On the Optimum Distribution of Money*

Ricardo de O. Cavalcanti† and Paulo K. Monteiro
University of Toronto and EPGE/FGV

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Abstract

Although the *quantity of money* is a concept deeply rooted in the history of economic thought, the truth is that money balances tend to disperse in models with idiosyncratic shocks, except when strong assumptions are imposed. This is so because monetary allocations are poor substitutes for perfect insurance. In this paper, we design the optimum money distribution when trade takes place according to simple pairwise meetings. The bad news is that the dimensionality of the problem is intimidating. The good news is that it is a common-sense generalization of the Ramsey problem in mainstream macroeconomics. The results are two: temporary incentive constraints suffice for defining the recursive problem without prior computation of the set of admissible states, and the optimum is attained when money holdings and aggregate shocks are discrete.

1 Introduction

The purpose of this article is twofold. First, to pose the problem of distributing money optimally in a random-matching economy, together with

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†Corresponding author. Please send correspondence to Department of Economics, 150 Saint George St., Toronto, Ontario M5S3G7, Canada.

conditions assuring that the optimum is attained. Second, to establish a connection between our analysis and that of Ramsey, a cornerstone formulation for a vast macroeconomics literature.

Models of medium of exchange become remarkably complex when general holdings and aggregate uncertainty are introduced. A connection with the Ramsey problem can provide tools and new insights for addressing substantive questions in monetary theory. That is the motivation why our mechanism-design problem is also described recursively. The similarities with the second-best problem of allocating intertemporal distortions shall become evident.

The result on the existence of an optimum distribution of money has, or course, its debt to recent work in matching models, following Kiyotaki and Wright [6]. It is in the attempt to find an algorithm for describing the solution that lies the connection to macroeconomics, in particular with seminal insights by Kydland and Prescott [7]. Their Ramsey problem has a single policymaker taxing capital and labor income, as well as providing public goods. Taxation, assumed distortionary, constitutes an externality that disconnects equilibrium and Pareto-optimal allocations. Consequently, the traditional recursive problem, indexed by *physical* state variables, no longer provides an algorithm for optimality. The remedy provided by [7] is a common-sense reinterpretation of states, which we shall extend to the money distribution problem.

The standard application of the Bellman's principle of optimality has a physical state, such as capital, summarizing the relevant past. In a competitive equilibrium without externalities, marginal rates of substitution and transformation can be equalized, and the physical state is a sufficient statistic for describing past outcomes. When taxation distorts marginal conditions, the planner has to choose among several outcomes, including intertemporal distortions, for the same choice of investment levels. Consistency with rational expectations requires, as a result, an additional state recording the selected distortion. The new variable, let us call it the *expectations* state, is a device to ensure that the effects of future planner's choices on individual's behavior in earlier periods are taken into account.

We claim that the problem of how to distribute money optimally can be posed recursively. We provide, in the next section of this paper, an informal presentation of the results that facilitates understanding the connection to Ramsey. We would like however to call attention up front for a difficulty that the approach in [7] apparently introduces, and how we react to it.

The difficulty is that, in order to use recursive methods, the correspondence mapping states into the set of feasible controls must be nonempty. However, preferences and technologies alone do not provide restrictions for ruling out emptiness. There is a concern about selecting current outcomes and expectations that cannot be sustained in the future by bounded expectations. The approach in [7] is to compute a set of admissible states *before* the recursive planner's problem is defined. They noticed that the set can be computed recursively as the largest fixed point of an operator defined on compact sets. Its fixed points have a striking resemblance with the self-generating sets of promises, arising in studies of sequential equilibria for repeated games, after [1].[1] In our approach, the set is computed simultaneously with optimal policies.

