Investment Cycles, Strategic Delay, and Self-Correcting Cascades*

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Abstract

We study investment cycles and information flows in a model of social learning in which investment returns fluctuate according to a Markov process. In our Waiting Game, agents observe the investment history and a private signal correlated with the current period's investment return. Agents then decide whether to invest in round 1 or to delay their decision to round 2 of the current period. Cascades in which everyone invests and no one invests eventually correct themselves. As compared to the No-Waiting Game with no opportunity for delay, the Waiting Game has shorter investment cascades, longer recessions, and shorter booms. The Waiting Game also has more underinvestment and less overinvestment.

1 Introduction

Some important features of investment markets are that agents are asymmetrically informed about investment returns, agents can choose to delay investment in order to learn from market activity, and investment returns fluctuate over time. We build a model capturing all three of these features, characterize the extent to which markets aggregate private information, and investigate the implications for investment cycles. We find that information cascades, in which either all agents invest or no agents invest, occur but eventually reverse themselves. Moreover, we find that the option to delay investment affects the dynamics of information aggregation and information cascades, which leads to shorter investment booms and longer recessions.

Our model extends the seminal work of Chamley and Gale (1994). Specifically, the investment state can be either high or low, and agents privately observe a binary signal that is correlated with the investment return. After observing their signal and the history of transactions, remaining agents simultaneously decide whether to invest right away or wait.[1] Our innovation is to introduce fluctuations in the investment return, which now evolves according to a Markov process. In each period, a new generation of agents is born, and each agent observes the history of previous investment decisions, receives a signal correlated with the current investment return, and decides whether to invest.

We consider two scenarios. In the No-Waiting Game, each period consists of one round, so agents have only one opportunity to invest. The No-Waiting Game is a benchmark that is related to the literature on herding with exogenous timing of moves.[2] We find that, depending on the beginning-of-period expected

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[1] See also the extension by Levin and Peck (2006) to two-dimensional signals, in which agents receive a common-value signal, either high or low, and a private-value signal (interpreted as the cost of investment), drawn from a continuous distribution.

[2] See, in particular, Banerjee (1992), Bikhchandani, Hirshleifer, and Welch (1992), Moscarini, Ottaviani, and Smith (1998), and Smith and Sorensen (2000).

investment return, the economy can be in one of three possible regimes in equilibrium: Regime 0 in which no agent invests, Regime 2 in which every agent invests, and Regime 1 in which only the agents who receive the high signal invest. Aggregate investment reveals no information in Regime 0 or 2, while it reveals all private information in Regime 1. The information revealed during period t, as well as the possibility that the investment state has changed from one period to the next, affect the initial beliefs for the next period. The economy randomly fluctuates between Regime 0 cascades and Regime 2 cascades. When a cascade ends, the economy moves into Regime 1 for one period, which aggregates information and initiates a new cascade.

In the Waiting Game, there is the possibility of strategic delay, because each period contains two rounds. An agent that does not invest in round 1 observes the history of investment in both rounds of all previous periods, and round 1 of the current period, before deciding whether to invest in round 2. Thus, the investment state and each agent's signal remains constant across the two rounds of the period in which an agent is alive. The payoff of investment in the second round is discounted, but the ability to observe market activity gives rise to option value of waiting. We find that, depending on the beginning-of-period expected investment return, the economy can be in one of four possible regimes in equilibrium. Besides Regimes 0, 1, and 2, there is Regime M, in which agents receiving the high signal mix, investing in round 1 with probability between zero and one, and those receiving the low signal wait. In Regime 1 and Regime M, investment activity in round 2 depends on the amount of investment in round 1.

Beliefs about the expected investment return determine which regime the economy is in. We find the following major differences between the No-Waiting game and the Waiting game. First, the range of beliefs giving rise to Regime 2 is narrower in the Waiting Game than in the No-Waiting Game, which implies that a Regime 2 cascade is shorter. Intuitively, Regime 2 ends when agents with the low signal do not invest. In the Waiting Game, these agents delay their investment as soon as the profits from investing immediately are outweighed by the option value of waiting; in the No-Waiting Game, these agents decide not to invest only when the profits from investing are negative. Second, the economy typically enters Regime M when it emerges from Regime 0 in the Waiting Game, but it enters Regime 1 in the No-Waiting Game. Intuitively, Regime 0 ends when investment first becomes profitable for agents with the high signal. Since these profits are typically small, not all of these agents can invest in the Waiting Game, because then a lot of information is learned by waiting. On the other hand, not all of these agents can wait, because then nothing is learned. Thus, agents with the high signal mix in the Waiting Game, but since profits are positive, they invest in the No-Waiting Game.

We characterize the long run properties for the large, patient, persistent economy, under the No-Waiting Game and the Waiting Game. The average length of a boom is shorter, and the average length of a recession is longer, in the Waiting Game than in the No-Waiting Game. The long run probability of being in recession is larger in the Waiting Game than in the No-Waiting Game. We also show that, for the Waiting Game, the probability of being in recession is at least one half. The Waiting Game has less overinvestment (investment when the return is low) and more underinvestment (lack of investment when the return is high) than the No-Waiting Game. When we depart from the large, patient, persistent economy, our simulations indicate that these long-run properties of the dynamics hold more generally.

