designer can appropriate all of the gains of trade.

The following theorem encapsulates all of our results

**Theorem 9.** *The revenues from an insurance transfer scheme yield a lower ex ante budget deficit (with respect to the true distribution  $\pi^*$ ) than any other payment scheme in the efficient bilateral trade mechanism. Further, the budget deficit is decreasing as agent ambiguity increases and revenues to the designer may be (weakly) positive.*

## 5.2. Ex-Ante Balanced Budget Bilateral Trade Mechanisms

Assume agent felicity functions are quasi-linear in payments with ambiguous belief correspondence  $\Delta(\theta_i)$ . Consider the bargaining mechanism  $x_B(\theta_B, \theta_S) = 1\{\theta_B - \theta_S \geq c\}$  where  $c$  is chosen so that the mechanism is ex-ante budget balanced with respect to  $\pi^*$ .<sup>24</sup> This necessitates a loss of efficiency in terms of efficient trades that are foregone, and we use  $c$  as a measure of the efficiency cost of ex ante budget balance.

As above we use insurance transfer schemes of the form

$$\begin{aligned} p_B^{INS}(\theta_S, \theta_B) &= \theta_B * 1\{\theta_B - \theta_S \geq c\} - U_B(\theta_B) \\ p_S^{INS}(\theta_S, \theta_B) &= \theta_S * 1\{\theta_B - \theta_S \geq c\} - U_S(\theta_S) \end{aligned}$$

<sup>24</sup>Ex ante budget balance requires that in equilibrium

$$E_{(\theta_B, \theta_S)}^{\pi^*} [p_B(\theta_B, \theta_S) + p_S(\theta_B, \theta_S)] = 0$$

To maximize revenues to the designer (and thus minimize the choice of  $c$  required to achieve budget balance), we must minimize the interim utility reaped by each agent. Using our payoff equivalence theorem, we have

$$\begin{aligned} U_S(\theta_S) &= \bar{\theta} - \int_{\theta_S}^{\bar{\theta}} E_B^{\pi_S(s)} 1\{\theta_B - c < s\} * ds \\ U_B(\theta_B) &= \int_{\underline{\theta}}^{\theta_B} E_S^{\pi_B(s)} 1\{s \geq \theta_S + c\} * ds \end{aligned}$$

Recollecting again that any measure is minimal under the insurance transfer schemes, to minimize the interim utility we set

$$\begin{aligned} \pi_S(\theta_S) &\in \arg \max_{\pi \in \Delta} E_B^{\pi} 1\{\theta_B - c < \theta_S\} = \arg \max_{\pi \in \Delta} F^{\pi}(\theta_S + c) \\ \pi_B(\theta_B) &\in \arg \min_{\pi \in \Delta} E_S^{\pi} 1\{\theta_B - c \geq \theta_S\} = \arg \min_{\pi \in \Delta} F^{\pi}(\theta_B - c) \end{aligned}$$

Denote the interim utilities given threshold  $c \geq 0$  and a set of priors  $\Delta$  as  $U_S(\theta_S; \Delta, c)$  and  $U_B(\theta_B; \Delta, c)$ . It is straightforward to show that  $c > c'$  implies  $U_S(\theta_S; \Delta, c) \leq U_S(\theta_S; \Delta, c')$  and  $U_B(\theta_B; \Delta, c) \leq U_B(\theta_B; \Delta, c')$  using the formula above.

Similar to our argument in Section 5.1, we have holding  $c$  fixed that

$$\begin{aligned} p_B^{INS}(\theta_B; \Delta) + p_S^{INS}(\theta_S; \Delta) &= \theta_S + (\theta_B - \theta_S) * 1\{\theta_B - \theta_S \geq c\} - U_B(\theta_B; \Delta, c) - U_S(\theta_S; \Delta, c) \\ &\leq \theta_S + (\theta_B - \theta_S) * 1\{\theta_B - \theta_S \geq c\} - U_B(\theta_B; \tilde{\Delta}, c) - U_S(\theta_S; \tilde{\Delta}, c) \\ &= p_B^{INS}(\theta_B; \tilde{\Delta}) + p_S^{INS}(\theta_S; \tilde{\Delta}) \end{aligned}$$

