

Approximation of Large Games with Applications to Uniform Price Auctions

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Abstract

We prove that the equilibria of a large interdependent values uniform price auction model where bidders have arbitrary preferences for multiple units can be approximated by a nonatomic exchange economy. We show that the uniform price auction is approximately efficient with a large number of participants and asymptotically aggregates idiosyncratic bidder information into the market price. More generally our analysis framework provides conditions justifying the use of nonatomic limit model approximations to analyze the large market behavior of otherwise intractable game-theoretic models. We demonstrate continuity requirements on the economic primitives sufficient for the equilibrium strategies of the two models to converge as the number of participants in the large finite game approaches infinity.

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1 Introduction

In a uniform price auction, bidders declare demand schedules to an auctioneer who mechanically aggregates the bids to calculate the aggregate demand curve. The auctioneer finds a market clearing price that equates supply and aggregate demand and allocates the units according to the bidders' stated demand curves at the uniform market clearing price. It is well known that uniform price auctions wherein agents have preferences for multiple units of a homogenous good are not generically efficient even in the private values case since agents have an incentive to withhold their demand for marginal units in order to drive down the price paid for the inframarginal units received (Back and Zender [8], Ausubel and Cramton [7]). Despite these disappointing theoretical properties, uniform price auctions have been widely implemented in both dynamic and static forms to allocate goods such as United States Treasury Bills, diamonds, corporate securities, and electricity provision rights.

We refer to a uniform price auction with a large, but finite, number of bidders and units for sale as a large finite (or simply a large) uniform price auction. The behavior of large uniform price auctions wherein agents demand multiple units and view successive units as complements or agent valuations have a common value component remains intractable when approached with game-theoretic techniques. In the context of securities auctions, bidders typically have demand for multiple units and the good has a common value component based on the stream of future returns that the security provides as well as the resale value. However, uniform price auctions with interdependent values have only been studied in the case of single unit demands (Reny and Perry [39], Pesendorfer and Swinkels [36]). Bidders in electricity procurement auctions generally desire to supply multiple units of power, and technologies with high fixed costs such as nuclear power stations induce complementary valuations for the right to provide successive units. The equilibria of large uniform price auctions with multi-unit demands have been studied only in the case where agents have declining private marginal valuations for successive units (for example, Swinkels [42]).

Our first contribution is to provide a limit game featuring a continuum of measure

0 agents intended to approximate the large uniform price auction. Intuition suggests (and we formally prove) that if bidders believe that their ability to influence the market clearing price is negligible, then the uniform price auction outcomes should resemble the predictions of an analytically tractable rational expectations equilibrium of an exchange economy. The epsilon equilibria of the nonatomic model reveal information about the common value through price and result in approximately efficient outcomes.

Our second contribution is to relate the large finite and nonatomic limit models through a series of approximation theorems that can be employed to analyze general large finite games. The intuition underlying our approximation results is that if an agent's utility is continuous in the aggregate distribution of the agents' actions,¹ then a single agent can treat economic aggregates as approximately exogenous to his own choice. If this intuition is valid, then game-theoretic equilibria converge to the equilibria of an analogous nonatomic game as the number of agents increases. However it has not been obvious under what conditions this intuition is valid in the context of complex market models such as large auctions.

Our approximation theorems describe formal conditions under which nonatomic models yield economic predictions that uniformly² approximate those of analogous game-theoretic models with a large finite set of agents.³ Our more important contribution is to emphasize the usage of these results to study the equilibria of otherwise intractable large games (e.g. our uniform price auction application) and state the conditions of these theorems in a fashion that renders them easily applicable to large market games of interest to mechanism designers, industrial organization economists, and applied theorists.

¹Given a set of agent actions $\{x_1, \dots, x_N\}$, the realized aggregate distribution of agent actions is an empirical cumulative distribution function defined as $F(x) = \sum_{i=1}^N 1\{x_i \leq x\}$ where $1\{x_i \leq x\}$ is an indicator function for the event $\{x_i \leq x\}$.

²Uniformly with respect to the type space of the agents. For example, Theorem 1 provides a uniform approximation of the equilibrium strategy set.

³Our framework incorporates a wide range of idiosyncratic and aggregate shock structures, making it particularly well-suited for studying markets with interdependent values. As discussed in Section 5, a lengthy literature exists in which results of a similar nature have been proven for a number of specialized cases of our setting.

Our first approximation theorem proves that the set of equilibrium strategies is upper hemicontinuous in the number of agents participating in the game. Therefore, for sufficiently large finite games, the equilibrium predictions of the large finite economy are approximately equal to the predictions of (a subset of) the equilibria of the nonatomic model. Our second and third theorems show that for any positive epsilon the equilibria of the nonatomic model are epsilon Bayesian-Nash equilibria of sufficiently large finite games. Our fourth theorem proves that for any positive epsilon the exact Bayesian-Nash equilibria of a sufficiently large finite economy are epsilon equilibria of the analogous nonatomic limit economy. The fourth theorem implies that the exact equilibria of the large finite uniform price auction must be approximately efficient and approximately reveal the state through price since the epsilon equilibria of the nonatomic limit model have these properties.

1.1 Outline of the Paper

Section 2 defines our model and the nonatomic limit games we use to approximate large finite games. Section 3 states and discusses the theorems of our analysis framework. Section 4 uses our theorems to analyze the equilibria of interdependent values uniform price auctions with a large number of agents who are allowed to have arbitrary preferences for multiple units of a homogenous good. Section 5 discusses our paper in the context of the literature on large games. Section 6 concludes the study. All proofs have been relegated to the appendix.

2 Model

Our results focus on the analysis of a sequence of games with the N^{th} game, played by N agents, defined as $(\mathcal{A}, \Omega \times \Theta^N, P_N, u_N)$. $(\mathcal{A}, d_{\mathcal{A}})$ denotes a metric space of actions, which is assumed to be identical for all games in the sequence. The metric space (Ω, d_{Ω}) denotes the aggregate shocks in the economy, and (Θ, d_{Θ}) is a metric space of characteristics for a single agent such as payoff relevant characteristics or private information. Let $\mathcal{A}, \Theta \subset \mathbb{R}^d$ for $d < \infty$. We refer to $\Omega \times \Theta^N$ as the space

of states of the world of the N -agent game.

The space of states of the world for each N -agent game is embedded in the measure space (Ψ, \mathcal{F}, P) where $\Psi = \Omega \times \Theta^N$, \mathcal{F} is the family of Borel sets on Ψ , and P is a probability measure. The distribution over states of the world in the N -agent game, P_N , is the marginal of P on $\Omega \times \Theta^N$. We assume that the characteristics of each agent are, conditional on $\omega \in \Omega$, independent and identically distributed (i.i.d) according to $\pi_0^\Theta(\omega)$.⁴

Throughout this work, we reserve the notation $\Delta(S)$ to refer to the space of probability measures over S that is measurable with respect to the Borel sets on S , which we denote $\mathcal{B}(S)$. Let $\Delta_N(\Theta)$ be the space of empirical probability measures over Θ that can be generated by N samples from Θ . $\theta \in \Theta$ and $\pi^\Theta \in \Delta(\Theta)$ denote generic elements of these spaces. Let $\Delta_N(\mathcal{A})$ be the analogous space of probability measures over \mathcal{A} . We endow $\Delta(\Theta)$ and $\Delta(\mathcal{A})$ with the weak-* topology unless otherwise noted. We use the notation $\delta_x \in \Delta(\mathcal{X})$ to denote a degenerate probability measure on the point $x \in \mathcal{X}$.

The function $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$ denotes a symmetric, semi-anonymous ex post utility function of the N agent game.⁵ We interpret the event $(\theta, \omega, a, \pi^{\mathcal{A}})$ as an agent of type θ taking action a in aggregate state $\omega \in \Omega$ with the actions of the agents generating empirical measure $\pi^{\mathcal{A}}$. By allowing the preferences to depend on ω , the agents' utilities may include a common value component. We assume that

⁴We can allow the agent characteristics to be independently but not identically drawn from a finite set of distributions $\{\pi_i^\Theta(\omega)\}_{i=1}^m$ over spaces $\{\Theta_i\}_{i=1}^m$ with the joint space defined as $\Theta = \times_{i=1}^m \Theta_i$. If we denote the number of agents drawn from the i^{th} space as N_i , we require $\frac{N_i}{N} \rightarrow \beta_i > 0$ as $N \rightarrow \infty$. This is, from the perspective of the agents, a process where the agent characteristics are drawn in two stages: (1) randomly choose the space Θ_i and (2) draw the characteristics from the distribution associated with the chosen space. Note that step (1) could be done deterministically. For example, the analyst may be concerned with models wherein the number of buyers and sellers are equal. In this case, buyers and sellers are defined using distinct characteristic spaces and enter the economy in (buyer, seller) pairs with characteristics for each agent in the pair determined independently.

⁵We define semi-anonymous utility functions directly. We could follow Kalai [24] and define $u_N : \Theta \times \Omega \times \mathcal{A}^N \rightarrow \mathbb{R}$ and then apply a symmetry requirement to obtain semi-anonymity. While semi-anonymity is not without loss of generality, it allows for models incorporating different roles for the agents (e.g. buyers and sellers), the possibility of non-participation (e.g. voters and abstainers), and random participation. See Kalai [24] for examples useful for reformulating games into this framework.

the agent preferences can be represented in expected utility form when considering stochastic outcomes, Bayesian-Nash equilibria, or equilibria in mixed strategies.

For a measurable function $f : \Delta(\Theta) \times \Omega \rightarrow \mathcal{X}$, we use the notation $E^\Psi[f(\pi^\Theta, \omega)]$ to denote an expectation with respect to both $\pi^\Theta \in \Delta_N(\Theta)$, $N < \infty$, and $\omega \in \Omega$.⁶ The notation $E^\Omega[f(\pi_0^\Theta(\omega), \omega)]$ refers to an expectation over the space of aggregate shocks Ω .

Let $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ refer to the utility in a nonatomic limit game. The following assumption relates the utility function of a sequence of large finite games to the utility function of a nonatomic limit game employed as an approximation of the large-finite games. Note that the definition accommodates the different domains of u_N and u and puts no restriction on convergence for $\pi \in \Delta(\mathcal{A})$ that are not contained in $\Delta_N(\mathcal{A})$ for any $N < \infty$.

Assumption 1. *For all $\varepsilon > 0$, there exists N^* such that for all $N > N^*$, all $\pi_N^A \in \Delta_N(\mathcal{A})$ and all $(\theta, \omega, a) \in \Theta \times \Omega \times \mathcal{A}$*

$$|u_N(\theta, \omega, a, \pi_N^A) - u(\theta, \omega, a, \pi_N^A)| < \varepsilon$$

The strategy space, denoted Σ , is the set of measurable maps from Θ into $\Delta(\mathcal{A})$. When required, we analyze the metric space (Σ, d_Σ) where for $\sigma, \sigma' \in \Sigma$

$$d_\Sigma(\sigma, \sigma') = \sup_{\theta \in \Theta} |\sigma(\theta) - \sigma'(\theta)|$$

Each strategy, when combined with a distribution of types, $\pi^\Theta \in \Delta(\Theta)$, generates a distribution of actions ex post. We define this distribution of actions for all $A \in \mathcal{B}(\mathcal{A})$ as

$$\pi^A[A] = \int_{\Theta} \sigma(\theta)[A] * \pi^\Theta[d\theta]$$

where we let $\sigma(\theta)[A]$ denote the probability an agent of type θ following strategy σ selects an action in A . Let $\pi_0^A(\omega)$ be the action distribution induced by a strategy

⁶Note that the measure P_N defines a measure over the space $\Delta_N(\Theta)$ for all $N < \infty$.

σ in conjunction with the true type distribution.⁷ Formally, for all $A \in \mathcal{B}(\mathcal{A})$ we have

$$\pi_0^A(\omega)[A] = \int_{\Theta} \sigma(\theta)[A] * \pi_0^{\Theta}(\omega)[d\theta]$$

We do not use σ as a parameter for either π^A or $\pi_0^A(\omega)$ as the relevant strategy is clear from context.

3 Approximation Framework

The primary theorems of our framework analyze the relationship between the equilibria of the large finite game and the equilibria of the corresponding nonatomic limit game. In section 3.1 we state and discuss Theorem 1, which provides sufficient conditions for the equilibrium strategy set to be upper hemicontinuous in the number of players. Section 3.2 discusses Theorems 2, 3, and 4, which provide results relating the exact equilibria of the limit game to epsilon equilibrium concepts of the large finite game and vice versa. These latter theorems require weaker continuity conditions than Theorem 1, but do not imply strategic convergence as Theorem 1 does.

3.1 Strategic Convergence

We first define the epsilon equilibrium notions we use for large finite and nonatomic games. The notation $\text{supp}[\pi]$ refers to the support of a measure π .