Last, but not least, we would like to highlight the relative contribution in our existence proof. We show that the optimum is attained when money holdings and aggregate shocks are discrete. A bound on output and the aggregate stock of money is required, but not on individual holdings. A contrast can be made with search models of money emphasizing bargaining theory. For instance, [9] assign to money holdings a small flow of utility in order to demonstrate the existence of steady states. They impose moreover a bargain rule keeping sellers indifferent between trading and not trading. They find a fixed point when pairs of current distributions and future expectations are mapped into pairs of future distributions and current expectations. This approach can only be suitable to steady-state analysis, and the special assumption on bargaining is necessary to assure convexity of the mapping to which the fixed-point theorem applies. Our mechanism-design approach, by contrast, dispenses with stationarity and ad-hoc bargaining restrictions, since allocations are chosen according to optimality.[2]

[1] This link can indeed be relevant for monetary models with imperfect monitoring. [8] provides a seminal analysis of second-best allocations in the context of markets and government transfers. [4] focus instead on the set of implementable allocations in the context of pairwise meetings and inside money (in the spirit of [3]). Lack of commitment by the planner can be introduced as in the voluminous literature of time-consistent public policies ([5] and references therein).

[2] [10] finds steady states by imposing the same bargaining and bounds on individual holdings as [9], but is able to dispense with the utility given to balances. References in [10] provide alternative formulations. In particular, an early attempt to introduce utility promises is [2] with 0-1 holdings and a continuum of money colors. An absolutely continuous distribution of utilities is found by solving a Volterra integral equation.

2 An informal exposition

In the Ramsey problem of [7] a value function for the planner is a real, bounded function w mapping the state triple (μ, v, z) into society's welfare. The variable μ is the physical state (capital holdings). The variable v is the expectations state (the expected marginal-utility term of the individual's savings decision). The variable z is the exogenous state (non controllable preferences or political shocks). For $x = (\mu, v, z)$, a choice of controllable states (μ', v') is temporarily incentive-feasible if it is an element of $\Gamma(x)$. The largest set X for which $\Gamma(X)$ is bounded is not known. Their approach is to find X recursively, according to the constraints defining the correspondence Γ. Only later in a second step, the society's period utility is considered, and optimal policies, together with w, are found. In a third step, $w(\mu_0, \cdot, z_0)$ is maximized over the set of admissible expectations v such that (μ_0, v, z_0) belong to X, for given initial μ_0 and z_0.

In our monetary problem μ is a probability measure describing the previous distribution of balances across individuals. Although individuals are heterogenous in terms of preferences, as in [6], only allocations that are symmetric over individual types are considered for simplicity. Money is fiat and in constant supply. As a result μ is also a physical state. The other two variables, v and z, have a similar interpretation as in [7]. The aggregate shock z has an exogenous law of motion, assumed *iid* for simplicity, and affects individual's preferences in a symmetric way. What really produces heterogeneity in the population is the fact that exchanges are monetary and according to uninsurable random meetings. More formally, we assume that individuals cannot commit to future actions (but the planner can) and cannot be monitored: all individual histories are private.

We are able to restore the representative-agent structure of Ramsey by assuming that an optimum is chosen according to average utility. More formally, μ induces a probability measure τ on the sets of pairwise meetings, with the understanding that a meeting type describes the holdings of two individuals, as well as their preferences. Also for simplicity we assume that holdings and preferences are observable by the two individuals in the meeting.

If x is the current state then our average utility criterion yields a society's period payoff $F(x, \mu', v')$, for a bounded function F, according to the pairwise trades allowed by μ. We now describe how choices for future states (μ', v') affect average utility. As in [6], the nontrivial meetings have a potential consumer, holding m units of money, and a potential producer, holding

n units of money. For each such a meeting, the planner recommends an allocation, which is allowed to vary according to the aggregate state x. The recommended allocation has a level of output y, in the form of a perishable consumption good, that the producer transfers to the consumer. It also has an after-trade holdings g for the consumer, as a result of the transfer of $m-g$ units to the producer. Seen as a function of (m,n), an allocation (y,g) defines a new distribution μ' (according to μ and g), as well as an average payoff (according to μ and y).