Therefore if

$$p_B^{INS}(\theta_B; \Delta) + p_S^{INS}(\theta_S; \Delta) = 0 < p_B^{INS}(\theta_B; \tilde{\Delta}) + p_S^{INS}(\theta_S; \tilde{\Delta})$$

then the designer can choose  $c' < c$  such that

$$0 = \theta_S + (\theta_B - \theta_S) * 1\{\theta_B - \theta_S \geq c'\} - U_B(\theta_B; \tilde{\Delta}, c') - U_S(\theta_S; \tilde{\Delta}, c')$$

Such a choice of  $c'$  then restores budget balance and improves efficiency by allowing more surplus enhancing exchanges to occur. As noted in Section 5.1, when  $\Delta$  admits any probability measure over  $\Theta$  the designer can set  $c = 0$  and achieve a weakly positive revenue surplus. These results are summarized in the Theorem below.

**Theorem 10.** *Interim incentive compatible payments from the buyer (to the seller) are weakly higher (lower) under greater ambiguity when the optimal transfer scheme is used in the ex ante budget balanced bilateral trade mechanism. Therefore, the ex ante budget balanced mechanism is more efficient as ambiguity increases and becomes fully efficient when  $\Delta$  admits any probability measure over  $\Theta$ .*

## 6. Conclusion

The primary contribution of this paper is the development of a payoff equivalence theorem for mechanisms with ambiguity averse participants. We show that monotonicity of the mechanism and the interim incentive compatibility first order conditions are sufficient for global incentive compatibility. The payoff equivalence result is used to derive the revenue maximizing auction for arbitrary, potentially type dependent ambiguous beliefs. The revenue maximizing auction incorporates an insurance transfer scheme, first studied by Bose et al. (2006), as well as a reservation value. The optimal reservation value depends on an ambiguous virtual valuation function similar to the virtual valuation derived by Myerson (1981) for SEU agents. We show that as ambiguity aversion increases, the revenue of the auctioneer increases due to increased bids by each type of bidder and a decrease in the optimal reservation value. The decreasing reservation value also renders ex post allocations increasingly efficient as bidder ambiguity increases.

We show that there is no general revenue ranking between the first and second price auctions when bidders have MEU preferences. When agents are risk averse and ambiguity neutral, first price auctions generate greater revenue than second price auctions (Maskin and Riley 1984). However, when the agents are ambiguity averse and risk neutral, the auction format that is revenue maximizing depends on the set of ambiguous beliefs entertained by the bidders. Our examples demonstrate that ambiguity and risk aversion have qualitatively different effects on agent behavior and that market designers who rely on intuitions based on risk aversion when considering instances of ambiguity aversion can be led astray in their analysis.

Our payoff equivalence result allows us to examine the behavior of agents bargaining over an object when the bargainers have ambiguous beliefs about the other agent's valuation. We study two bilateral trade mechanisms. The first is an ex post efficient bilateral trade mechanism styled after Myerson and Satterthwaite (1983). We derive a revenue maximizing transfer scheme and show that increases in the degree of participant ambiguity lead to decreases in the ex ante budget deficit. Our second bilateral trade mechanism is ex ante budget balanced, but prevents some instances of efficient trade where the buyer's valuation is not sufficiently greater than the seller's. We show that when a revenue maximizing transfer scheme is employed, increases in participant ambiguity increase the set of types that can trade efficiently ex post while retaining ex ante budget balance. Therefore, regardless of whether efficiency or ex ante budget balance is of greater concern, increasing agent ambiguity leads to unequivocally improved outcomes in bilateral trade mechanisms that use an optimal transfer scheme.

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## Appendix A. Saddle Point Conditions

Consider the problem facing a mechanism participant of type  $\theta_i$

$$U(\theta_i) = \max_{\hat{\theta}_i \in \Theta_i} \min_{\pi \in \Sigma, \lambda \in [0, c]} E_{\hat{\theta}_i}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})] + \lambda g(\pi, \theta_i) \quad (\text{P2})$$

It is not obvious that the solution to (P2) forms a saddle point, but this is required to apply the Milgrom-Segal Envelope Theorem to our problem.

We prove that the solution to the following problem is a saddle point.

$$\max_{\sigma \in \Sigma} \min_{\pi \in \Delta(\theta_i)} \int E_{\hat{\theta}_i}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})] * \sigma(d\hat{\theta}_i) \quad (*)$$

recollecting that  $\Sigma$  represents the space of all possible measures over  $\Theta$ . Problem (\*) is equivalent to (P2) when the agent's strategy space is extended to mixtures over  $\Theta$ . If a pure strategy equilibrium of (\*) exists and is a saddle point, then the optimizers are obviously a saddle point equilibrium of (P2) given the restricted strategy space of (P2).