Definition 1. *Given an N -agent game with utility function $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$, the strategy $\sigma^N : \Theta \rightarrow \Delta(\mathcal{A})$ is an **Ex Post ε -Nash Equilibrium** at*

⁷As the true type distribution, $\pi_0^{\Theta}(\omega)$, is a function of the state $\omega \in \Omega$, the induced distribution of actions, $\pi_0^A(\omega)$, is also a function of ω .

state $(\omega, \theta_1, \dots, \theta_N)$ if for all agents i , all $a' \in \mathcal{A}$ and all $a^* \in \text{supp}[\sigma^N(\theta_i)]$ we have⁸

$$u_N(\theta_i, \omega, a^*, \pi^{\mathcal{A}}(\varphi)) + \varepsilon \geq u_N(\theta_i, \omega, a', \pi^{\mathcal{A}}(\varphi) + \frac{1}{N}[\delta_{a'} - \delta_{a^*}])$$

where for all $A \in \mathcal{B}(\mathcal{A})$

$$\pi^{\mathcal{A}}(\varphi)[A] = \int_{\Theta} \sigma^N(\theta)[A] * \pi^{\Theta}(\varphi)[d\theta]$$

The ex post aspect of the equilibrium requires that agent actions be approximately optimal at the state $\varphi \in \Psi$ once all of the agent actions and the aggregate state $\omega \in \Omega$ are common knowledge. Note that a deviation of an agent from $a^* \in \text{supp}[\sigma^N(\theta_i)]$ to $a' \in \mathcal{A}$ changes the agents' utilities indirectly by altering the ex post empirical distribution of actions in the game, which is captured by the term $\frac{1}{N}[\delta_{a'} - \delta_{a^*}]$.

The following epsilon equilibrium notion is applied to our nonatomic games.

Definition 2. An ε -Bayesian-Nash equilibrium (ε -BNE) strategy $\sigma^\infty : \Theta \rightarrow \Delta(\mathcal{A})$ of the nonatomic game defined by utility function $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$ satisfies for all $a^* \in \text{supp}[\sigma^\infty(\theta)]$, $a' \in \mathcal{A}$, $\theta \in \Theta$

$$E^\Omega[u(\theta, \omega, a^*, \pi_0^{\mathcal{A}}(\omega))|\theta] + \varepsilon \geq E^\Omega[u(\theta, \omega, a', \pi_0^{\mathcal{A}}(\omega))|\theta]$$

where for all $A \in \mathcal{B}(\mathcal{A})$ ⁹

$$\pi_0^{\mathcal{A}}(\omega)[A] = \int_{\Theta} \sigma^\infty(\theta)[A] * \pi_0^{\Theta}(\omega)[d\theta]$$

Note that deviations from the strategy by a single agent do not affect the aggre-

⁸The dependence of $\pi^\Theta(\varphi)$ and $\pi^{\mathcal{A}}(\varphi)$ on $\varphi \in \Psi$ indicates that $\pi^\Theta(\varphi)$ and $\pi^{\mathcal{A}}(\varphi)$ are distributions associated with a particular state of the N -agent game, $\varphi = (\omega, \theta_1, \dots, \theta_N)$. This is in contrast to the type and action distributions ($\pi_0^\Theta(\omega)$ and $\pi_0^{\mathcal{A}}(\omega)$ respectively) induced by the true probability measure of types, which depend only on the realized aggregate shock.

⁹In this definition $\pi_0^{\mathcal{A}}(\omega)$ is generated by $\pi_0^\Theta(\omega)$ in combination with the ε -BNE strategy σ^∞ . We continue to use the notation $\pi_0^{\mathcal{A}}(\omega)$ for parsimony, but note that the earlier definition referred to an action distribution induced by $\pi_0^\Theta(\omega)$ and an arbitrary strategy.

gate distribution of actions $\pi_0^A(\omega)$. Therefore in the nonatomic game the choice of action by a single agent has no effect on other agents' outcomes. Also note that the agents' expectations account for the fact that the distribution of agent actions, $\pi_0^A(\omega)$, can reflect information about the state of the economy ω . As we will see in our application to interdependent values uniform price auctions, the agents use knowledge of the equilibrium to make inferences incorporating information conveyed by market price.

Let the Bayesian-Nash equilibrium correspondence be denoted $\mathcal{E} : \mathbb{N} \rightrightarrows \Sigma$, where $\mathcal{E}(N)$ is the set of Bayesian-Nash equilibrium strategies in the N agent game, and let $\mathcal{E}^\infty = \lim_{N \rightarrow \infty} \mathcal{E}(N)$.¹⁰ Denote the set of equilibrium strategies of the nonatomic limit game as \mathcal{E}^{NA} . Our first approximation theorem provides a continuity condition sufficient for the upper hemicontinuity of the equilibrium correspondence. This implies that the equilibria of the nonatomic limit model approximately circumscribe the admissible strategies of any equilibrium of the large finite game with sufficiently many agents.

Theorem 1. *Endow $\Delta(\mathcal{A})$ with the weak-* topology. The correspondence \mathcal{E} is upper hemicontinuous as $N \rightarrow \infty$ ($\mathcal{E}^\infty \subset \mathcal{E}^{NA}$) **if** the family $\{E^\Omega[u(\theta, \omega, \cdot, \cdot) | \theta]\}_{\theta \in \Theta}$ is uniformly equicontinuous in $\mathcal{A} \times \Delta(\mathcal{A})$ and uniformly bounded.*

Theorem 1 implies that any convergent sequence of exact equilibria of the large finite game converges to some equilibrium of the nonatomic model. Suppose such a convergent sequence did not have a limit point in \mathcal{E}^{NA} . This would imply that if the agents in the nonatomic game played the limit strategy, some positive measure of these agents would have a profitable deviation. Theorem 4 (section 3.2) suggests a contradiction, and our proof goes through the formal steps of showing this to be the case.

Throughout this work we take the existence of equilibria in the large finite and nonatomic game as given. If we combine the continuity assumption of Theorem 1 with an added assumption that \mathcal{A} is compact, equilibrium existence in the nonatomic

¹⁰ $\lim_{N \rightarrow \infty} \mathcal{E}(N)$ denotes the set of limits of convergent sequences $\{\sigma_N\}_{N=1}^\infty$ where $\sigma_N \in \mathcal{E}(N)$.

limit game is not problematic.¹¹ Proving existence of equilibria for the large finite game may be difficult in cases where the large finite game is discontinuous. We show equilibria of our large finite auction exist in Theorem 5.

The convergence rate is important for the practical application of our theorems.¹² In practice once the nonatomic equilibrium strategies have been computed, δ can be found for a particular N -agent game by calculating the optimal deviation for an agent given that all of the other agents follow the nonatomic limit game equilibrium strategy. Computing such an optimal deviation is (relatively) tractable since this is the result of a single person decision problem.

Ideally, we would like to prove that the equilibrium correspondence is also lower hemicontinuous as N goes to infinity. As the following example shows, lower hemicontinuity of $\mathcal{E}(N)$ does not hold in general.

Example 1. *Consider a game wherein each agent names a number $m \in [0, 1]$ and receives a transfer equal to the average of the agents' announcements. Assume agent utility functions are increasing in monetary payments. For any $N < \infty$ the unique equilibrium is for each agent to announce $m = 1$. However, any strategy is an equilibrium of the nonatomic game as the individual agents cannot affect the payment.*

In order to make predictions about the equilibrium strategies of the finite game using the nonatomic equilibria as an approximation, it is crucial that the set of equilibria of the nonatomic game be small relative to the size of the strategy space or that the equilibrium properties of interest (e.g. efficiency) be fixed across the set of nonatomic equilibria. In the example the set of nonatomic equilibria is equal to the entire strategy space, so no conclusions about the behavior of the agents in the exact equilibria of the finite game can be made.

¹¹Mas-Colell [28] provides a proof for the case without aggregate shocks. Bergin and Bernhardt [9] provide more extensive results for dynamic nonatomic games with aggregate shocks.

¹²Proving a rate of convergence requires placing conditions on the rate of change of u near the equilibrium outcome. Bonnans and Shapiro [11] provides references to the relevant techniques.

3.2 Epsilon Equilibria

The theorems of this section use continuity arguments, similar to those used in the proof of Theorem 1, to show that exact equilibria of the nonatomic limit game are epsilon equilibria of the large finite game (Theorems 2 and 3) and that exact equilibria of the large finite game are epsilon equilibria of the limit game (Theorem 4). Note that if we assume that u is Lipschitz continuous, then the convergence rate of ε in Theorems 2, 3, and 4 can be shown to be $O(\sqrt{N})$ using Corollary 3.

Our first result proves that equilibria of the nonatomic limit game are approximate equilibria of the large finite game. Our continuity conditions are weaker than those required of Theorem 1 in that we do not require continuity of u with respect to \mathcal{A} and employ a finer topology over $\Delta(\mathcal{A})$. Both of these weakenings are crucial to handle the discontinuities present in our uniform price auction application.

Theorem 2. *Fix $\varepsilon > 0$ and endow $\Delta(\mathcal{A})$ with the Kolmogorov topology.¹³ For each Bayesian-Nash equilibrium $\sigma^\infty : \Theta \rightarrow \Delta(\mathcal{A})$ of the nonatomic game $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$, there exists an N^* such that for all $N > N^*$, $\sigma^\infty(\circ)$ is an ε -Bayesian-Nash equilibrium of the N -agent game $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$ **if** the family $\{E^\Omega[u(\theta, \omega, a, \cdot) | \theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded and uniformly equicontinuous in a relatively open set of $\Delta(\mathcal{A})$ containing $\cup_{\omega \in \Omega} \{\pi_0^{\mathcal{A}}(\omega)\}$*

The proof relies on using continuity of the limit game, u , and Assumption 1 to show that the optimization problem facing an agent in a large finite game is close to the decision problem of an agent in the nonatomic limit game. Since the agent types represent independent and identically distributed draws from the distribution $\pi_0^\Theta(\omega)$, the empirical cumulative distribution function (CDF) of agent types in the large finite economy almost surely converges to $\pi_0^\Theta(\omega)$ in the Kolmogorov metric as $N \rightarrow \infty$. If the agents follow the nonatomic equilibrium strategy σ^∞ , then the distribution of actions is in a small neighborhood of $\pi_0^{\mathcal{A}}(\omega)$ with high probability for large N . Since the limit game utility function, u , is continuous over this set and u_N converges to u as $N \rightarrow \infty$, the problem facing the agents in the large finite game is

¹³For the definition of the Kolmogorov topology, please see Appendix A.1. For many purposes it suffices to note that the Kolmogorov topology is a refinement of the weak-* topology.

approximately the same as the one facing the agents in the nonatomic limit game equilibrium. Therefore, σ^∞ is an ε -equilibrium of the large finite game.

Theorem 3. *Assume $\Omega = \{\omega\}$ and fix $\varepsilon > 0, \rho \in (0, 1]$. Endow $\Delta(\mathcal{A})$ with the Kolmogorov topology. For each Nash equilibrium $\sigma^\infty : \Theta \rightarrow \Delta(\mathcal{A})$ of the nonatomic game $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$, there exists an N^* such that for all $N > N^*$, $\sigma^\infty(\circ)$ is an ex post ε -Nash equilibrium of the N -agent game $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$ with probability $1 - \rho$ **if** the family $\{u(\theta, \omega, a, \cdot)\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly equicontinuous in a relatively open set of $\Delta(\mathcal{A})$ containing $\pi_0^{\mathcal{A}}(\omega)$*

Our assumption that $\Omega = \{\omega\}$ rules out the case of aggregate shocks, and hence the nonatomic limit game is a complete information game. As $N \rightarrow \infty$ the individual agents in the large finite game can make accurate ex ante predictions about the distribution of characteristics of the agents participating in the game and the effect of their actions on ex post outcomes. Therefore, the outcome of the large finite game with incomplete information is approximated by the complete information nonatomic limit game as the number of agents approaches infinity.

Proving an analogous result for deviations by fixed finite coalitions of agents within the economy is straightforward.¹⁴

Corollary 1. *Consider any fixed finite set of K agents in the economy denoted with indices $\mathcal{I} = \{i_1, \dots, i_K\} \subset \mathbb{N}$. Given the conditions of Theorem 2 hold, for any $\varepsilon > 0$ there is an $N^* \geq i_K$ that that for all $N > N^*$, $i \in \mathcal{I}$ and $a^* \in \sigma^\infty(\theta_i)$ the following holds*

$$E^\Psi[u_N(\theta_i, \omega, a^*, \pi^{\mathcal{A}}(\varphi)) | \theta_i] + \varepsilon \geq \max_{a' = (a'_1, \dots, a'_K) \in \mathcal{A}^K} E^\Psi[u_N(\theta_i, \omega, a'_i, \pi^{\mathcal{A}}(\varphi) + \frac{1}{N} \sum_{i=1}^K [\delta_{a'_i} - \delta_{a^*}]) | \theta_i]$$

Intuitively any finite set of deviations has a small impact on the $\Delta_N(\mathcal{A})$ term for large N . This implies that finite groups of agents cannot collude to significantly alter the outcome of the game. Corollary 1 can be viewed as a coalition-proofness

¹⁴Formal proof available from the author.

equilibrium refinement. For the purposes of market design, collusion by a small group of agents within a large market is often feasible, and the analyst can draw comfort that under the conditions outlined one need not worry about impact of this form of deviation.