An allocation is implementable if it satisfies participation constraints dictated by the fact that anonymous traders can leave a meeting with the initial holdings brought to the meeting. This is the notion of weak implementability used by [4], which here requires an assignment of continuation values v'. Rational expectations, requires that the previous v be consistent with expectations about v' (also according to μ, g and y). Because the distribution of money does not collapse into a trivial one, v has to be defined as a function expressing promises to all possible types, contingent on realizations z. For each z, the domain of v is that of money holdings whose Borel sets are measured according to μ.

As the paragraphs above suggest, we avoid the formidable task of seeking marginal conditions representing individual rationality. We simply impose an expectations function as a state. In summary, in our model of heterogenous agents, the physical state in [7] is extended to a distribution, and the expectations state is extended to a (continuation value) function. The perspective of a single decision maker is still valid in this high-dimensional setting. We now explain the main results.

We derive an upper bound K for expectations v provided by discounting and bounded output. We then define the correspondence Γ by imposing temporary constraints. Next we define the return function F_K, with image on the extended reals $\mathbb{R}_+ \cup \{-\infty\}$, by using $F(x,\mu',v')$ for $(\mu',v') \in \Gamma(x)$ and $\|v'\|_\infty \leq K$, and $-\infty$ otherwise. We then use $\sup\{F_K(x,\cdot) + \beta E^z w(\cdot)\}$, where the supremum is taken with respect to controls, and the expectation with respect to shock realizations, in order to define a mapping on the space welfare functions w, bounded from above. We then show that, starting with any bounded w_0, the mapping defines a sequence $\{w_i\}$ converging to a unique fixed point w^*. The support of w^* is the set of admissible promises X. Convergence of $\{w_i\}$, restricted to X, takes place according to the contraction mapping theorem. Since no continuity assumption is actually used, nonemptiness requires a separate argument. But that follows from the fact that simple trades like those in [6] with 0-1 holdings are trivially implementable.

With regards to the issue of attaining the optimum by a specific initial distribution of money, we proceed by recalling that the equivalence between this recursive formulation and the corresponding sequential problem holds. Then, in a corollary of a lemma about the first moments of any feasible $\{\mu_t\}$, we show that w^* has the analytical solution $\int v d\mu$ on the set of admissible states.[3] We finally show that when holdings and discrete, and bounded at the aggregate level, the product topology can be used to assure that the optimum is attained by some $\{\mu_t\}$. The required bound on aggregate holdings can be set arbitrarily large, but it cannot be dispensed with. The planner would like to use unbounded quantities of indivisible money in order to approach the welfare supported by divisible money, as divisibility weakens feasibility constraints.[4]

The rest of this paper is organized as follows. The monetary environment is described in section 3. The lemma characterizing feasible sequences of distributions is presented in section 4. The corollary about average utility is presented in section 5. That w^* satisfies a functional equation with contracting properties is proved in the theorem of section 5. That the optimum is attained with discrete holdings, as the maximization of a continuous function over a compact set, is proved in section 6.

3 The monetary environment

Time is discrete and the horizon is infinite. The economy is populated by a continuum of individuals symmetrically divided according to an integer number of types N, where $N \geq 3$. There are N consumption goods per date assumed divisible and perishable. People meet once every period according

[3] This is so under the assumption that discounted utilities are averaged according to μ_0, as if individuals are initially assigned to holdings according to the initial distribution of money. If the support for holdings is just 0-1 then it is possible to consider a scalar promise, instead of (v_0, v_1). The dimensionality of the problem would be the same as that in [7], but the analytical form for w^* is lost with this change of variables.