**Lemma 2.** *Assume condition (A2), (A5) and (A7) hold. Then the solution to (\*) exists and forms a saddle point.*

**Proof.** Problem (\*) can be thought of as a zero-sum game between the agent (choosing  $\hat{\theta}_i \in \Sigma$ ) and nature (choosing  $\pi \in \Delta(\theta_i)$ ). From the continuity

and convexity<sup>25</sup> of the objective of (\*) and the compactness of  $\Delta(\Theta)$  and  $\Theta$ , the theorem of the maximum implies that the optimizers  $\widehat{\theta}(\pi)$  and  $\pi(\widehat{\theta})$  exist and are convex-valued and upper-hemicontinuous in their respective arguments. Then there exists a fixed point to our system of optimality problems (Aliprantis and Border 2006, Corollary 17.55). This in turn implies that the solution to (\*) is a saddle-point. ■

There are choices of economic primitives that render it straightforward to find the minimizing measure  $\pi(\theta_i)$ .

**Lemma 3.** *Suppose for some  $\theta_i \in \Theta_i$  there exists  $\bar{\pi} \in \Delta(\theta_i)$  such that  $\bar{\pi}$  first order stochastically dominates all  $\pi \in \Delta(\theta_i)$ . In addition, suppose  $v(x_i(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i})$  is decreasing in  $\theta_{-i}$ . Then  $\bar{\pi} \in \pi(\theta_i)$ .*

**Proof.** Since  $\bar{\pi}$  first order stochastically dominates all  $\pi \in \Delta(\theta_i)$ , this implies that the  $(N - 1)$ -fold product measure  $(\bar{\pi} \times \bar{\pi} \times \dots \times \bar{\pi})(\circ)$  dominates (in the strong stochastic order) the  $(N - 1)$ -fold product measure  $(\pi \times \pi \times \dots \times \pi)(\circ)$  for any choice  $\pi \in \Delta(\theta_i)$ . Since  $v(x_i(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i})$  is decreasing in  $\theta_{-i}$ , we then have that

$$E_{\theta_{-i}}^{\bar{\pi}} [v(x_i(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i})] \leq E_{\theta_{-i}}^{\pi} [v(x_i(\theta_i, \theta_{-i}), \theta_i) - p_i(\theta_i, \theta_{-i})]$$

■

We now provide an alternative proof of the saddle point result without requiring (A8)

**Theorem 11.** *Suppose that for each  $\theta_i \in \Theta_i$  there exists  $\bar{\pi}(\theta_i) \in \Delta(\theta_i)$  such that  $\bar{\pi}$  first order stochastically dominates all  $\pi \in \Delta(\theta_i)$ . In addition, suppose  $v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})$  is decreasing in  $\theta_{-i}$ . Then the solution to (P1) exists and forms a saddle point if there is a maximizer of*

$$\max_{\widehat{\theta}_i \in \Theta_i} E_{\theta_{-i}}^{\bar{\pi}(\theta)} [v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})]$$

**Proof.** Since  $v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})$  is decreasing in  $\theta_{-i}$  for any type declaration  $\widehat{\theta}_i$ , we know from the lemma above that we can take  $\bar{\pi}(\theta_i)$  as the minimal measure regardless of the agent's choice of declaration. The agent's choice of type declaration then reduces to

$$\max_{\widehat{\theta}_i \in \Theta_i} E_{\theta_{-i}}^{\bar{\pi}(\theta)} [v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})]$$

which obviously has a solution from the compactness of the type space.

Further, since the minimal measure is independent of the agent's type declaration, the order of choice (measure followed by type declaration or vice versa)

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<sup>25</sup>The convexity of equation (\*) is due to the fact we have extended the agent's action space to mixtures over possible type declarations.

is irrelevant and we have

$$\begin{aligned} \max_{\widehat{\theta}_i \in \Theta_i} \min_{\pi \in \Delta(\theta_i)} E_{\widehat{\theta}_{-i}}^\pi [v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})] = \\ \min_{\pi \in \Delta(\theta_i)} \max_{\widehat{\theta}_i \in \Theta_i} E_{\widehat{\theta}_{-i}}^\pi [v(x_i(\widehat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \theta_{-i})] \end{aligned}$$