Theorem 2 shows that the set of nonatomic limit game equilibria is a subset of the ε -Bayesian-Nash equilibria of the large finite game in the limit as $N \rightarrow \infty$. We now provide a partial converse in the sense of proving that the exact Bayesian-Nash equilibria of the large finite game are ε -Bayesian-Nash equilibria of the nonatomic limit game.

Theorem 4. *Fix $\varepsilon > 0$ and endow $\Delta(\mathcal{A})$ with the Kolmogorov topology. There exists an N^* such that for all $N > N^*$ and any exact symmetric Bayesian-Nash equilibrium $\sigma^N : \Theta \rightarrow \Delta(\mathcal{A})$ of the N agent game, σ^N is an ε -Bayesian-Nash equilibrium of the nonatomic game **if** the family $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded and uniformly equicontinuous in a relatively open set of $\Delta(\mathcal{A})$ containing $\cup_{\omega \in \Omega} \{\pi_\infty^N(\omega)\}$ where for all $A \in \mathcal{B}(\mathcal{A})$ ¹⁵*

$$\pi_\infty^N(\omega)[A] = \int_{\Theta} \sigma^N(\theta)[A] * \pi_0^\Theta(\omega)[d\theta]$$

The principle difficulty in proving this theorem is to show that as $N \rightarrow \infty$, the optimization problem facing the agents in the large finite game is dominated by the high probability event that the distribution of actions is close to $\pi_\infty^N(\omega)$ in the Kolmogorov metric. Corollary 3 (appendix) demonstrates a uniform bound on the probability that the empirical CDF of actions is more than δ in distance from $\pi_\infty^N(\omega)$ in the Kolmogorov metric and that this probability approaches 0 as $N \rightarrow \infty$. Denote as event E states of the world where the empirical CDF is more than δ in distance from $\pi_\infty^N(\omega)$ in the Kolmogorov metric. Since the utility functions are bounded, the difference in utility between the best and worst action contingent on E occurring is bounded. For N sufficiently large, the probability of E can be made arbitrarily small. Therefore, for any choice of $\delta > 0$, the incentive for an agent to consider

¹⁵Due to the use of a sequence of strategies in this theorem, we use the notation $\pi_\infty^N(\omega)$ in this context to refer to the distribution of actions induced by $\pi_0^\Theta(\omega)$ and σ^N .

states of the world wherein the empirical and true measures of actions differ by more than δ vanishes as $N \rightarrow \infty$.

From the continuity of the nonatomic limit game utility function and the uniform pointwise convergence of u_N to u (assumption 1), if the empirical CDF of actions and $\pi_\infty^N(\omega)$ are sufficiently close (the event $\Psi \setminus E$) and N sufficiently large, then the agents' optimization problem in the large finite game is approximately the same as that facing the agents in the nonatomic game. This implies that for sufficiently large N , σ^N approximately solves the optimization problem for agents in the nonatomic game when all other nonatomic agents follow σ^N . Therefore, σ^N is an ε -Nash equilibrium of the nonatomic limit game.

4 Large Auctions with Complementary Preferences

Intuition suggests that a large uniform price auction may behave similarly to a general equilibrium model with the auction serving as a price discovery and information aggregation mechanism. We use our nonatomic approximation techniques to show that under general preference distributions, the intuitive relationship between large uniform price auctions and general equilibrium models is valid. Our work can be interpreted as providing a linkage between the nonstrategic models in the spirit of Aumann [6] and Hildenbrand [21] and the game-theoretic literature represented by Swinkels [42].

Prior analyses of multi-unit demand uniform price auctions have employed restrictions on agent preferences, such as insisting the agents are risk neutral and have decreasing private marginal valuations for the goods (Swinkels [42]), in order to apply game-theoretic analysis techniques and discover the equilibrium strategies and outcomes in the limit as $N \rightarrow \infty$. In the context of electricity auctions, consider an electricity generator with a large fixed and low marginal cost of production. The average cost for the generator is declining in the number of units produced, which means the profit function has the complements property with respect to the number of units provided. We provide a tractable analysis of uniform price auctions with complementary preferences, a crucial application of auction theory that has, to

this point, resisted solution. We are able to show that uniform price auctions are asymptotically efficient in the private values case even with complementary demand for successive units.

Our framework allows us to assume that the agent valuations admit a common value component. Interdependent agent valuations are crucial for the analysis of securities auctions, goods which have a common value component in terms of the resale price and the stream of payouts the holders of the security receive. We show that the multi-unit demand uniform price auction mechanism aggregates information into the market price in the limit as $N \rightarrow \infty$. Therefore, our analysis contributes to the literature on game theoretic foundations for rational expectations equilibria. Although the question had been previously answered for single and double auction models wherein the bidders have unit demand (Reny and Perry [39]), the extension to the multi-unit demand case remained an open problem. Finally, we show that the auction is asymptotically efficient.

4.1 Model

We consider a uniform price auction where the agents have preferences that are quasi-linear in price for up to $K < \infty$ homogeneous units. Agent types $\theta \in \Theta = [0, \bar{v}]^K$ represent signals about marginal valuation, and we assume that the types of the agents are conditionally i.i.d. given an aggregate random variable $\omega \in \Omega \subset \mathbb{R}$ where Ω is finite. Denote the probability mass function of ω as $g(\omega)$. We denote the conditional distribution of agent types as $\pi_0^\Theta(\omega)$ and assume $\pi_0^\Theta(\omega)$ has a continuous conditional probability density function $f(\theta|\omega)$. By allowing the distribution of agent types to depend on ω , we can model both the affiliated private values setting and utilities with a common value component.

Agent valuation of the goods received is determined by a bounded function¹⁶ $v : \Theta \times \Omega \times \{0, \dots, K\} \rightarrow \mathbb{R}_+$. The ex post utility of an agent given an allocation of

¹⁶We place an upper bound on valuations to rule out economies with infinite prices in equilibrium.

k units at a price of p per unit is then

$$v(\theta, \omega, k) - p * k$$

At the ex ante stage of the nonatomic limit game, the agents infer the value of ω from their own type and the equilibrium market price function $p : \Omega \rightarrow \Delta(\mathbb{R}_+)$. As we prove below, the nonatomic limit game equilibrium is a rational expectations equilibrium (REE) of an exchange economy.

In order to prove that the nonatomic limit game admits only fully revealing REE, we prove that the agent bids are monotone in type in the nonatomic limit game equilibrium. For this we require the following

Assumption 2. For all $k \in \{1, \dots, K\}$, $v(\theta, \omega, k)$ is continuous in θ , bounded, and strictly increasing in (θ, ω, k) for $\theta \in (0, \bar{v})^K$

Assumption 3. $v(\theta, \omega, k)$ has strictly increasing differences in (θ, ω, k)

Assumption 4. For all $k \in \{1, \dots, K\}$, if $h_k(\theta, \omega) = v(\theta, \omega, k) - v(\theta, \omega, k - 1)$, then $h_k(0, \omega) = 0$ and $h_k(\bar{v}, \bar{v}, \dots, \bar{v}, \omega) = \bar{v}$

Assumption 5. $f(\theta|\omega)$ is positive and continuous over Θ and strictly log supermodular in (θ, ω)

Assumption 4 insures that the equilibrium demand is not constant across all types for any equilibrium price. This assumption can be weakened at the cost of a significant increase in notational complexity.¹⁷ Lemma 4 proves that these properties are sufficient to ensure that equilibrium bids in the nonatomic game are strictly increasing in type. We prove that this monotonicity implies nonatomic limit game equilibria are fully revealing.

Given an agent's type θ , the true interim demand of the agent in the nonatomic limit game is

$$D_\theta(p) = \arg \max_{k \in \{0, 1, \dots, K\}} E^\Omega[v(\theta, \omega, k)|\theta, p] - p * k$$

¹⁷We require that for any equilibrium price p there exist types $\theta > \theta'$ such that $D_\theta(p) > D_{\theta'}(p)$. Assumption 4 is sufficient, but not necessary, for this to hold.

where the expectation is taken over ex post utility given the realization of the endogenous equilibrium market price, p , and the agent's type θ . The dependence of demand on market price allows the agents to incorporate whatever information is contained in prices into their assessment of the optimal bid.

A bid $q \in \mathcal{A}$ denotes a demand schedule $q : [0, \bar{v}] \rightarrow \{0, \dots, K\}$ for prices within $[0, \bar{v}]$, and we require each $q \in \mathcal{A}$ to be representable as the sum of d step functions where $(K + 1) * |\Omega| \leq d < \infty$.¹⁸ We endow the space \mathcal{A} with the L^1 norm and the space $\Delta(\mathcal{A})$ with the Kolmogorov topology. Given a pure strategy $\sigma : \Theta \rightarrow \mathcal{A}$, let the declared demand of an agent of type θ at price p be $\sigma(p; \theta)$. Given a distribution of declared demand schedules, $\pi^{\mathcal{A}}$, the auction makes a stochastic assignment to an agent who declared $q \in \mathcal{A}$ equal to $x(\pi^{\mathcal{A}}, q) \in \Delta(\{0, \dots, K\})$ at a price of $p(\pi^{\mathcal{A}}) \in \Delta([0, \bar{v}])$ per unit.¹⁹

In order to insure our model is continuous, we add noise to the supply side of the model. Let r denote a random per-capita supply of the homogenous good for auction in the nonatomic limit game, which may reflect noise traders who suffer aggregate shocks to demand, a reserve price in oil lease or Federal Reserve security auctions that is not disclosed to bidders, or a shock to the measure of participants.²⁰ We assume that r is distributed according to nonatomic measure ν over $(\bar{r} - \varkappa, \bar{r} + \varkappa) \subset (0, K)$.

First we define the market clearing price for the large finite and nonatomic limit games. Let r denote the aggregate per capita supply in the nonatomic limit game, and $M_N = \text{floor}(N * r)$ be the supply in the N agent uniform price auction.²¹ The ex post market clearing price in the N agent auction conditional on M_N , $p_N(\pi^{\mathcal{A}}; M_N)$, and the ex post market clearing price in the nonatomic limit game conditional on r , $p(\pi^{\mathcal{A}}; r)$, are (where s.t. refers to "such that")

¹⁸ \mathcal{A} can be embedded in $[0, \bar{v}]^d$. We require a strategy space rich enough so that the agents can declare their demand truthfully in a fully revealing equilibrium, but not so rich that our asymptotic convergence results fail to hold. These conditions can be met with a number of model structures - we choose one that is particularly intuitive in this setting.

¹⁹We use the Euclidean norm over $\Delta(\{0, \dots, K\})$.

²⁰The noise in the aggregate supply (r) is crucial for eliminating the market-power of individual bidders in large markets, which insures that the price distribution in the large finite games converges to the price distribution of the nonatomic limit game.

²¹ $\text{floor}(N * r)$ denotes the largest integer less than $N * r$.

$$\text{For } \pi^{\mathcal{A}} \in \Delta_N(\mathcal{A}), p_N(\pi^{\mathcal{A}}; M_N) = \sup p \text{ s.t. } \int q(p) * \pi^{\mathcal{A}}(dq) > \frac{M_N}{N} \quad (4.1)$$

$$\text{For } \pi^{\mathcal{A}} \in \Delta(\mathcal{A}), p(\pi^{\mathcal{A}}; r) = \sup p \text{ s.t. } \int q(p) * \pi^{\mathcal{A}}(dq) > r$$

Our uniform pointwise convergence assumption holds since p_N and p are outcome equivalent over $\Delta_N(\mathcal{A}) \times (0, K)$.²² These price functions correspond to the highest losing bid in the auction. Given these ex post market clearing price functions, define the interim market clearing price functions for $P \in \mathcal{B}([0, \bar{v}])$

$$\text{For } \pi^{\mathcal{A}} \in \Delta_N(\mathcal{A}), p_N(\pi^{\mathcal{A}})[P] = \nu(\{r : p_N(\pi^{\mathcal{A}}; \text{floor}[N * r]) \in P\})$$

$$\text{For } \pi^{\mathcal{A}} \in \Delta(\mathcal{A}), p(\pi^{\mathcal{A}})[P] = \nu(\{r : p(\pi^{\mathcal{A}}; r) \in P\})$$

where $p_N(\pi^{\mathcal{A}})[P]$ refers to the probability that $p_N(\pi^{\mathcal{A}}; M_N) \in P$. We endow the range of p and p_N with the Kolmogorov topology.