[4] This property is also found in [2]: a continuum of money colors improve welfare when asset holdings are restricted to 0-1. [2] has consumers and producers swapping holdings whenever doing so is incentive-feasible. The outcome is thus a lower bound for $\int v d\mu$. We thank Neil Wallace for calling our attention to this property. This lower bound on welfare can be used in applications by setting $F_K(x, \mu', v') = -\infty$, whenever $\int v' d\mu'$ falls below the lower bound, as a device for ignoring suboptimal states.

to random, pairwise meetings, as in [6], and consumption by one person can only be provided by the meeting partner, and only if a coincidence of types occurs. More specifically, type i enjoys good i but can only produce good $i+1$, modulo N. Since $N > 2$ no double-coincidence in which two individuals consume can occur.

If $c \in \mathbb{R}_+$ is produced by type $i-1$ and consumed by type i then the period utility for the consumer is $u_z(c)$, and the period utility for the producer is $-c$, independently of i, where z, the realization of an aggregate shock, is common to all types in a period. Individuals discount future utilities according to the common discount factor β, where $\beta \in (0, 1)$. For our existence result, the support of the aggregate shock is assumed discrete. Although they could be derived under the assumption that shocks follow a first-order Markov process, we find it easier to expose the results under the assumption of *iid* shocks. For each realization z, the function $u_z(\cdot)$ is defined on $\mathbb{R}_+$ and assumed continuous and increasing. Output is assumed bounded by a positive constant $y_{\max}$, and units are normalized so that $u_z(0) = 0$ for all z. We assume moreover that $u_z(y_{\max}) - y_{\max} \leq 0$ for all z.

People cannot commit to future actions and their histories are private. The set of possible money holdings is $\mathbb{R}_+$. We assume that holdings and types are observable by participants in a meeting. We follow [4] in describing allocations that are symmetric over types but we do not impose stationarity. We also follow their concept of implementability. In a single-coincidence meeting (type i meets type $i-1$), in which the potential consumer holds m and the potential producer holds n, an allocation is a recommendation $(y(m, n), g(m, n)) \in \mathbb{R}^2_+$ of output and after-trade holdings for the consumer. It is physically feasible if $y \leq y_{\max}$ and $g \leq m$, and is incentive-feasible if it satisfies two participation constraints. The constraints are dictated by the option that each individual has of walking away from the meeting with the holdings brought to the meeting.

People in the meeting play an *agree-disagree* response to the recommendation. For the proper generalization to contingent sequences of meetings, an implementable allocation is a recommendation for all periods, meeting types, and shock realizations, such that *agree* in all meetings is a subgame perfect equilibrium. The participation constraints, one for the producer,

$$y(m, n) \leq \beta E^z \left[v'(m + n - g(m, n)) - v'(n) \right],$$

and one for the consumer,

$$u_z\left(y\left(m,n\right)\right) \geq \beta E^z\left[v'\left(m\right) - v'\left(g\left(m,n\right)\right)\right],$$

written with continuation values v' and expectation operator E^z, to be defined below, guarantee that an allocation is a subgame perfect equilibrium of some game.

4 Distribution dynamics

We denote by $\mathcal{M}$ be the set of Borelean probability measures on $\mathbb{R}_+$. If $\mu \in \mathcal{M}$ we write $\tau_\mu := \mu \times \mu$ for the product measure. Let $(Z, \mathcal{Z}, P)$ be a probability space. The set of exogenous shocks that may occur at each date is Z. In what follows, an integral of a function, such as ϕ, with respect to a measure, such as τ, can be written either as $\int \phi d\tau$ or $\int \phi\tau(d(m,n))$, depending whether we want to call attention to the dummy variables of the sets on which the measure is defined. Let

$$\begin{aligned} V &= \{v : \mathbb{R}_+ \to \mathbb{R}_+,\ v \text{ is increasing.}\}; \\ Y &= \left\{g : \mathbb{R}_+^2 \to \mathbb{R}_+;\ g \text{ is measurable}\right\}. \end{aligned}$$

If $g \in Y$ and $\lambda \in \mathbb{R}_+$ we define the set

$$[g \leq \lambda] := \left\{(m,n) \in \mathbb{R}_+^2; g\left(m,n\right) \leq \lambda\right\}.$$