Therefore the solution to (P1) exists and forms a saddle point. ■

## Appendix B. Proofs

### Appendix B.1. Proofs of Section 3

**Theorem 1 1.** (Payoff Equivalence) Assume (A2), (A5), (A7), and (A8). Then the utility function,  $U(\theta_i)$ , is differentiable almost everywhere with a derivative equal to

$$\frac{\partial}{\partial \theta_i} U(\theta_i) = U'(\theta_i) = E_{\theta_{-i}}^{\pi(\theta_i)} v_\theta^i(x_i(\theta_i, \theta_{-i}); \theta_i) + \lambda(\theta_i) * g_\theta(\pi(\theta_i), \theta_i) \quad (\text{ICFOC})$$

for some multiplier  $\lambda : \Theta_i \rightarrow \mathbb{R}_+$

**Proof.** We must show that the solution to (P2) is a saddle point, which follows from Lemma 2 once we note that a pure strategy saddle-point solution to (\*) is a saddle-point solution to (P2). Note that we have assumed that  $v_\theta(x_i; \circ)$  is continuous for fixed  $x_i$ . Therefore,  $E_{\widehat{\theta}_{-i}}^\pi v_\theta(x_i(\widehat{\theta}_i, \widehat{\theta}_{-i}), \circ)$  is continuous and integrable (assumption A2). This implies that  $E_{\widehat{\theta}_{-i}}^\pi [v(x_i(\widehat{\theta}_i, \widehat{\theta}_{-i}), \theta_i) - p_i(\widehat{\theta}_i, \widehat{\theta}_{-i})]$  is absolutely continuous in  $\theta_i$  for fixed  $(\pi, \widehat{\theta}_i)$ . Noting that  $g_\theta(\pi, \circ)$  is integrable (Assumption A7) and that  $\lambda \in [0, c]$  is compact implies that

$$\{E^\pi v(x_i(\widehat{\theta}_i, \theta_{-i}), \circ) - p_i(\widehat{\theta}_i, \widehat{\theta}_{-i}) + \lambda g_\theta(\pi, \circ)\}_{(\widehat{\theta}_i, \pi, \lambda)}$$

is equidifferentiable and absolutely continuous. Noting that  $v_\theta$  and  $g_\theta$  are bounded (assumptions A2 and A7) in  $\Theta_i$  for fixed  $\widehat{\theta}_i \in \Theta_i$  and  $\pi \in \Delta(\theta_i)$  and that  $\lambda \in [0, c]$  is compact implies that there exists an integrable function  $b(\widehat{\theta}_i)$  such that for all  $\widehat{\theta} \in \Theta$  and  $\pi$  we have

$$b(\widehat{\theta}_i) > |E^\pi v_\theta(x_i(\widehat{\theta}_i, \widehat{\theta}_{-i}), \circ) + \lambda g_\theta(\pi, \theta_i)|$$

The last form of continuity required is our assumption (assumption A8) that

$$E_{\widehat{\theta}_{-i}}^\pi [v_\theta(x_i(\widehat{\theta}_i, \theta_{-i}); \theta_i) + \lambda g_\theta(\pi, \theta_i)]$$

is continuous in  $\widehat{\theta}_i$  and  $\pi$  for fixed  $\theta_i$  (assumption A2). Therefore, the conditions of Theorem 4 of Milgrom and Segal are satisfied.

Theorem 4 of Milgrom and Segal (2002) implies that

$$U(\theta_i) = U(\theta) + \int_{\theta}^{\theta_i} [E_{\widehat{\theta}_{-i}}^\pi v_\theta(x_i(\widehat{\theta}_i, \theta_{-i}), \widehat{\theta}_i) + \lambda(\widehat{\theta}_i) g_\theta(\pi, \widehat{\theta}_i)] d\widehat{\theta}_i$$

In addition, Theorem 4 yields that  $U(\theta_i)$  is absolutely continuous and hence differentiable almost everywhere. Therefore, where the derivative exists it is equal to

$$U'(\theta_i) = E_{\theta_{-i}}^{\pi(\theta_i)} v_{\theta}^i(x_i(\theta_i, \theta_{-i}); \theta_i) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i)$$

■

**Theorem 2 1.** *Assume assumptions (A1), (A3), and (A4) hold. Given (IC-FOC) and (M) hold, the mechanism is truthful and globally incentive compatible.*