The allocation functions for agent i in the N -agent²³ and nonatomic limit games conditioned on the ex post realization of aggregate supply, r , are defined as

$$\text{For } \pi^{\mathcal{A}} \in \Delta_N(\mathcal{A}), x_N^i(\pi^{\mathcal{A}}, q; M_N) = \lim_{\delta \rightarrow 0} q(p_N(\pi^{\mathcal{A}}; M_N) + \delta)$$

$$\text{For } \pi^{\mathcal{A}} \in \Delta(\mathcal{A}), x^i(\pi^{\mathcal{A}}, q; r) = \lim_{\delta \rightarrow 0} q(p(\pi^{\mathcal{A}}; r) + \delta)$$

Note that the highest losing bidder, whose marginal bid sets the lower bound of the support of the market clearing price, does not receive a good for that bid. Define the interim allocation functions for $k \in \{0, \dots, K\}$ as

$$\text{For } \pi^{\mathcal{A}} \in \Delta_N(\mathcal{A}), x_N^i(\pi^{\mathcal{A}}, q)[k] = \nu(\{M_N : x_N^i(\pi^{\mathcal{A}}, q; M_N) = k\})$$

$$\text{For } \pi^{\mathcal{A}} \in \Delta(\mathcal{A}), x^i(\pi^{\mathcal{A}}, q)[k] = \nu(\{r : x^i(\pi^{\mathcal{A}}, q; r) = k\})$$

²²Alternately we could assume $\frac{M_N}{N} \rightarrow r$ almost surely and $p_N \rightarrow p$ uniformly. We have not found general conditions to insure $p_N \rightarrow p$ uniformly.

²³Assume $i \leq N$ in the N -agent game. By convention, $x_N^i = 0$ for $i > N$.

The utility functions in the N -agent and nonatomic limit games are then²⁴

$$\begin{aligned} u_N(\theta, \omega, q, \pi_N^A) &= E^{\Psi \times [0, K]}[v(\theta, \omega, x_N^i(\pi_N^A, q; M_N))] - p_N(\pi_N^A; M_N) * x_N^i(\pi_N^A, q; M_N) | \theta \\ u(\theta, \omega, q, \pi^A) &= E^{\Omega \times [0, K]}[v(\theta, \omega, x^i(\pi^A, q; r))] - p(\pi^A; r) * x^i(\pi^A, q; r) | \theta \end{aligned}$$

In the case of the N -agent game we take expectations over the valuations (and hence bids) of the other agents as well as the common value component ω and the aggregate supply shock r . In the nonatomic limit game the types of the other agents are determined by the law of large numbers, but the expectation is taken over all remaining variables. For the remainder of this work we suppress the agent indices.

We use the following notion of approximate efficiency evaluated at the ex post stage of the N -agent auction.

Definition 3. Consider an N -agent auction with supply M_N . The set of feasible allocations is

$$\mathcal{J} = \{x : \{1, \dots, N\} \rightarrow \{1, \dots, K\} \text{ such that } \sum_{i=1}^N x(i) \leq M_N\}$$

An equilibrium allocation of the N -agent game, $x_N : \Delta(\mathcal{A}) \times \mathcal{A} \rightarrow \Delta(\{0, \dots, K\})$ with strategy σ^N , is **ex post** (ε, ρ) -**efficient** if

$$\int v_N(\theta, \omega, x_N(\pi^A, \sigma^N(\theta))) * \pi^\Theta(d\theta) + \varepsilon \geq \max_{x^* \in \mathcal{J}} \frac{1}{N} \sum_{i=1}^N v_N(\theta_i, \omega, x^*(i))$$

with probability at least $1 - \rho$ with respect to measure P_N over $\Omega \times \Theta^N$.

If the set of equilibria is empty in the large finite game, then our approximation theorems are of little use. However, it is straightforward to prove the existence of equilibria using the techniques of McAdams [30] and Perry [38].^{25,26}

²⁴The state space $\Psi \times [0, K]$ refers to the realization of $(\omega, \theta_1, \dots, \theta_N, \dots)$ and per-capita supply r . The state space $\Omega \times [0, K]$ refers to the realization of ω and per-capita supply r .

²⁵The technical innovation in our result over McAdams [30] is the inclusion of an unobserved type of nature dictating the common value component.

²⁶We thank an insightful referee for suggesting this additional result.

Theorem 5. *There exists a monotone pure strategy equilibrium for all N in the large-finite game.*

4.2 Applying our Framework

To use our theorems, we need to find the equilibrium of the nonatomic limit game. Given that price cannot be affected by the declarations of individual agents in the nonatomic limit game and since agents receive an optimal allocation given their declared demand schedules and the market clearing price, all agents choose the bid that reflects their optimal allocation given the market clearing price. This suggests equilibria are efficient and information aggregates into prices, which we prove formally below.

We show in Lemma 4 that since $v(\theta, \omega, k)$ has strictly increasing differences in (θ, ω, k) and the conditional distribution of types is strictly log supermodular, the equilibrium bids declared by agents in the limit game are strictly increasing in θ . This implies that the ex post price function $p(\omega, r)$ is strictly monotone in ω . Therefore, $p = p(\omega, r)$ is sufficient to discover ω ex post if either r is known or if $\omega \neq \omega'$ implies the supports of $p(\omega)$ and $p(\omega')$ are non-overlapping. For large \varkappa , the distributions of $p(\omega)$ and $p(\omega')$ identify the state of the world, but a single price realization is not sufficient. We refer to an equilibrium as *fully revealing* if $\omega \neq \omega'$ implies that the supports of $p(\omega)$ and $p(\omega')$ are non-overlapping.

Lemma 1. *There exists $\bar{\varepsilon}, \bar{\varkappa} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$ and $\varkappa < \bar{\varkappa}$ such that any ε -BNE of the nonatomic limit game is fully revealing.*

Given that the REE is fully revealing, it is straightforward to show that the allocation of goods sold is ex post efficient.

Lemma 2. *For any $\bar{\varkappa} > 0$ sufficiently small, the allocation of goods sold is ex post $(0, 0)$ -efficient in any 0-BNE of the nonatomic limit game. For any $\gamma > 0$, we can choose $\bar{\varepsilon} > 0$ such that for $\varepsilon < \bar{\varepsilon}$ any ε -BNE of the nonatomic limit game is ex post $(\gamma, 0)$ -efficient.²⁷*

²⁷In an earlier version of this study we restricted agents to using decreasing bids, and as a result

To show that the nonatomic equilibrium outcomes approximate the equilibria of the large finite auction, we need to show that the nonatomic limit game is continuous. The market clearing price is set by the declaration of an agent who claims to be indifferent between two bundles at that price, and it is straightforward to show that the market clearing price changes only slightly if the distribution of bids changes only slightly in the Kolmogorov topology. Given the continuity of the market clearing price, it is immediate that the allocation for any declared demand is continuous. Since agent preferences are also continuous in allocation and price, we have the requisite continuity condition for our approximation theorems to hold.

Lemma 3. *Consider any strategy, σ , played in the nonatomic limit game. Then for $\chi > 0$ sufficiently small, $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded and uniformly equicontinuous in a relatively open set of $\Delta(\mathcal{A})$ containing $\cup_{\omega \in \Omega} \{\pi_0^A(\omega)\}$ where for all $A \in \mathcal{B}(\mathcal{A})$*

$$\pi_0^A(\omega)[A] = \int_{\Theta} \sigma(\theta)[A] * \pi_0^\Theta(\omega)[d\theta]$$

Our continuity result is sufficient to employ Theorem 4 to argue that for any $\varepsilon > 0$ any exact equilibria of the large finite auction with sufficiently many players is an ε -BNE of the nonatomic limit game. Lemma 1 and 2 then imply that the 0-BNE of sufficiently large finite auctions are revealing and approximately efficient with high probability when played as ε -BNE in the nonatomic limit game. By characterizing the ε -BNE of the nonatomic limit game, Theorem 4 implies that we have indirectly characterized the 0-BNE of the large finite auction with sufficiently many agents.

Theorem 6. *The uniform price auction asymptotically aggregates information since $p_N(\omega) : \Omega \rightarrow \Delta([0, \bar{v}])$ converges to the fully revealing price $p(\omega)$ almost surely. Further, for any $\varepsilon, \rho > 0$ and \varkappa sufficiently small, we can choose N^* such that for $N > N^*$ the equilibrium of the N -agent auction is ex post (ε, ρ) -efficient.*

were unable to show efficiency of the nonatomic equilibria in all settings. Since agents can issue nonmonotone bids in the current version, we can show efficiency in all cases.

Analyses of the equilibria of large finite auctions are complicated by the strategic interactions of the bidders, which in the case of the private values uniform price auction involves bidders withholding demand to lower the market clearing price. Our analysis is simplified by the fact that the nonatomic form of the auction reduces to a stochastic Walrasian economy that can be analyzed without the preference restrictions required to employ game theoretic techniques. Characterizing the full set of ε -BNE of the nonatomic limit game is relatively tractable, and reveals that any such equilibrium approximately aggregates information and is approximately ex post efficient.

Once the nonatomic uniform price auction is shown to be continuous, Theorem 4 lets us employ the fact that 0-BNE of the large finite game are approximate rational expectations equilibria of the nonatomic game to translate properties of the approximate limit equilibria to properties of the exact equilibria of the large finite auction. The nonatomic equilibria are efficient, which implies ex-post (ε, ρ) -efficiency of the large finite auction equilibria. In the interdependent values case, the uniform price auction market clearing price can serve as a consistent estimator of the state of the world.

4.3 Extension - Risk Preferences and Private Values

Consider the private values formulation, in which case the nonatomic limit game is a private values general equilibrium exchange economy. The only condition we require on the agent utility functions is that they be continuous in the allocation and price that the agent receives. We could extend our analysis in the private value case to:

- Expected utility preferences without a common prior²⁸
- Risk averse preferences

²⁸If the agents have uncommon priors but all agents believe the equilibrium prices are positive probability events, then truthful declaration of demand is an equilibrium as $N \rightarrow \infty$. If a positive measure of agents believes a set of prices realized with positive probability in equilibrium is a null event, then they have no incentive to declare truthfully for these prices, the set of equilibrium outcomes expands, and efficiency is no longer assured in equilibrium.

- Agent valuations drawn from a (finite) set of asymmetric valuation distributions
- Agents with loss averse preferences

For each of these preference relations, truthful revelation of demand remains an equilibrium of the private values nonatomic limit game. Therefore, equilibria of the large finite game are approximately efficient.

5 Previous Literature

Kalai [24] provides a study of semi-anonymous large games in order to show the equilibria of games with finite action and type spaces are robust to modifications of the game form. The closest point of overlap between our papers is Kalai’s Theorem 1, which shows that for any choice of $\varepsilon, \rho > 0$, the equilibria of the game approaches an (ε, ρ) -ex post Nash equilibrium as the number of players increases to infinity. None of the theorems presented here provides an extension of this result, although it is clear that the techniques underlying Theorems 2 and 4, once specialized to the case of no aggregate shocks ($\Omega = \{\omega\}$), could extend Kalai’s Theorem 1 to the case of subsets of finite dimensional Euclidean spaces.

The prior literature on the relationship between large finite and nonatomic games generally focuses on games of complete information and does not emphasize the use of nonatomic games as a framework for analyzing the equilibria of game-theoretic models (examples include Housman [22], Khan and Sun [25], Carmona [13], and Carmona and Podczeck [14] and [15] amongst others). In addition some of these works focus on weaker approximate equilibrium concepts (e.g. the (ε, η) -equilibrium used by Carmona and Podczeck [14]), and it is not clear that these concepts yield our uniform equilibrium convergence results. In an unpublished contemporaneous work, Yang [44] develops a similar result to Theorem 2.

Particularly relevant prior works for our study include Green [18] and [19] and Sabourian [41], which provide sufficient conditions for the set of nonatomic subgame perfect equilibria (SPE) and the SPE of the analogous large finite dynamic game

to be approximately the same.²⁹ Al-Najjar ([1], [2]) has studied the relationship between large finite economies and their nonatomic equivalents to clarify difficult technical issues that have plagued the literature on generic nonatomic games. Al-Najjar and Smorodinsky [3] prove that the fraction of players who are α -pivotal has a bound of the form $O\left(\frac{1}{\sqrt{N}}\right)$, but allow a potentially large number of players to retain significant strategic choices for large N .³⁰

The large literature on Walrasian games addresses the question of whether finite games taken to represent strategic foundations for Walrasian equilibrium in fact achieve Walrasian equilibria in the limit as the number of agents increases (some examples of this literature include Hildenbrand [20] and [21], Roberts and Postlewaite [40], Otani and Sicilian [35], Jackson and Manelli [23]). In these games the agents in the economy declare a demand schedule, and the market price is defined by the price that clears the market given the aggregate declared demand. Jackson and Manelli provide the analysis closest in spirit to this paper. Jackson and Manelli prove that when agents believe they have little influence on the market-clearing price, then the agents' declared demands must converge to the Walrasian equilibrium demands as the economy grows in size. Our continuity condition, based on the exogenous economic primitives, is similar to the endogenous small influence condition of Jackson and Manelli.