We define for each $\mu \in \mathcal{M}$, $g \in Y$ and $h = m+n-g$, the probability measure $\hat{\mu} = \hat{\mu}\left(g\right)$, by

$$\hat{\mu}\left([0,\lambda]\right) = \left(1 - \frac{2}{N}\right)\mu\left([0,\lambda]\right) + \frac{\tau_\mu\left([g \leq \lambda]\right) + \tau_\mu\left([h \leq \lambda]\right)}{N} \tag{1}$$

for each $\lambda \in \mathbb{R}_+$. The equation (1) gives the next measure, $\hat{\mu}$, after μ, when the after-trade holdings of the consumer is given by g in single-coincidence meetings, which takes place with probability $\frac{1}{N}$. The measure $\hat{\mu}$ is unique. An immediate consequence of (1) is the

Lemma 1 *For every measurable $\phi : \mathbb{R}_+ \to \mathbb{R}_+$,*

$$\int \phi d\hat{\mu} = \left(1 - \frac{2}{N}\right)\int \phi d\mu + \frac{1}{N}\int \left(\phi \circ g + \phi \circ h\right) d\tau_\mu. \tag{2}$$

In particular $\int m\hat{\mu}\left(dm\right) = \int m\mu\left(dm\right)$.

Proof. An usual application of the monotone convergence theorem in (2) shows that it suffices to prove the lemma for simple functions. And from the linearity of (2) it is enough to prove the lemma for characteristic functions χ_A, A Borelean. Thus let

$$\mathcal{C} = \{A \in \mathcal{B};\ (2) \text{ is true for } \phi = \chi_A\}.$$

Since μ and τ_μ are probability measure if $A \in \mathcal{C}$ then $A^c \in \mathcal{C}$. Also if $A^n \in \mathcal{C}$, $A^n \uparrow A$ then the monotone convergence theorem implies that $A \in \mathcal{C}$. Also if $B \subset A$ and $A, B \in \mathcal{C}$ then since $\chi_{A \setminus B} = \chi_A - \chi_B$, $A \setminus B \in \mathcal{C}$. Therefore $\mathcal{C}$ is a $\sigma-$algebra. Since by the definition of $\hat{\mu}$, $[0, \lambda] \in \mathcal{C}$ for every $\lambda \geq 0$, we have that $\mathcal{C} = \mathcal{B}$. Finally if $\phi(m) = m$ then $\phi \circ g + \phi \circ h \equiv m + n$ and therefore the right-hand side of (2) is $\int m\mu(dm)$. □

5 The monetary payoffs

Let $S^b = \mathcal{M} \times V^Z \times Y \times Y$ be the set of endogenous states and balance-adjustment variables. The augmented set of states is $S^a = S^b \times Z$. Thus if $(\mu, v, g, y, z) \in S^a$ we have that $v \in V^Z$. That is, for every $\omega \in Z$, $v_\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is the individual value function at state ω, and $v_\omega(m)$ is the expected utility of holding $m \geq 0$ if the realized state is ω. And $g(m, n)$ is the quantity of money left for the consumer with $m \geq 0$ after a meeting with a producer with $n \geq 0$ who supplies $y(m, n)$ of product in this meeting. The measure μ is the distribution of money endowments. Let $\tilde{\beta} := \beta\left(1 - \frac{2}{N}\right)$. We now describe the constraints on choices by the correspondence $\Gamma^b : \mathcal{M} \times V^Z \times Z \to S^b$. For each $(\mu, v) \in \mathcal{M} \times V^Z$, and for every current shock $z \in Z$, define the set $\Gamma^b(\mu, v, z)$ of functions $v' : Z \to V$, and $(\mu', v', g, y) \in S^b$ such that for every $\omega \in Z$, $m, n \geq 0$:

$$\begin{aligned} v_z(m) &= \tilde{\beta} E^z[v'(m)] + \frac{1}{N}\int (u_z(y(m,n)) - y(n,m))\,\mu(dn) \\ &\quad + \beta\frac{1}{N}\int E^z[v'(g(m,n)) + v'(h(n,m))]\,\mu(dn); \qquad (3) \\ 0 &\leq g(m,n) \leq m; \\ y(m,n) &\leq \beta E^z[v'(m+n-g(m,n)) - v'(n)]; \\ u_z(y(m,n)) &\geq \beta E^z[v'(m) - v'(g(m,n))]; \\ \mu' &= \hat{\mu}(g), \end{aligned}$$

where $\hat{\mu}(g)$ is defined by (1) above, and the expectation is taken on $\omega \in Z$: $E^z[v'(m)] = \int v'_\omega(m) P(d\omega)$. Define $k_z(m,n) = u_z(y(m,n)) - y(m,n)$.

Corollary 2 *If* $(\mu', v', g, y) \in \Gamma^b(\mu, v, z)$ *and* $\hat{\mu} = \hat{\mu}(g)$ *then*

$$\int v_z(m)\,\mu(dm) = \frac{1}{N}\int k_z(m,n)\,\tau_\mu(d(m,n)) + \beta\int E^z[v'(m)]\,\hat{\mu}(dm). \quad (4)$$

Proof. Define $q_z(m,n) = q_z(y) = u_z(y(m,n)) - y(n,m)$. From (3) we have that

$$\int v_z d\mu = \frac{1}{N}\int q_z d\tau_\mu + \tilde{\beta}\int E^z[v']\,d\mu + \frac{\beta}{N}\int E^z[v'\circ g + v'\circ h]\,d\tau_\mu.$$

From (2) applied for $\phi = v'_\omega$ we have that

$$\frac{\beta}{N}\int (v'_\omega\circ g + v'_\omega\circ h)\,d\tau_\mu = \beta\int v'_\omega d\hat{\mu} - \tilde{\beta}\int v'_\omega d\mu.$$

Thus taking expectation in Z:

$$\tilde{\beta}\int E^z[v']\,d\mu + \frac{\beta}{N}\int E^z[v'\circ g' + v'\circ h']\,d\tau_\mu = \beta\int E^z[v']\,d\hat{\mu}.$$

Therefore

$$\int v_z d\mu = \frac{1}{N}\int q_z(y)\,d\tau_\mu + \beta\int E^z[v']\,d\hat{\mu}.$$

Finally, since $\int\left(\int y(n,m)\,\mu(dn)\right)\mu(dm) = \int y(n,m)\,\mu(dn)\,\mu(dm)$, which is the same as $\int y(m,n)\,\mu(dm)\,\mu(dn)$, the proof is finished. $\square$

We have assumed that output and utilities are bounded, with $u_z(y_{\max}) - y_{\max} \leq 0$ for every $z \in Z$. Define $Y^* = \{y \in Y; y \leq y_{\max}\}$.

6 Reduced form

Since production is bounded, the maximum utility a consumer gets in a meeting does not exceed $\overline{U} := \sup_{z\in Z} u_z(y_{\max})$. Let $K = \frac{\overline{U}}{1-\beta}$. We restrict value functions by K. Define for $x = (\mu, v)$ and $(x,z) \in \mathcal{M}\times V^Z \times Z$ the correspondences

$$\Gamma_\infty(x,z) = \left\{x' \in \mathcal{M}\times V^Z; \exists (y,g) \in Y^*\times Y, (x',g,y) \in \Gamma^b(x,z)\right\}$$

and

$$\Gamma_K(x,z) = \{(\mu',v') \in \Gamma_\infty(x,z); \|v'\|_\infty \le K\}.$$

Define now for $(x,x') \in \left(\mathcal{M} \times V^Z\right)^2$ and $z \in Z$ the set

$$A(x,x',z) = \{y \in Y^*; \exists g \in Y, (x',g,y) \in \Gamma(x,z)\}.$$

Define also

$$F_K(x,x',z) = \begin{cases} \sup_{y \in A(x,x',z)} \int (u_z(y) - y)\, d\tau_\mu & \text{if } x' \in \Gamma_K(x,z); \\ -\infty & \text{if } x' \notin \Gamma_K(x,z). \end{cases}$$

Let $H := \mathcal{M} \times V^Z$ and $\tilde{H} := H \times Z$.