**Proof.** Note that the agent's problem in the mechanism can be rewritten in terms of choosing the optimal type for his announcement. Let

$$\tilde{U}(\hat{\theta}_i; \theta_i) = \min_{\pi \in \Delta} E_{\theta_{-i}}^{\pi} [v(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i) - p_i(\hat{\theta}_i, \theta_{-i})]$$

and consider the problem  $\max_{\theta \in \Theta} \Phi(\hat{\theta}_i, \tilde{\theta}_i; \theta_i)$  where

$$\Phi(\hat{\theta}_i, \tilde{\theta}_i; \theta_i) = \tilde{U}(\hat{\theta}_i; \theta_i) - \tilde{U}(\tilde{\theta}_i; \theta_i)$$

Using Theorem 1 of Milgrom and Segal (2002) we find the following first order conditions for  $\tilde{\theta}_i = \theta_i$

$$\begin{aligned} \Phi_{\theta_i}(\hat{\theta}_i, \theta_i; \theta_i) &= \frac{\partial}{\partial \theta_i} \tilde{U}(\hat{\theta}_i; \theta_i) - \frac{\partial}{\partial \theta_i} \tilde{U}(\theta_i; \theta_i) \\ &= \{E_{\theta_{-i}}^{\tilde{\pi}} v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i)\} - \{E_{\theta_{-i}}^{\pi} v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} \end{aligned}$$

Note that the distribution in each term is different. The first term is the result of a selection of  $\pi \in \Delta(\theta_i)$  when the outcome function is  $x_i(\hat{\theta}_i, \theta_{-i})$ , whereas the second term has outcome function  $x_i(\theta_i, \theta_{-i})$ .

For  $\theta_i \geq \hat{\theta}_i$ , we have

$$\begin{aligned} E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i)\} - \{E_{\theta_{-i}}^{\pi} v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} &\leq \\ E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} & \end{aligned}$$

Given that  $v(x_i; \theta_i)$  is supermodular in  $\theta_i$  and  $x_i(\theta_i, \theta_{-i})$  is monotone increasing in  $\theta_i$ , we have  $v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i) \leq 0$ . Therefore

$$E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} \leq 0$$

and hence  $\Phi_{\theta_i}(\hat{\theta}_i, \theta_i; \theta_i) \leq 0$  if  $\theta_i \geq \hat{\theta}_i$ .

Now consider the  $\theta_i \leq \hat{\theta}_i$  case. Then we have

$$\begin{aligned} E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i)\} - \{E_{\theta_{-i}}^{\pi} v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} &\geq \\ E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} & \end{aligned}$$

Again, noting that  $x_i(\hat{\theta}_i, \theta_{-i}) \geq x_i(\theta_i, \theta_{-i})$  we have  $v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i) \geq 0$ . Hence

$$E_{\theta_{-i}}^{\tilde{\pi}} \{v_{\theta}(x_i(\hat{\theta}_i, \theta_{-i}); \theta_i) - v_{\theta}(x_i(\theta_i, \theta_{-i}); \theta_i)\} \geq 0$$

Therefore  $\Phi_{\theta_i}(\hat{\theta}_i, \theta_i; \theta_i) \geq 0$  if  $\theta_i \leq \hat{\theta}_i$ . This implies that  $\Phi(\hat{\theta}_i, \theta_i; \theta_i)$  is maximized at  $\theta_i = \hat{\theta}_i$  and the mechanism is truthful. ■

Appendix B.2. Proofs of Section 4.1

**Theorem 3 1.** *Given a mechanism  $(\Theta, x, p)$  such that the (M) and (ICFOC) hold, the full-insurance mechanism  $(\Theta, x, p^{INS})$  is globally incentive compatible and generates weakly greater revenues.*

**Proof.** Let  $U(\theta_i)$  be the interim utility from  $(\Theta, x, p)$ . The insurance auction payment scheme for an agent of type  $\theta_i$  is  $p_i^{INS}(\theta_i, \theta_{-i}) = x_i(\theta)v(\theta_i) - U(\theta_i)$ , so the agent utility obviously meets the (ICFOC). Since the outcome function meets (M) by assumption, the mechanism  $(\Theta, x, p^{INS})$  is globally incentive compatible according to Theorem 2.