Mas-Colell and Vives [29] discuss implementation theory for exchange economies with a continuum of agents. The theoretical techniques employed are similar to ours in that they utilize approximation theorems to argue that implementations in continuum economies yield approximate Bayesian-Nash implementations in large finite exchange economies. Our theorems are more readily applied outside of the context of exchange economies to models such as large auctions for indivisible goods and matching markets. Implementation problems similar in spirit to those tackled by Mas-Colell and Vives could be approached using our framework, although this issue is left for future work.

²⁹These papers focus on complete information games and assume convergence results that we prove from economic primitives.

³⁰The results of Al-Najjar and Smorodinsky [3] do not require a continuity condition, which is the force that drives the uniform limit on the pivotality of the players in our framework.

A growing literature employs the techniques of game-theoretic analysis to analyze the performance of large markets (Pesendorfer and Swinkels [36], Swinkels [42], Cripps and Swinkels [16], McLean and Postlewaite [31], Budish [12], Kojima and Pathak [27], and Fudenberg et al. [17] amongst many others). Most of the works in this literature proceed by studying the properties of game-theoretic equilibria as the number of agents increases to infinity.

6 Conclusion

We have demonstrated continuity and semi-anonymity restrictions sufficient for nonatomic limit games to approximate the outcomes of their large finite analogs. Further, we are able to show that as the number of agents increases, the approximation improves in both the strategy and payoff space of the model. Therefore, nonatomic limit games can be used as approximations of otherwise intractable game theoretic models.

One reason that the study of the limit behavior of large finite games has been of increasing interest to microtheorists is the need to design markets. For example, the uniform price auction is an ubiquitous choice for allocating goods ranging from US Treasury Bills to electricity provision rights. Since small uniform price auctions can be inefficient, it is important to determine whether these negative aspects of the mechanism disappear as the number of participants increases. However, the game theoretic analysis of this mechanism has been limited due to the difficulty of determining the equilibria outside of specialized settings. For settings where the good has a common value component, such as securities auctions, or bidders can view successive units as complements, such as electricity auctions, the prior literature on multi-unit demand uniform price auctions cannot tell us whether information is aggregated into prices or the auction outcomes are efficient as $N \rightarrow \infty$.

We use our framework to analyze the multi-unit demand uniform price auction and allow for both a common value component and complementary valuations of successive units. Once we prove that a nonatomic general equilibrium economy approximates the equilibria of the large finite uniform price auction, the price taking

behavior of the general equilibrium economy lends us analytical tractability for deriving the rational expectations equilibrium outcome of our nonatomic limit economy. We can then use our approximation theorems to translate the properties of the fully revealing rational expectation equilibria into properties of the large uniform price auction equilibria. We contribute to the literature on the strategic foundations of rational expectations equilibria by proving that information aggregates into market price as $N \rightarrow \infty$. We also show that the equilibria of the auction are approximately efficient with the efficiency loss vanishing as $N \rightarrow \infty$.

Interesting topics for future work include further exploring the effects of market equilibria that become ex post in the limit as $N \rightarrow \infty$. In terms of applications, the framework outlined above provides a powerful tool for analyzing large matching markets, competition between market mechanisms, and large market games with after markets such as auctions with resale. The study of strategic foundations of rational expectations models could be extended outside of the context of auctions and double auctions into more general models of markets with competing producers. Finally, these techniques may be fruitfully extended to dynamic games of interest to econometricians, macroeconomists, and structural labor and industrial organization economists.

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A Proofs

A.1 Proofs from Section 3

We begin this section with some useful results from the theory of the weak convergence of empirical processes. Let $d_K : \Delta(\mathcal{S}) \times \Delta(\mathcal{S}) \rightarrow \mathbb{R}_+$ denote the Kolmogorov metric that metricizes the Kolmogorov topology where $\pi, \xi \in \Delta(\mathbb{R}^d)$, $d < \infty$, and $F_\pi(\circ), F_\xi(\circ)$ are the respective CDFs

$$d_K(\pi, \xi) = \sup_{y \in \mathbb{R}^d} |F_\pi(y) - F_\xi(y)|$$

Let $d_{LP} : \Delta(\mathcal{S}) \times \Delta(\mathcal{S}) \rightarrow \mathbb{R}_+$ denote the Lévy-Prokhorov metric, which metricizes the weak-* topology. We reserve the notation d_K^S and d_{LP}^S for when clarity requires us to denote the measures to which the metric applies.

Theorem 7. *Consider a random variable $X : \Omega \rightarrow \mathbb{R}^d$, $d < \infty$, with measure π_0 and associated CDF $F(y) = \int_\Omega 1\{x \leq y\} * \pi_0(dx)$. For N i.i.d. realizations, $\{X_1, \dots, X_N\}$ drawn from π_0 we have*

$$d_K(\pi_N, \pi_0) = \sup_{y \in \mathbb{R}^d} |F_N(y) - F(y)| \rightarrow 0 \text{ almost surely as } N \rightarrow \infty$$

Proof. (Proof of Theorem 7) Follows from van der Vaart et al. [43], p. 135 and noting that the sets of the form $\{x : x \leq y\}$ for $y \in \mathbb{R}^d$ are lower contours and form a VC Class. \square

Corollary 2. *Define the empirical measure generated by the counting measure over $\{X_1, \dots, X_N\}$ as π_N . Then $\pi_N \rightarrow \pi_0$ almost surely in the weak-* topology over $\Delta(\mathbb{R}^d)$*

Proof. (Proof of Corollary 2) From Billingsley (p. 18, [10]) we have that $F_N(y) \rightarrow F(y)$ at continuity points of F implies $\pi_N \rightarrow \pi_0$ almost surely in the weak-* topology. Since we have uniform convergence $F_N(y) \rightarrow F(y)$ for all y almost surely, we have $\pi_N \rightarrow \pi_0$ in the weak-* topology. \square

Corollary 3. *Consider a random variable $X : \Omega \rightarrow \mathbb{R}^d$, $d < \infty$, and associated CDF $F(y)$. Denote the N realization empirical CDF as $F_N(y)$. Then*

$$\Pr\{\sqrt{N} \sup_{y \in \mathbb{R}^d} |F_N(y) - F(y)| > t\} \leq C * e^{-2t^2}$$

where the constant $C > 0$ depends only on the dimension d .

Proof. (Proof of Corollary 3) This result follows directly from Theorems 2.6.7 and 2.14.9 of van der Vaart and Wellner [43]. \square

We now restate and prove theorems 1, 2, 3, and 4. Due to the repeated use of sequences of strategies in our proofs, we employ the notation $\pi_\infty^N(\omega)$ to refer to the distribution of actions induced by $\pi_0^\Theta(\omega)$ and σ^N . $\pi_\infty(\omega)$ denotes the distribution of actions induced by a limit strategy, σ^∞ , and $\pi_0^\Theta(\omega)$. Written formally, for all $A \in \mathcal{B}(\mathcal{A})$ let

$$\pi_\infty^N(\omega)[A] = \int_{\Theta} \sigma^N(\theta)[A] * \pi_0^\Theta(\omega)[d\theta] \tag{A.1}$$

$$\pi_\infty(\omega)[A] = \int_{\Theta} \sigma^\infty(\theta)[A] * \pi_0^\Theta(\omega)[d\theta] \tag{A.2}$$

Theorem 1. *Endow $\Delta(\mathcal{A})$ with the weak-* topology. The correspondence \mathcal{E} is upper hemicontinuous as $N \rightarrow \infty$ ($\mathcal{E}^\infty \subset \mathcal{E}^{NA}$) **if** the family $\{E^\Omega[u(\theta, \omega, \cdot, \cdot)|\theta]\}_{\theta \in \Theta}$ is uniformly bounded and uniformly equicontinuous in $\mathcal{A} \times \Delta(\mathcal{A})$.*

Proof. Consider a sequence of exact Bayesian-Nash equilibrium strategies, $\{\sigma^N : \Theta \rightarrow \Delta(\mathcal{A})\}_{N=1}^\infty$ where $\sigma^N \in \mathcal{E}(N)$. Suppose $\sigma^N \rightarrow \sigma^\infty$ in the supremum norm. For any $\delta > 0$ we can choose N large enough that $d_{LP}^A(\pi_\infty^\omega, \pi_\infty^N(\omega)) < \delta$ uniformly over $\omega \in \Omega$.

Suppose our theorem is false and $\sigma^\infty \notin \mathcal{E}^{NA}$. Then there exists a set $\mathcal{S} \subset \Theta$ of measure $\rho > 0$ (under $\pi_0^\Theta(\omega)$) such that for all $\theta \in \mathcal{S}$, $a_\infty^* \in \text{supp}[\sigma^\infty(\theta)]$ and some $\underline{\varepsilon} > 0$

$$E^\Omega[u(\theta, \omega, a_\infty^*, \pi_\infty^\omega(\omega))|\theta] + \underline{\varepsilon} < \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^\omega(\omega))|\theta] \quad (\text{A.3})$$

Since $\{E^\Omega[u(\theta, \omega, \cdot, \cdot)|\theta]\}_{\theta \in \Theta}$ is uniformly equicontinuous in the weak-* topology over $\Delta(\mathcal{A})$, we have that for any $\varepsilon > 0$ we can choose N sufficiently large that

$$\sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^N(\omega))|\theta] + \varepsilon \geq \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^\omega(\omega))|\theta] \quad (\text{A.4})$$

From Theorem 4 for sufficiently large N we have for $a_N^* \in \text{supp}[\sigma^N(\theta)]$

$$E^\Omega[u(\theta, \omega, a_N^*, \pi_\infty^N(\omega))|\theta] + 4\varepsilon \geq \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^N(\omega))|\theta] \quad (\text{A.5})$$

But from the uniform equicontinuity of $\{E^\Omega[u(\theta, \omega, \cdot, \cdot)|\theta]\}_{\theta \in \Theta}$ and $\sigma^N \rightarrow \sigma^\infty$ we have for large enough N that

$$E^\Omega[u(\theta, \omega, a_\infty^*, \pi_\infty^\omega(\omega))|\theta] + \varepsilon \geq E^\Omega[u(\theta, \omega, a_N^*, \pi_\infty^N(\omega))|\theta] \quad (\text{A.6})$$

But this entails

$$\begin{aligned} E^\Omega[u(\theta, \omega, a_\infty^*, \pi_\infty^\omega(\omega))|\theta] + 6\varepsilon &\geq E^\Omega[u(\theta, \omega, a_N^*, \pi_\infty^N(\omega))|\theta] + 5\varepsilon \\ &\geq \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^N(\omega))|\theta] + \varepsilon \\ &\geq \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_\infty^\omega(\omega))|\theta] \end{aligned} \quad (\text{A.7})$$

Since we can choose ε arbitrarily small (in particular $6\varepsilon < \underline{\varepsilon}$) by selecting N sufficiently large, this contradicts the assertion that $\sigma^\infty \notin \mathcal{E}^{NA}$. Therefore it must be

that $\sigma^\infty \in \mathcal{E}^{NA}$. Since this holds for all such sequences of strategies, we have from Theorem 17.16 of Aliprantis and Border [4] that the equilibrium correspondence is upper hemicontinuous. \square

Theorem 2. *Fix $\varepsilon > 0$ and endow $\Delta(\mathcal{A})$ with the Kolmogorov topology. For each Bayesian-Nash equilibrium $\sigma^\infty : \Theta \rightarrow \Delta(\mathcal{A})$ of the nonatomic game $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$, there exists an N^* such that for all $N > N^*$, $\sigma^\infty(\circ)$ is an ε -Bayesian-Nash equilibrium of the N -agent game $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$ **if** the family $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded and uniformly equicontinuous in $\Delta(\mathcal{A})$ in a relatively open set of $\Delta(\mathcal{A})$ containing $\cup_{\omega \in \Omega} \{\pi_0^A(\omega)\}$.*

Proof. Denote the empirical distribution of agent actions in the N -agent game $\pi_N^A(\varphi)$. From Theorem 7, we have $\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$ uniformly over ω almost surely (with respect to P) in the Kolmogorov metric as $N \rightarrow \infty$. Consider only $\varphi \in \Psi$ such that $\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$.