Theorem 3 *There exists the largest function $w^* : \tilde{H} \to \mathbb{R} \cup \{-\infty\}$ such that*

$$w^*(x,z) = \sup_{x' \in H} F_K(x,x',z) + \beta E^z [w^*(x',\cdot)]. \tag{5}$$

Proof. We define an operator $\Phi : \mathcal{F}\left(\tilde{H}, \mathbb{R} \cup \{-\infty\}\right) \to \mathcal{F}\left(\tilde{H}, \mathbb{R} \cup \{-\infty\}\right)$. For any $w : \tilde{H} \to \mathbb{R} \cup \{-\infty\}$ and for every $x = (\mu, v)$,

$$[\Phi w](x,z) = \sup_{x' \in H} F_K(x,x',z) + \beta E^z [w(x',\cdot)].$$

If w_0:$\tilde{H} \to \mathbb{R}$ is bounded we define the sequence of iterations, $w_1 = \Phi(w_0)$, $w_{n+1} = \Phi(w_n)$. The set U_1 is defined as

$$U_1 = \left\{(x,z) \in \tilde{H},\ \Gamma_K(x,z) = \emptyset\right\} = [w_1 = -\infty].$$

The set U_2 is then defined as

$$\begin{aligned} U_2 &= \left\{(x,z) \in \tilde{H}; \exists z' \in Z, \forall (x',z') \in \Gamma_K(x,z), \Gamma_K(x',z') = \emptyset\right\} \\ &= [w_2 = -\infty]. \end{aligned}$$

and so on,

$$\begin{aligned} U_{j+1} &= \left\{(x,z) \in \tilde{H}; \forall (x_l)_{l=1}^k; x_l \in \Gamma_K(x_{l-1}, z_{l-1}), \Gamma_K(x_l, z_l) = \emptyset\right\} \\ &= [w_{j+1} = -\infty]. \end{aligned}$$

The sequence is increasing, $U_j \subset U_{j+1}$. Let $U = \cup_{j=1}^{\infty} U_j$. In the complement $X = U^c$, the sequence $(w_j|_{U^c})_{j=1}^{\infty}$is a contraction. Just use Blackwell's theorem to get

$$|[\Phi w](x,z) - [\Phi \hat{w}](x,z)| \leq \beta \sup_{x',z'} |w(x',z') - \hat{w}(x',z')|.$$

The rest follows from well known properties of contractive functionals. □

7 Existence of optima with indivisible money

In this section, we prove that there exists an optimal distribution of money, with respect to the welfare criteria of average payoff, for the case in which the sets of money holdings and shock realizations are discrete. We denote by M the average money holdings. We suppose that there is a $\overline{M}$ such that $M \leq \overline{M}$. The equivalence between recursive and sequential planner's problems is standard. The social planner problem can therefore be described as that of choosing a sequence of consumption plans $(y^t)_{t=1}^{\infty}$, $y^t : Z \to Y$ a sequence of monetary expectations $(v^t)_{t=0}^{\infty}$ a sequence of monetary exchanges $(g^t, h^t)_{t=1}^{\infty}$, $g^t, h^t : Z \to Y$, and a sequence of measures $(\mu^t)_{t=0}^{\infty}$ such that $\mu^{t+1} = \widehat{\mu^t}(g_{z^t}^t)$ and $\int m\mu(dm) \leq \overline{M}$, that maximizes

$$E \sum_{t=0}^{\infty} \beta^t \int \left(u\left(y^t\right) - y^t\right) d\tau^t.$$