Since  $x_i(\theta)v(\theta_i) - p_i^{INS}(\theta)$  is constant across realizations of  $\theta_{-i}$ , we have that any  $\pi \in \Delta(\theta_i)$  is minimal under the full-insurance mechanism. Therefore

$$\begin{aligned} U(\theta_i) &= \min_{\pi \in \Delta(\theta_i)} E_{\theta_{-i}}^{\pi} [x_i(\theta)v(\theta_i) - p_i^{INS}(\theta)] \\ &= E_{\theta_{-i}}^{\pi^*} [x_i(\theta)v(\theta_i) - p_i^{INS}(\theta)] \\ &= v(\theta_i) * E_{\theta_{-i}}^{\pi^*} [x_i(\theta)] - E_{\theta_{-i}}^{\pi^*} p_i^{INS}(\theta) \end{aligned}$$

Therefore  $E_{\theta_{-i}}^{\pi^*} p_i^{INS}(\theta) = F^*(\theta_i)v(\theta_i) - U(\theta_i)$ . To see the optimality of the insurance auction transfer scheme, note that

$$\begin{aligned} U(\theta_i) &= \min_{\pi \in \Delta(\theta_i)} E_{\theta_{-i}}^{\pi} [x_i(\theta)v(\theta_i) - p_i(\theta)] \\ &\leq v(\theta_i) * E_{\theta_{-i}}^{\pi^*} x_i(\theta) - E_{\theta_{-i}}^{\pi^*} p_i(\theta) \\ &= F^*(\theta_i)v(\theta_i) - E_{\theta_{-i}}^{\pi^*} p_i(\theta) \end{aligned}$$

Therefore, we have <sup>26</sup>

$$E_{\theta_{-i}}^{\pi^*} p(\theta) \leq F^*(\theta_i)v(\theta_i) - U_i(\theta_i) = E_{\theta_{-i}}^{\pi^*} p_{INS}(\theta)$$

■

**Corollary 1 1.** *Suppose  $\tilde{\Delta}(\theta) \subseteq \Delta(\theta)$ . Then the expected revenues from the insurance transfer scheme are higher under  $\tilde{\Delta}(\theta)$ .*

**Proof.** Note that for any  $(\Theta, x, p)$

$$\begin{aligned} U^{\tilde{\Delta}(\theta_i)}(\theta_i) &= \min_{\pi \in \tilde{\Delta}(\theta_i)} E_{\theta_{-i}}^{\pi} [x_i(\theta)v(\theta_i) - p_i(\theta)] \\ &\geq \min_{\pi \in \Delta(\theta_i)} E_{\theta_{-i}}^{\pi} [x_i(\theta)v(\theta_i) - p_i(\theta)] = U^{\Delta(\theta_i)}(\theta_i) \end{aligned}$$

<sup>26</sup>The expectations are taken over  $\theta_{-i}$  for a given value of  $\theta_i$ , so we have shown the stronger result that for each value of  $\theta_i$  expected revenues from an agent of type  $\theta_i$  are higher under the full-insurance mechanism than under any other transfer scheme. Taking an expectation over  $\theta_i$  leads to the conclusion that  $p^{INS}$  is optimal over the set of all incentive compatible payment schemes.

Let  $p_{\Delta(\theta_i)}^{INS}$  and  $p_{\tilde{\Delta}(\theta_i)}^{INS}$  denote the insurance transfer for agent  $i$  under the ambiguous beliefs  $\Delta(\theta_i)$  and  $\tilde{\Delta}(\theta_i)$  respectively. From the definition of the insurance transfer scheme

$$\begin{aligned} E_{\theta_{-i}}^{\pi^*} p_{\tilde{\Delta}(\theta_i)}^{INS}(\theta) &= F^*(\theta_i) v(\theta_i) - U_i^{\tilde{\Delta}(\theta_i)}(\theta_i) \\ &\leq F^*(\theta_i) v(\theta_i) - U_i^{\Delta(\theta_i)}(\theta_i) = E_{\theta_{-i}}^{\pi^*} p_{\Delta(\theta_i)}^{INS}(\theta) \end{aligned}$$

As argued above, taking an expectation over  $\theta_i$  on each side shows that the revenues are greater under  $\Delta(\theta_i)$ . ■

**Theorem 4 1.** *The revenue maximizing auction uses the insurance transfer scheme and allocation function associated with the mechanism  $(\Theta, x, p)$  that maximizes*

$$\sum_{i=1}^N \int_{\underline{\theta}}^{\bar{\theta}} \left( v(\theta_i) - [1 - F^*(\theta_i)] * \frac{v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i)}{E_{\theta_{-i}}^{\pi^*} [x_i(\theta)]} \right) * E_{\theta_{-i}}^{\pi^*} [x_i(\theta)] * d\theta_i$$

subject to the constraints

1.  $\sum_{i=1}^N x_i(\theta) \in [0, 1]$
2.  $x_i(\theta)$  is increasing in  $\theta_i$
3.  $\pi(\theta_i) \in \arg \min_{\pi \in \Delta(\theta_i)} v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i)$

**Proof.** Our proof proceeds in two steps. First we identify the interim incentive compatible mechanism  $(\Theta, x, p)$  that minimizes the interim utility. Second, we use insurance transfers to maximize revenues from that allocation function.