From the uniform pointwise convergence of u_N to u , for any $\varepsilon > 0$ we can choose N sufficiently large that

$$\sup_{(\theta, \omega, a, \pi_N^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A})} |u_N(\theta, \omega, a, \pi_N^A) - u(\theta, \omega, a, \pi_N^A)| < \varepsilon \quad (\text{A.8})$$

Consider $a^* \in \text{supp}[\sigma^\infty(\theta)]$. By definition

$$E^\Omega[u(\theta, \omega, a^*, \pi_0^A(\omega))|\theta] = \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi_0^A(\omega))|\theta] \quad (\text{A.9})$$

From the uniform equicontinuity of $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$, we have that for all $\varepsilon > 0$ there exists a $\delta > 0$ so that for any $\pi^A(\varphi)$ such that $d_K(\pi^A(\varphi), \pi_0^A(\omega)) < \delta$ we have³¹

$$E^\Omega[u(\theta, \omega, a^*, \pi^A(\varphi))|\theta] + \varepsilon \geq \sup_{a \in \mathcal{A}} E^\Omega[u(\theta, \omega, a, \pi^A(\varphi))|\theta] \quad (\text{A.10})$$

To close our proof, note that we have assumed that u is uniformly bounded, so

³¹Formally $\pi^A(\varphi)$ is a map from the state space Ψ into $\Delta(\mathcal{A})$.

there exists $\infty > M > 0$ such that

$$M > \sup_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A})} u(\theta, \omega, a, \pi^A) - \inf_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A})} u(\theta, \omega, a, \pi^A) \quad (\text{A.11})$$

$\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$ for some ω almost surely as $N \rightarrow \infty$ implies that for any $\delta, \rho > 0$ we can choose N sufficiently large that $d_K(\pi^A(\varphi), \pi_0^A(\omega)) < \delta$ with probability at least $1 - \rho$. In the event $d_K(\pi^A(\varphi), \pi_0^A(\omega)) < \delta$, A.10 implies that a^* is an $\frac{\varepsilon}{2}$ optimal action for sufficiently small δ . Therefore we can write

$$E^\Psi[u(\theta, \omega, a^*, \pi^A(\varphi))|\theta] + \frac{\varepsilon}{2} + \rho * M \geq \sup_{a \in \mathcal{A}} E^\Psi[u(\theta, \omega, a^*, \pi^A(\varphi))|\theta] \quad (\text{A.12})$$

If we choose $\rho = \frac{\varepsilon}{2M}$ then we have

$$E^\Psi[u(\theta, \omega, a^*, \pi^A(\varphi))|\theta] + \varepsilon \geq \sup_{a \in \mathcal{A}} E^\Psi[u(\theta, \omega, a^*, \pi^A(\varphi))|\theta] \quad (\text{A.13})$$

Using our uniform pointwise convergence result we have

$$E^\Psi[u_N(\theta, \omega, a^*, \pi^A(\varphi))|\theta] + 2\varepsilon \geq \sup_{a \in \mathcal{A}} E^\Psi[u_N(\theta, \omega, a^*, \pi^A(\varphi))|\theta] \quad (\text{A.14})$$

Therefore, σ^∞ is an ε -BNE of the N agent game. \square

Theorem 3. *Assume $\Omega = \{\omega\}$ and fix $\varepsilon > 0, \rho \in (0, 1]$. Endow $\Delta(\mathcal{A})$ with the Kolmogorov topology. For each Nash equilibrium $\sigma^\infty : \Theta \rightarrow \Delta(\mathcal{A})$ of the nonatomic game $u : \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A}) \rightarrow \mathbb{R}$, there exists an N^* such that for all $N > N^*$, $\sigma^\infty(\circ)$ is an ex post (ε, ρ) -Nash equilibrium of the N -agent game $u_N : \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A}) \rightarrow \mathbb{R}$ **if** the family $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly equicontinuous in $\Delta(\mathcal{A})$ in a relatively open set of $\Delta(\mathcal{A})$ containing $\pi_0^A(\omega)$*

Proof. Denote the empirical distribution of agent actions as $\pi_N^A(\varphi)$. From Theorem 7, we have $\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$ almost surely in the Kolmogorov Metric as $N \rightarrow \infty$. Consider only $\varphi \in \Psi$ such that $\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$ for some ω .

From the uniform pointwise convergence of u_N to u , for any $\varepsilon > 0$ we can choose

N sufficiently large that

$$\sup_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A})} |u_N(\theta, \omega, a, \pi^A) - u(\theta, \omega, a, \pi^A)| < \varepsilon \quad (\text{A.15})$$

Consider $a^* \in \text{supp}[\sigma^\infty(\theta)]$. From the uniform equicontinuity of $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$, we have that for all $\varepsilon > 0$ there exists a $\delta > 0$ so that if $d_K(\pi^A(\varphi), \pi_0^A(\omega)) < \delta$ we have

$$\sup_{a \in \mathcal{A}} u(\theta, \omega, a, \pi^A(\varphi)) \leq u(\theta, \omega, a^*, \pi^A(\varphi)) + \varepsilon \quad (\text{A.16})$$

Combining these relations yields

$$u_N(\theta, \omega, a^*, \pi^A(\varphi)) + \varepsilon \geq \sup_{a \in \mathcal{A}} u_N(\theta, \omega, a, \pi^A) \quad (\text{A.17})$$

To close our proof, note that $\pi_N^A(\varphi) \rightarrow \pi_0^A(\omega)$ almost surely as $N \rightarrow \infty$ implies that for any $\rho > 0$ we can choose N sufficiently large that $d_K(\pi^A(\varphi), \pi_0^A(\omega)) < \delta$ with probability at least $1 - \rho$. Therefore, σ^∞ is an ex post (ε, ρ) -Nash equilibrium of the N agent game. \square

Theorem 4. *Fix $\varepsilon > 0$ and endow $\Delta(\mathcal{A})$ with the Kolmogorov topology. There exists an N^* such that for all $N > N^*$ and any exact symmetric Bayesian-Nash equilibrium $\sigma^N : \Theta \rightarrow \Delta(\mathcal{A})$ of the N agent game, σ^N is an ε -Bayesian-Nash equilibrium of the nonatomic game **if** the family $\{E^\Omega[u(\theta, \omega, a, \cdot)|\theta]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded and uniformly equicontinuous in $\Delta(\mathcal{A})$ in a relatively open set of $\Delta(\mathcal{A})$ containing $\cup_{\omega \in \Omega} \{\pi_\infty^N(\omega)\}$ where for all $A \in \mathcal{B}(\mathcal{A})$*

$$\pi_\infty^N(\omega)[A] = \int_{\Theta} \sigma^N(\theta)[A] * \pi_0^\Theta(\omega)[d\theta]$$

Proof. By the same logic employed in the proof of Theorem 2, we know that the K draw empirical distribution of actions induced by the strategy $\sigma^N(\circ)$ combined with the type distribution $\pi_0^\Theta(\omega)$, denoted $\pi_K^N(\varphi)$, converges uniformly over $\omega \in \Omega$ and almost surely to the distribution $\pi_\infty^N(\omega)$ in the Kolmogorov metric as $K \rightarrow \infty$.³²

³²Note that we are not asserting that σ^N remains an equilibrium when we take the number of

Consider $\varphi \in \Psi$ such that $\pi_K^N(\varphi) \rightarrow \pi_\infty^N(\omega)$ for some ω as $K \rightarrow \infty$. From Corollary 3 we know

$$P(\sqrt{N} * d_K(\pi_N^N(\varphi), \pi_\infty^N(\omega)) > \lambda) \leq C e^{-2\lambda^2} \quad (\text{A.18})$$

with the choice of C independent of the equilibrium strategy.³³

From the uniform pointwise convergence of u_N to u , for any $\varepsilon > 0$ we can choose N sufficiently large that

$$\sup_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta_N(\mathcal{A})} |u_N(\theta, \omega, a, \pi^A) - u(\theta, \omega, a, \pi^A)| < \varepsilon \quad (\text{A.19})$$

Let

$$M = \sup_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A})} u(\theta, \omega, a, \pi^A) - \inf_{(\theta, \omega, a, \pi^A) \in \Theta \times \Omega \times \mathcal{A} \times \Delta(\mathcal{A})} u(\theta, \omega, a, \pi^A) \quad (\text{A.20})$$

which is finite from the boundedness of the utility function.

From the definition of Bayesian-Nash equilibrium we have for all $\tilde{a} \in \mathcal{A}$ and for all $a^* \in \text{supp}[\sigma^N(\theta)]$

$$E^\Psi[u_N(\theta, \omega, a^*, \pi_N^N(\varphi)) - u_N(\theta, \omega, \tilde{a}, (\pi_N^N(\varphi) + \frac{1}{N}[\delta_{\tilde{a}} - \delta_{a^*}]])|\theta] \geq 0 \quad (\text{A.21})$$

Using our uniform pointwise convergence result we have

$$E^\Psi[u(\theta, \omega, a^*, \pi_N^N(\varphi))|\theta] + 2\varepsilon \geq \sup_{\tilde{a} \in \mathcal{A}} u(\theta, \omega, \tilde{a}, (\pi_N^N(\varphi) + \frac{1}{N}[\delta_{\tilde{a}} - \delta_{a^*}]])|\theta]$$

For any $\varepsilon, \rho > 0$, the uniform convergence of $\pi_K^N(\omega) \rightarrow \pi_\infty^N(\omega)$ as $K \rightarrow \infty$, and the boundedness of u imply for sufficiently large N

$$\begin{aligned} E^\Psi[u(\theta, \omega, a^*, \pi_N^N(\varphi))|\theta] &\leq E^\Omega[u(\theta, \omega, a^*, \pi_\infty^N(\omega))|\theta] + \varepsilon + \rho * M \\ E^\Psi[u(\theta, \omega, \tilde{a}, (\pi_N^N(\varphi) + \frac{1}{N}[\delta_{\tilde{a}} - \delta_{a^*}]])|\theta] &\geq E^\Omega[u(\theta, \omega, \tilde{a}, \pi_\infty^N(\omega))|\theta] - \varepsilon - \rho * M \end{aligned} \quad (\text{A.22})$$

agent to ∞ .

³³The uniformity over Σ is crucial to our argument since the equilibrium strategy, σ^N , and the measures $\pi_N^N(\varphi)$ and $\pi_\infty^A(\omega)$ change with N as we take the limit $N \rightarrow \infty$.

For ρ sufficiently small we have

$$\begin{aligned} E^\Psi[u(\theta, \omega, a^*, \pi_N^N(\varphi))|\theta] &\leq E^\Omega[u(\theta, \omega, a^*, \pi_\infty^N(\omega))|\theta] + 2\varepsilon \text{(A.23)} \\ E^\Psi[u(\theta, \omega, \tilde{a}, (\pi_N^N(\varphi) + \frac{1}{N}[\delta_{\tilde{a}} - \delta_a]))|\theta] &\geq E^\Omega[u(\theta, \omega, \tilde{a}, \pi_\infty^N(\omega))|\theta] - 2\varepsilon \end{aligned}$$

Substituting this into the definition of the BNE we have

$$E^\Omega[u(\theta, \omega, a^*, \pi_\infty^N(\omega))|\theta] + 6\varepsilon \geq E^\Omega[u(\theta, \omega, \tilde{a}, \pi_\infty^N(\omega))|\theta] \quad (\text{A.24})$$

Therefore $\sigma^N(\theta)$ is an ε -Bayesian-Nash Equilibrium of the nonatomic game generated by the strategy $\sigma^N(\theta)$. \square

A.2 Proofs from Section 4

We begin this section by proving the existence of an equilibrium in the large-finite game. Our result follows from techniques used to prove existence of monotone pure strategies in Bayesian games with well-behaved discontinuities.

Theorem 5. *There exists a monotone pure strategy equilibrium for all N in the large-finite game.*

Proof. (of Theorem 5) To incorporate ω , we introduce Nature as a player. We also discretize the action space as an arbitrary finite sublattice of the space of bids. At the end of the proof we use techniques based on better-reply-security to insure our existence results holds in the limit as the bid space takes the connected form assumed in our model.

Assumptions 1 through 3 of McAdams [30] are straightforward. Assumptions 4 and 5, quasi-supermodularity and single-crossing of interim utility, follow from Lemma 4 once we note that supermodularity in (k, θ) implies both quasi-supermodularity and single-crossing in type and bid.³⁴ Theorem 1 of McAdams [30] then proves the existence of a monotone pure strategy equilibrium in an arbitrary finite sublattice

³⁴Note that k , the number of goods received, is monotone in the bid submitted regardless of the bids of the other agents.

of the space of bids. Denote a sequence of such finitely gridded bid spaces $\{B_k\}_{k=1}^\infty$ and a monotone equilibrium strategy on B_k as σ^k .