Let $\boldsymbol{Z} = (\mathcal{M} \times V^Z \times Y^2 \times Z)^{\mathbb{N}}$. To be more precise, define for each $(\mu, v, g, y) \in S^b$, the set $\mathbf{S}(\mu, v, g, y)$ of sequences $(\mu^t, v^t, g^t, y^t, z^t)_{t=1}^{\infty} \in \boldsymbol{Z}$ such that $(\mu^t, v^t, g^t, y^t) \in \Gamma^b(\mu^{t-1}, v^{t-1}, \mu^{t-1}, z^{t-1})$ and $\mu^t = \widehat{\mu^t}(g^t)$. By defining also

$$\mathbf{S} = \left\{(\mu, v, g, y, s); s \in \mathbf{S}(\mu, v, g, y), \int m d\mu \leq \overline{M}\right\},$$

the planner's problem, for the average-payoff criteria, can be written as

$$\max_{(\mu,v,g,y,s)\in\mathbf{S}} E \sum_{t=1}^{\infty} \beta^t \int k_z(y^t) d\tau^{t-1} \tag{6}$$

with the understanding that $k_z(y_t(m,n)) = u_z(y_t(m,n)) - y_t(m,n)$. If we use induction in (4) we obtain

$$\int v d\mu = E \sum_{t=0}^{\infty} \beta^t \int \left(u\left(y^t\right) - y^t\right) d\tau^t + \lim_{t\to\infty} \beta^t E \int v^{t+1} d\mu^{t+1}. \tag{7}$$

Since (v^t) is bounded we conclude that

$$\int v d\mu = E \sum_{t=0}^{\infty} \beta^t \int \left(u\left(y^t\right) - y^t\right) d\tau^t. \tag{8}$$

Thus the planner's problem is thus that of maximizing $\int v d\mu$ amongst the feasible ones. From now on we suppose that $m \in \mathbb{Z}_+$. We use the product topology on the set of feasible paths. This set is compact:

Lemma 4 **S** *is compact in the product topology.*

Claim 5 *The function* $\int v d\mu$ *is continuous.*

Proof. Suppose $v^l \to v$ and $\mu^l \to \mu$. Since $\int m d\mu^l(m) \leq \overline{M}$ we have that for any $\epsilon > 0$, $\mu^l([\frac{\overline{M}}{\epsilon}, \infty)) \leq \epsilon$.Since v^l is uniformly bounded, it suffices to show therefore that $\int v^l d\mu$ converges to $\int v d\mu$. This now follows from Lebesgue's dominated convergence theorem. It remains to be proven that the set of feasible paths is closed. But this is easy since $g^t(m, n) \in \mathbb{Z}_+$. □

We have thus proved the following

Theorem 6 *If money is indivisible, there is an optimal sequence of distributions of money.*

The following theorem may be useful for computations.

Theorem 7 *Let us, for each integer p, maximize* $\int v d\mu$ *in the space of sequences*

$$(\mu^t, v^t, g^t, y^t, z^t)_{t=1}^p$$

such that $(\mu^t, v^t, g^t, y^t, z^t) \in \Gamma^b(\mu^{t-1}, v^{t-1}, g^{t-1}, y^{t-1}, z^{t-1})$ *and* $\mu^t = \hat{\mu^t}(g^t)$. *Let* α_p *be this maximum. Then* $\lim_{p\to\infty} \alpha_p$ *is the monetary optimum.*

Proof. It shall be clear that α_p is decreasing. Thus $\lim_{p\to\infty} \alpha_p = \inf_p \alpha_p$. The same reasoning that implies the compactness of $\boldsymbol{S}$ guarantee the compactness, for each p, of the feasible set of (μ, v). The sequence (μ^p, v^p) has a converging subsequence and the limit of this subsequence is an element of $\boldsymbol{S}$. Therefore the monetary optimum is greater than or equal to $\lim_{p\to\infty} \alpha_p$. But since α_p is greater than or equal to the monetary optimum (since α_p is obtained from a set with less restrictions) the proof is finished. □

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