We first perform some elementary manipulations of the designer's problem to clarify our analysis. Using the definition of  $p_i^{INS}(\theta)$

$$\sum_{i=1}^N \int_{\Theta^N} p_i^{INS}(\theta) * F^*(d\theta) = \sum_{i=1}^N \int_{\Theta^N} [v(\theta_i) * x_i(\theta) - U(\theta_i)] * F^*(d\theta) \quad (\text{B.1})$$

Noting that the agent type distributions are ex ante identical and independent yields:

$$\int_{\Theta^N} [x_i(\theta) v(\theta_i) - U(\theta_i)] * F^*(d\theta) = \int_{\underline{\theta}}^{\bar{\theta}} [v(\theta_i) * E_{\theta_{-i}}^{\pi^*} [x_i(\theta)] - U(\theta_i)] * F^*(d\theta_i)$$

Using Theorem 1

$$\begin{aligned} \int_{\underline{\theta}}^{\bar{\theta}} U(s) * F^*(ds) &= \int_{\underline{\theta}}^{\bar{\theta}} \left[ \int_{\underline{\theta}}^s \left( v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i) \right) d\theta_i \right] * F^*(ds) \\ &= U(\bar{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} F^*(\theta_i) \left( v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i) \right) d\theta_i \\ &= \int_{\underline{\theta}}^{\bar{\theta}} [1 - F^*(\theta_i)] \left( v_{\theta}(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) + \lambda(\theta_i) * g_{\theta}(\pi(\theta_i), \theta_i) \right) d\theta_i \end{aligned}$$

Note that we are implicitly setting  $U(\underline{\theta}) = 0$  since this type never wins the auction. For notational purposes, denote the virtual valuation as

$$V(\theta_i) = v(\theta_i) - [1 - F^*(\theta_i)] * \frac{v_\theta(\theta_i) * E_{\theta_{-i}}^{\pi(\theta_i)}[x_i(\theta)] + \lambda(\theta_i) * g_\theta(\pi(\theta_i), \theta_i)}{E_{\theta_{-i}}^{\pi^*}[x_i(\theta)]}$$

Recollect that under  $p_{INS}$  any  $\pi \in \Delta(\theta_i)$  is minimal, so we can maximize profit by using the minimal measure that minimizes the virtual valuation of the agents, which is the measure described by condition (3). Inserting our formula for  $\int_{\underline{\theta}}^{\bar{\theta}} U(\theta_i) * F^*(d\theta)$  into B.1 and reordering yields

$$\sum_{i=1}^N \int_{\Theta^N} p_i^{INS}(\theta) * F^*(d\theta) = \sum_{i=1}^N \int_{\underline{\theta}}^{\bar{\theta}} V(\theta_i) * E_{\theta_{-i}}^{\pi^*}[x_i(\theta)] * d\theta_i$$

Substituting this as the objective of the seller's mechanism design problem yields the objective function in our Theorem.

We prove that our interim utility minimizing mechanism exists by appealing to Theorem 2 and noting that (M) is satisfied by assumption (2) and (ICFOC) is satisfied from our use of Theorem 1 to define  $U'(\theta_i)$ .