From the compactness of \mathcal{A} , any sequence $\{\sigma^k\}_{k=1}^K$ has a convergent subsequence with a limit σ^* .³⁵ The argument of corollary 5.2 of Reny [38] proves better-reply-security of our model with a sufficiently finely-gridded space of bids. This then implies that σ^* is an equilibrium of our original large-finite game with action space, \mathcal{A} (see remark 3.1 of Reny [37]). \square

We now describe the ε -BNE of the nonatomic limit game through a series of lemmas, in the process showing that any ε -BNE must be approximately efficient and fully revealing for ε sufficiently small. We then prove continuity results sufficient to employ Theorem 4 to argue that the 0-BNE of the large finite game are such ε -BNE of the nonatomic limit game and thus must possess these desired properties (at least when played as strategies of the limit game). Theorem 6 is proven by showing that if a 0-BNE of the large finite game is approximately efficient and fully revealing in the limit game, it must also have these properties as a 0-BNE of the large finite game.

Lemma 4. *Consider an arbitrary price process, $p(\omega) \in \Delta([0, \bar{v}])$, with associated Radon-Nikodym derivative $g(p|\omega)$.³⁶ Any agent's exact best response in the nonatomic limit game is outcome equivalent to truthfulness. Furthermore, it must be the case that the agents declare bids that are increasing in θ , so $\theta > \theta'$ implies for all p realized in equilibrium that $\sigma(\theta; p) \geq \sigma(\theta'; p)$.*

Proof. Since a single agent's bid in the nonatomic limit game cannot affect the market clearing price, it is obvious that a bid equal to the expected demand at each price is strictly optimal for prices realized under $p(\omega)$ and weakly optimal for all other prices. What remains is to show that these declarations are monotone in θ .

Since the market clearing price is exogenous to a single agent's decision, we have

³⁵For the purposes of this result all we require is sequential compactness. Since \mathcal{A} is compact, the space of maps from Θ to \mathcal{A} is compact under pointwise-convergence by Tychonoff's theorem.

³⁶The measure with respect to which $g(p|\omega)$ is absolutely continuous is well-defined, but depends on the measure $p(\omega)$.

that θ and p are independent conditional on ω .³⁷ Therefore we can write

$$g(\omega|\theta, p) = \frac{f(\theta|\omega) * g(p|\omega) * g(\omega)}{\sum_{\omega \in \Omega} f(\theta|\omega) * g(p|\omega) * g(\omega)}$$

Since we have assumed $f(\theta|\omega)$ is continuous in θ and strictly log-supermodular by assumption 5, we have that $g(\omega|\theta, p)$ is continuous in θ and strictly log-supermodular in (θ, ω) .

Since v is continuous in θ by assumption 2 and has strictly increasing differences by assumption 3, for $k > k'$, $\theta > \theta'$ and $\omega > \omega'$

$$\begin{aligned} v(\theta, \omega, k) - v(\theta, \omega, k') &> v(\theta', \omega, k) - v(\theta', \omega, k') \\ v(\theta, \omega, k) - v(\theta, \omega, k') &> v(\theta, \omega', k) - v(\theta, \omega', k') \end{aligned}$$

From the strict log-supermodularity of $g(\omega|\theta, p)$ we have

$$\begin{aligned} \sum_{\omega \in \Omega} [v(\theta, \omega, k) - v(\theta, \omega, k')] * g(\omega|\theta, p) &> \tag{A.25} \\ &\sum_{\omega \in \Omega} [v(\theta', \omega, k) - v(\theta', \omega, k')] * g(\omega|\theta', p) \end{aligned}$$

Therefore it must be the case that agents bids are increasing in θ , so $\sigma(\theta; p) \geq \sigma(\theta'; p)$. \square

Note also that from Assumption 4 we have that for any p in the support of an arbitrary $p(\omega)$ it must be that there exists $\theta > \theta'$ such that $\sigma(\theta; p) > \sigma(\theta'; p)$. Importantly, this rules out constant bids across types.

Lemma 5. *For any $\delta > 0$, there is an $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$ we have in any ε -BNE that all but a measure δ (or less) of agents are truthful for all prices realized in equilibrium.*

Proof. Given an arbitrary ε -BNE of the nonatomic limit game, consider the associated distribution of market clearing price $p(\omega)$. If an agent of type θ has true demand

³⁷Recollect the agent types are independent conditional on ω .

k at price p in the support of $p(\omega)$, then an agent can declare demand $k' \neq k$ in an ε -BNE of the limit game only if

$$\sum_{\omega \in \Omega} [v(\theta, \omega, k) - v(\theta, \omega, k')] * g(\omega|\theta, p) < (k - k') * p + \varepsilon \quad (\text{A.26})$$

From equation A.25, the quantity

$$\sum_{\omega \in \Omega} [v(\theta, \omega, k) - v(\theta, \omega, k')] * g(\omega|\theta, p)$$

is strictly increasing in θ . Since $\pi_0^\Theta(\omega)$ is atomless, this implies that for any $\delta > 0$ we can choose ε sufficiently small that equation A.26 holds for at most a measure $\frac{2\delta}{(K+1)K}$ agents. Summing over all pairs $k > k'$ we then have that equation A.26 holds for a measure δ (or less) of the agents.

We note that our claim of truthfulness holds strictly only for prices realized in equilibrium since there is no incentive to declare a particular demand (optimal or otherwise) for prices that are never observed. \square

Let $\Theta(x, p) = \{\theta : \sigma(\theta; p) \geq x\}$ denote a demand level set in a 0-BNE of the nonatomic limit game with equilibrium strategy σ . $\Theta(x, p)$ is an upper-contour set since $\theta \geq \theta'$ implies $\sigma(\theta; p) \geq \sigma(\theta'; p)$. Denote aggregate demand in state $\omega \in \Omega$ as

$$D(p, \omega) = \sum_{x=1}^K \pi_0^\Theta(\omega)[\Theta(x, p)]$$

Using the assumption that $f(\theta|\omega)$ is strictly log supermodular, we have that at any price p realized in equilibrium and $\omega > \omega'$

$$\begin{aligned} D(p, \omega) &= \sum_{x=1}^K \pi_0^\Theta(\omega)[\Theta(x, p)] \\ &> \sum_{x=1}^K \pi_0^\Theta(\omega')[\Theta(x, p)] = D(p, \omega') \end{aligned}$$

Lemma 6. $D(p, \omega)$ is continuous in p for all p realized in equilibrium.

Proof. A discontinuity in $D(p, \omega)$ can only result from a positive measure of agents declaring bids that are discontinuous at p . From Lemma 5 all agents declare their true demand in any 0-BNE, but then an agent declares a bid that is discontinuous at p only if he is indifferent between bundles of goods. Since $\pi_0^\Theta(\omega)$ is atomless, the set of agents indifferent between two bundles at any given price is a null set.³⁸ Therefore $D(p, \omega)$ is continuous. \square

Let $\Theta_\varepsilon(x, p) = \{\theta : \sigma_\varepsilon(\theta; p) \geq x\}$ denote a demand level set in a ε -BNE with equilibrium strategy σ_ε . Denote the associated aggregate demand in state $\omega \in \Omega$ as

$$D_\varepsilon(p, \omega) = \sum_{x=1}^K \pi_0^\Theta(\omega) [\Theta_\varepsilon(x, p)]$$

Lemma 7. Consider any ε -BNE with price process $p(\omega)$ and associated demand $D_\varepsilon(p, \omega)$. Let $D_T(p, \omega)$ denote the demand generated by exact best responses by all agents to $p(\omega)$.³⁹ For any $\delta > 0$ there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$ we have

$$|D_\varepsilon(p, \omega) - D_T(p, \omega)| < \frac{\delta}{2}$$

for all p in the support of $p(\omega)$.

Proof. From Lemma 5 we have that for any $\delta > 0$ we can choose $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$ we have⁴⁰

$$|\pi_0^\Theta(\omega) [\Theta_\varepsilon(x, p)] - \pi_0^\Theta(\omega) [\Theta(x, p)]| < \frac{\delta}{(K+1)K}$$

³⁸See the proof of Lemma 5 for an argument that such a set of indifferent agents is a null set under $\pi_0^\Theta(\omega)$.

³⁹ D_T is not necessarily a 0-BNE demand function. Note that when the agents bid their true expected demand, the agents condition on the realized price based on the price process of the ε -BNE.

⁴⁰The maximum deviation in demand is generated by a measure δ of agents in each of the $K+1$ level sets declaring either 0 or K unit demands. This inequality weakly bounds the change in measure of each level set generated by such a deviation.

for all $x \in \{0, \dots, K\}$. Therefore we have⁴¹

$$|D_\varepsilon(p, \omega) - D_T(p, \omega)| < \frac{\delta}{(K+1)K} * (K+1) * K = \delta$$

□

Note that the logic of Lemma 6 implies that $D_T(o, \omega)$ is continuous.⁴² Combining this fact with Lemma 7 implies that $D_\varepsilon(o, \omega)$ admits vanishing discontinuities as $\varepsilon \rightarrow 0$.

Now we move on to prove the lemmas in Section 4.

Proof. (proof of Lemma 1) Let $\bar{D}_\varepsilon(p; \omega)$ refer to the smallest monotone decreasing function such that $\bar{D}_\varepsilon(p; \omega) \geq D_\varepsilon(p; \omega)$.⁴³ Note that our market clearing condition (equation 4.1) holds if and only if for all $\gamma > 0$

$$\bar{D}_\varepsilon(p - \gamma; \omega) > r, \bar{D}_\varepsilon(p + \gamma; \omega) \leq r \tag{A.27}$$

We work with the downward sloping $\bar{D}_\varepsilon(p; \omega)$ for analytical ease.

Let $P(\omega) = \{p : p \in \text{supp}[p(\omega)]\}$ denote the support of the ε -BNE price distribution $p(\omega)$ in state $\omega \in \Omega$. Suppose there exists an equilibrium that is not fully revealing. Then there exists $\omega, \omega' \in \Omega$ such that $\omega > \omega'$ and some $p \in P(\omega) \cap P(\omega')$. Our market clearing conditions for any ε -BNE require that there exists $r \geq r' \in \text{supp}[v]$ such that for all $\gamma > 0$

$$\begin{aligned} \bar{D}_\varepsilon(p - \gamma; \omega) &> r, \bar{D}_\varepsilon(p - \gamma; \omega') > r' \\ \bar{D}_\varepsilon(p + \gamma; \omega) &\leq r, \bar{D}_\varepsilon(p + \gamma; \omega') \leq r' \end{aligned} \tag{A.28}$$

⁴¹The factor of $(K+1) * K$ reflects summing over $K+1$ level sets with a maximal change in demand of K resulting from the mass $\frac{\delta}{K(K+1)}$ added to that level set.

⁴²Note that Lemma 6 did not employ the fact that the prices, $p(\omega)$, were part of an equilibrium of any kind.

⁴³Formally

$$\bar{D}_\varepsilon(p; \omega) = \sup\{D_\varepsilon(p'; \omega) : p' \geq p\}$$

Note that equation A.28 implies

$$\lim_{\gamma \rightarrow 0} \overline{D}_\varepsilon(p + \gamma; \omega) - \overline{D}_\varepsilon(p - \gamma; \omega') \leq r - r'$$

Let D_T denote the demand generated by exact best responses by all agents to $p(\omega)$, and let $\overline{D}_T(p; \omega)$ refer to the smallest upper bound of $D_T(p; \omega)$. Using the assumption that $f(\theta|\omega)$ is strictly log supermodular and that strategies are monotone (Lemma), we have that at any price $p \in \cup_{\omega \in \Omega} P(\omega)$, $D_T(p, \omega) > D_T(p, \omega')$. Combined with the continuity of $D_T(p, \omega)$, we have that $\overline{D}_T(p, \omega) > \overline{D}_T(p, \omega')$.

From $\overline{D}_T(p, \omega) > \overline{D}_T(p, \omega')$ and Lemma 7 we have for χ, ε sufficiently small (alternatively r and r' sufficiently close and δ sufficiently small) that

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \overline{D}_\varepsilon(p + \gamma; \omega) - \overline{D}_\varepsilon(p - \gamma; \omega') &\geq \overline{D}_T(p; \omega) - \overline{D}_T(p; \omega') - \delta \\ &> \varkappa > r - r' \end{aligned}$$

From this contradiction, we conclude that it must be the case that all equilibria are fully revealing. \square

We now prove the approximate efficiency of the exact equilibria.

Proof. (of Lemma 2) We prove our lemma for the 0-BNE case and then employ a perturbation argument to prove the claim for any ε -BNE.

0-BNE: From Lemma 1 we know that the 0-BNE are fully revealing for $\chi > 0$ sufficiently small. Lemma 4 implies that any 0-BNE strategy of the nonatomic limit game is outcome equivalent to truthful declaration of ex-post demand. The First Welfare Theorem for exchange economies then implies the outcome is Pareto efficient. Pareto efficiency at the ex-post stage implies the equilibrium is ex-post (0, 0)-efficient.

ε -BNE: From Lemma 5, for any $\delta > 0$ we can choose $\varepsilon > 0$ sufficiently small that at least a measure $1 - \delta$ of the agents in any ε -BNE are truthful. From the

boundedness and continuity of v we can choose $M < \infty$ such that for all $\theta \in \Theta, \omega \in \Omega$

$$v(\theta, \omega, K) - v(\theta, \omega, 0) < M$$

This then implies that if a measure δ of agents declares a nontruthful bid, the maximal number of misallocated units is $K * \delta$ with an associated maximal efficiency loss of $M * \delta$ in any state $\omega \in \Omega$. For any $\gamma > 0$ we can choose $\bar{\varepsilon} > 0$ sufficiently small that $M * \delta < \gamma$ in any ε -BNE where $\varepsilon < \bar{\varepsilon}$. This then implies the allocation is ex-post $(\gamma, 0)$ -efficient in any ε -BNE. \square

We now prove the continuity result crucial for tying together the approximate nonatomic equilibria and the exact equilibria of the large finite game.

Proof. (Proof of Lemma 3) We break our proof into two successive arguments. We first show the continuity of market clearing price and use this fact to show continuity of the allocation function. The equicontinuity of the utility functions follows immediately. Note that $\{E^\Omega[v(\theta, \omega, x) - p \cdot x | \theta, \sigma]\}_{\theta \in \Theta, a \in \mathcal{A}}$ is uniformly bounded as a result of the boundedness of $v(\theta, \omega, x)$ and $p(\pi^A)$.

Continuity of the Market Clearing Price: Recollect that we are assessing continuity near the following distribution of bids

$$\pi_0^A(\omega)[A] = \int_{\Theta} \sigma(\theta)[A] * \pi_0^\Theta(\omega)[d\theta]$$

where σ is an arbitrary strategy. Consider any sequence of $\{\pi_k^A\}_{k=1}^\infty$ such that $d_K^A(\pi_k^A, \pi_0^A) = \beta_k$ and $\beta_k \rightarrow 0$. Define the aggregate demand function given a bid distribution π_k^A as

$$D(p; \pi_k^A) = \int q(p) \pi_k^A(dq) = \sum_{x=1}^K \pi_k^A(\{q : q(p) \geq x\})$$

As in the proof of Lemma 1, we will work with the smallest monotone decreasing function that is greater than $D(p; \pi_k^A)$ and denote this smallest upper bound as $\bar{D}(p; \pi_k^A)$. Since $d_K^A(\pi_k^A, \pi_0^A) \rightarrow 0$ it must be the case that $\pi_k^A(\{q : q(p) \geq x\}) \rightarrow$

$\pi_0^A(\{q : q(p) \geq x\})$ uniformly over p , which in turn implies that $\bar{D}(p; \pi_k^A) \rightarrow \bar{D}(p; \pi_0^A)$ uniformly over p . Therefore, we have

$$\begin{aligned}\bar{D}(p; \pi_k^A) + K * \beta_k &\geq \bar{D}(p; \pi_0^A) \\ \bar{D}(p; \pi_0^A) + K * \beta_k &\geq \bar{D}(p; \pi_k^A)\end{aligned}$$

Using our market clearing equations (equation A.28), the first relation implies that for all r , $p(\pi_k^A; r) \geq p(\pi_0^A; r + K * \beta_k)$. The second relation implies that for all r , $p(\pi_0^A; r) \geq p(\pi_k^A; r + K * \beta_k)$, which can be rewritten as $p(\pi_0^A; r - K * \beta_k) \geq p(\pi_k^A; r)$. Together we have

$$p(\pi_0^A; r + K * \beta_k) \leq p(\pi_k^A; r) \leq p(\pi_0^A; r - K * \beta_k)$$

Since $p(\pi_0^A, \circ)$ is decreasing, $p(\pi_0^A, \circ)$ is almost everywhere continuous with respect to r .⁴⁴ Therefore for almost all r ,

$$\lim_{k \rightarrow \infty} p(\pi_0^A; r + K * \beta_k) = \lim_{k \rightarrow \infty} p(\pi_0^A; r - K * \beta_k) = p(\pi_0^A; r)$$

We then have that for almost all r , $\lim_{k \rightarrow \infty} p(\pi_k^A; r) = p(\pi_0^A; r)$. Therefore, at the interim stage we have $\lim_{k \rightarrow \infty} p(\pi_k^A) = p(\pi_0^A)$ in the weak-* topology over the range of $p : \Delta(\mathcal{A}) \rightarrow \Delta(\mathbb{R})$.

Note that since the distribution of r is atomless, $p(\pi_0^A)$ admits atoms if and only if $D(p; \pi_0^A)$ has a discontinuity at some p . Denote the set of all such discontinuities as P_D . We can only have such a discontinuity in $D(p; \pi_0^A)$ if π_0^A places positive probability on a subdimensional subset of \mathcal{A} . A more economically intuitive way of stating this condition is to note that a bid may be described as a set of up to d prices where a step function begins and the size of the associated step (where the size is an element of $\{-K, \dots, K\}$) with the restriction that the bid is nonnegative at all prices. Any distribution π^A can then be interpreted as a measure over these price-size pairs, denoted as points $(p, x) \in \mathbb{R}_+ \times \{-K, \dots, K\}$. The set of points generating

⁴⁴Almost everywhere with respect to the Lebesgue measure (or equivalently v).

a discontinuity in $D(p; \pi_0^A)$ at price $p \in P^D$ is then the finite set $\{(p', x) : p' = p\}$, and π_0^A places atoms of positive probability on members of $\{(p', x) : p' = p\}$.

Since π_k^A converges to π_0^A in the Kolmogorov topology, this implies that for k sufficiently large π_k^A must contain atoms at the same (p, x) pairs as well. Furthermore, the masses on each atom of π_k^A uniformly converge to the masses of the corresponding atoms of π_0^A as $k \rightarrow \infty$.⁴⁵ This implies that for sufficiently large k , $D(p; \pi_k^A)$ has discontinuities at prices $p \in P_D$ and that the size of these discontinuities approaches the size of the corresponding discontinuities in $D(p; \pi_0^A)$ as $k \rightarrow \infty$. Therefore $p(\pi_k^A)$ admits atoms at the same prices as $p(\pi_0^A)$. Furthermore, the masses at each atom of $p(\pi_k^A)$ converges uniformly to the masses at the corresponding atoms of $p(\pi_0^A)$. Therefore we have $p(\pi_k^A) \rightarrow p(\pi_0^A)$ in the Kolmogorov topology.

$\{E^\Omega[v(\theta, \omega, x) - p \cdot x | \theta]\}_{\theta \in \Theta}$ is uniformly equicontinuous and uniformly bounded: Consider any sequence of $\{\pi_k^A\}_{k=1}^\infty$ such that $d_K^A(\pi_k^A, \pi_0^A) = \beta_k$ and $\beta_k \rightarrow 0$. Since any $q \in \mathcal{A}$ can be described as a step function with at most d steps, $p(\pi_k^A) \rightarrow p(\pi_0^A)$ in the Kolmogorov topology implies

$$\lim_{k \rightarrow \infty} \Pr\{q(p(\pi_k^A)) = i\} = \Pr\{q(p(\pi_0^A)) = i\}, i \in \{0, \dots, K\}$$

As the Kolmogorov topology insures the uniform convergence of the probability of each such event, we have that $\lim_{k \rightarrow \infty} x(\pi_k^A, q) = x(\pi_0^A, q)$ uniformly over q .⁴⁶

From the finiteness of $\{0, \dots, K\}$ and the linearity of utility with respect to lotteries in x , we immediately have equicontinuity of $\{E^\Omega[v(\theta, \omega, x) - p \cdot x | \theta]\}_{\theta \in \Theta, q \in \mathcal{A}}$. Uniformity follows from noting that \mathcal{A} , and hence $\Delta(\mathcal{A})$, are compact and that the Heine-Cantor theorem allows us to strengthen continuity to uniform continuity. \square

⁴⁵At this point we use the fact that convergence in the Kolmogorov topology is a refinement of the weak-* topology. If π_0^A admits atoms, convergence in the weak-* topology only implies that π_k^A places significant measure near those atoms for large enough k . The Kolmogorov topology implies the stronger conclusion that there must be an atom at the same point for large enough k .

⁴⁶Recollect that $x^i(\pi^A, q)$ is a $K + 1$ element vector

$$x_k^i(\pi^A, q) = \Pr\{q(p(\pi^A)) = k\}$$

We now use our continuity result to prove Theorem 6.

Proof. (of Theorem 6) Lemma 3 proves the requisite continuity condition to apply Theorem 4, which implies that for any $\varepsilon > 0$ we can choose N^* such that for $N > N^*$ any 0–BNE of the N agent game is an ε –BNE of the nonatomic limit game. We now prove each component of the theorem separately.

$p_N(\varphi) \rightarrow p(\omega)$ **almost surely:** Theorem 7 implies that for some $\omega \in \Omega$, $\pi^A(\varphi) \rightarrow \pi_\infty^N(\omega)$ and $\pi^\Theta(\varphi) \rightarrow \pi_0^\Theta(\omega)$ almost surely in the Kolmogorov topology where for all $A \in \mathcal{B}(\mathcal{A})$ we have

$$\begin{aligned}\pi^A(\varphi)[A] &= \int_{\Theta} \sigma^N(\theta)[A] * \pi^\Theta(\varphi)[d\theta] \\ \pi_\infty^N(\omega)[A] &= \int_{\Theta} \sigma^N(\theta)[A] * \pi_0^\Theta(\omega)[d\theta]\end{aligned}$$

and $\pi^A(\varphi)$ reflects the dependence of π^A on the realized $\varphi = (\omega, \theta_1, \dots, \theta_N, \dots)$. Since $p(\pi^A)$ is continuous in the Kolmogorov topology (see proof of Lemma 3) and $p_N(\pi^A)$ converges to $p(\pi^A)$ uniformly pointwise, we have $p_N(\varphi) \rightarrow p(\omega)$ almost surely in equilibrium.

Ex post (ε, ρ) –efficiency: Fix $\varepsilon > 0$ and $\rho \in (0, 1]$. Theorem 4 and Lemma 2 imply that for any N sufficiently large we have that σ^N is an ε –BNE that is $(\frac{\varepsilon}{2}, 0)$ –efficient in the nonatomic limit game.

From the continuity of x with respect to π^A (Lemma 3), the continuity of v with respect to θ and the convergence of $\pi^A(\varphi) \rightarrow \pi_\infty^N(\omega)$ and $\pi^\Theta(\varphi) \rightarrow \pi_0^\Theta(\omega)$, for any $\varepsilon > 0$ we can choose N^* such that for all $N > N^*$ with probability at least $1 - \rho$

$$\begin{aligned}\sum_{k=0}^K \int v(\theta, \omega, k) * \pi_0^\Theta(\omega)[d\theta] * x(\pi_\infty^N(\omega), \sigma^N(\theta))[k] \\ \leq \sum_{k=0}^K \int v(\theta, \omega, k) * \pi^\Theta(\varphi)[d\theta] * x(\pi^A(\varphi), \sigma^N(\theta))[k] + \frac{1}{6}\varepsilon\end{aligned}$$

where $x(\pi_\infty^N(\omega), \sigma^N(\theta))[k]$ and $x(\pi^A(\varphi), \sigma^N(\theta))[k]$ denote the probability that k units are allocated to the agent given his bid σ^N and respective bid distributions $\pi_\infty^N(\omega)$

and $\pi^A(\varphi)$. Since x_N converges to x uniformly pointwise, we have for large enough N

$$\begin{aligned} \sum_{k=0}^K \int v(\theta, \omega, k) * \pi_0^\Theta(\omega)[d\theta] * x(\pi_\infty^N(\omega), \sigma^N(\theta))[k] \\ \leq \sum_{k=0}^K \int v(\theta, \omega, k) * \pi^\Theta(\varphi)[d\theta] * x_N(\pi^A(\varphi), \sigma^N(\theta))[k] + \frac{1}{3}\varepsilon \end{aligned} \tag{A.29}$$

Note also that the continuity of $p(\circ)$ and the uniform convergence of p_N to p implies for sufficiently large N that $|p_N(\pi^A(\varphi)) - p(\pi_\infty^N(\omega))| < \frac{\varepsilon}{6}$ with probability at least $1 - \rho$. Combining this with equation A.29, we have that with probability at least $1 - \rho$ the total utility earned in the N agent auction is within $\frac{\varepsilon}{2}$ of the ex post utility earned in the nonatomic limit game wherein each agent plays σ^N .

Since σ^N played by all agents in the nonatomic limit game is ex-post $(\frac{\varepsilon}{2}, 0)$ -efficient, this implies that for N sufficiently large any 0-BNE of the large finite auction is (ε, ρ) -efficient in the N -agent auction. \square