Now that we have identified the interim expected utility minimizing incentive compatible allocation function  $x$ , we generate the revenue maximizing auction by combining our result on the optimality of  $p^{INS}$  to yield the fully optimal allocation mechanism  $(\Theta, x, p^{INS})$ .<sup>27</sup> ■

**Theorem 5 1.** *Assume that  $\Delta(\theta_i) = \Delta$ . The revenue maximizing auction sets  $x : \Theta^N \rightarrow [0, 1]^N$  to maximize*

$$\sum_{i=1}^N \int_{\underline{\theta}}^{\bar{\theta}} \left( v(\theta_i) - [1 - F^*(\theta_i)] * v_\theta(\theta_i) * \frac{E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta)}{E_{\theta_{-i}}^{\pi^*}[x_i(\theta)]} \right) * E_{\theta_{-i}}^{\pi^*}[x_i(\theta)] * d\theta_i$$

subject to the constraints

1.  $\sum_{i=1}^N x_i(\theta) \in [0, 1]$
2.  $x_i(\theta)$  is increasing in  $\theta_i$
- 3'  $\pi(\theta_i) \in \arg \min_{\pi \in \Delta} E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta)$

**Proof.** The objective function and constraints (1) and (2) are identical to those in Theorem 4 once we note that  $g_\theta = 0$ . The optimal auction then attains the minimum of  $E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta)$  consistent with  $(\Theta, x, p)$  satisfying (M) and (ICFOC), which is the minimal measure in both the first price or all pay auctions. ■

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<sup>27</sup>We can explicitly compute the interim utilities (and hence the insurance transfer payments) using our payoff equivalence theorem and our choice of minimal measure from condition (3).

**Corollary 2 1.** *Assume the ambiguity averse virtual valuation,  $V(\theta)$ , is increasing under the type independent sets of priors  $\Delta \subset \tilde{\Delta}$ . Then the optimal reservation value is lower for  $\tilde{\Delta}$*

**Proof.** The formula (\*) above allocates the item to the bidder with the highest ambiguity averse virtual valuation conditional on any bidder having a non-zero virtual valuation. The monotonicity assures us that there exists a threshold type  $\theta_0$  such that:

$$v(\theta_0) - [1 - F^*(\theta_0)] * v_\theta(\theta_0) * \frac{E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta_0, \theta_{-i})}{E_{\theta_{-i}}^{\pi^*} [x_i(\theta_0, \theta_{-i})]} = 0$$

Let  $r^\Delta = \theta_0$  denote the reservation value for ambiguous beliefs  $\Delta$  where

$$x_i(\theta) = \begin{cases} 1 & \text{if } \theta_i = \theta^{(1)} \\ 0 & \text{otherwise} \end{cases}$$

Denote the respective minimal measure correspondences  $\pi^\Delta(\theta)$  and  $\pi^{\tilde{\Delta}}(\theta)$  where

$$\begin{aligned} \pi^\Delta(\theta) &\in \arg \min_{\pi \in \Delta} E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) \\ \pi^{\tilde{\Delta}}(\theta) &\in \arg \min_{\pi \in \tilde{\Delta}} E_{\theta_{-i}}^{\pi(\theta_i)} x_i(\theta) \end{aligned}$$

Then we have that  $E_{\theta_{-i}}^{\pi^\Delta(\theta_i)} [x_i(\theta_0, \theta_{-i})] \geq E_{\theta_{-i}}^{\pi^{\tilde{\Delta}}(\theta_i)} [x_i(\theta_0, \theta_{-i})]$ . But this implies that

$$\begin{aligned} v(r^\Delta) - [1 - F^*(r^\Delta)] * v_\theta(r^\Delta) * \frac{E_{\theta_{-i}}^{\pi(r^\Delta)} [x_i(r^\Delta, \theta_{-i})]}{E_{\theta_{-i}}^{\pi^*} [x_i(r^\Delta, \theta_{-i})]} &= 0 \\ &\leq v(r^\Delta) - [1 - F^*(r^\Delta)] * v_\theta(r^\Delta) * \frac{E_{\theta_{-i}}^{\pi^{\tilde{\Delta}}(r^\Delta)} [x_i(r^\Delta, \theta_{-i})]}{E_{\theta_{-i}}^{\pi^*} [x_i(r^\Delta, \theta_{-i})]} \end{aligned}$$

Since the virtual valuation is monotone, the value  $r^{\tilde{\Delta}}$  that solves

$$v(r^{\tilde{\Delta}}) - [1 - F^*(r^{\tilde{\Delta}})] * v_\theta(r^{\tilde{\Delta}}) * \frac{E_{\theta_{-i}}^{\pi^{\tilde{\Delta}}(r^{\tilde{\Delta}})} [x_i(r^{\tilde{\Delta}}, \theta_{-i})]}{E_{\theta_{-i}}^{\pi^*} [x_i(r^{\tilde{\Delta}}, \theta_{-i})]} = 0$$

entails  $r^{\tilde{\Delta}} \leq r^\Delta$ . ■