The differences in the long run properties across games are driven by the differences in the lengths of regimes described above. Booms are shorter in the Waiting Game, because a Regime 2 cascade is shorter, so information about agents' signals is revealed more frequently. This reduces the average length of a boom, and also decreases the probability of overinvestment. Recessions are longer in the Waiting Game, because the economy transitions to Regime M, while it transitions directly into Regime 1 in the No-Waiting Game. Thus, it takes longer for the economy to learn that the investment state has improved in the Waiting Game, which prolongs the recession and increases the probability of underinvestment.

The previous papers on herding with endogenous timing assume that the investment return remains constant over time. Chamley and Gale (1994) and Levin and Peck (2006) motivate their analysis with a discussion of how recessions can be prolonged, when many firms receive signals that the investment climate has improved to the point of being profitable, but they delay investment in an attempt to improve their information by observing whether other firms invest. While this motivation is legitimate, there are no cycles in those models. In large economies when the investment state is high, either the economy manages to take off and reveal the investment state, or investment dries up forever. In Caplin and Leahy (1994), the equilibrium features three regimes: "business as usual" where no information is revealed, a regime in which some agents take actions that reveal the state of demand, and "wisdom after the fact" where agents operate under full information (no cycles). The economy eventually leaves the business as usual regime, not because the state could have changed, but because agents accumulate more private information over time. In our model, on the other hand, the economy leaves Regime 0 or Regime 2 because agents become *less* sure of the investment state over time, to the point that different types might choose different actions.[3] Other papers in this literature include Gul and Lundholm (1995) and Chari and Kehoe (2004).

There has been some discussion of self-correcting cascades in the literature on herding with exogenous timing. Moscarini, Ottaviani, and Smith (1998) study a model that is essentially the same as our No-Waiting Game with one agent per period, and show that cascades are self-correcting. The extension of this idea to the case of n agents per period, and to the Waiting Game, is fairly obvious. Our contribution is to embed a changing investment state and self-correcting cascades into an endogenous timing model, while still being able to characterize the equilibrium. In addition, we derive results about long run dynamics. Goeree, Palfrey, Rogers, and McKelvey (2006) present experimental evidence that subjects manage to escape cascades. It is common knowledge that the investment state remains constant, so Nash equilibrium implies permanent cascades, but quantal response equilibrium allows for cycles of cascade formation and collapse.

Our model focuses purely on the flow of asymmetric information, and there is no payoff externality across agents. Gale (1996) develops a dynamic model that incorporates both delay and cycles. However, the incentive to delay is generated by payoff externalities rather than informational externalities. Chamley (1999) and Yang (2006) consider dynamic coordination games, and show that equilibrium cycles and regime switches can occur. Information is asymmetric, but the payoff externality greatly affects the analysis. The papers by Zeira (1994), Van Nieuwerburgh and Veldkamp (2004), and Veldkamp (2005) look at informational cycles in a setting of symmetric information. For example, in Veldkamp (2005), the knowledge that an agent has invested is not informative, but the investment activity automatically provides information to the market. As a result, more investment activity is more informative, and leads to cycles exhibiting a slow buildup of activity followed by a sudden crash. In our model, investment does not automatically release a new signal to the market. No information is revealed when everyone invests. Instead, information is revealed when only agents with the high signal are willing to invest, and a small amount of investment can be very informative. We believe that the presence of asymmetric information, and the tendency of agents with favorable information to wait for confirmation, plays an important role in actual business cycles. We hope that future work will add timing features, payoff externalities, and other signals, in order to better match up with empirical findings. Although it may be difficult to derive analytical results when new features are added to the model, computing an equilibrium trajectory for given parameters is likely to be straightforward.

The rest of the paper is laid out as follows. Section 2 contains the benchmark of the No-Waiting Game, including the equilibrium characterization. In Section 3, the Waiting Game is presented and equilibrium is characterized. Section 4 contains a discussion about the long-run patterns of boom and recession, along with simulation results. Section 5 contains analytical results about long-run dynamics of both the No-Waiting Game and the Waiting Game, for the case of the large, patient, persistent economy. Section 6 contains some brief concluding remarks. Some of the more technical proofs are contained in the Appendix.

[3] As a technical matter, their assumption of a continuum of firms precludes the possibility of some, but not all, information being revealed in equilibrium. This possibility plays a crucial role in our Waiting Game.

2 The No-Waiting Game

In our benchmark game, each agent has only one opportunity to invest. Time is discrete, $t = 1, 2, ...,$ and in each period, there are n agents or potential investors, who live for a single period. The investment return in period t is common to all investors, and is normalized to be either zero or one. We denote the investment state in period t by S^t, and assume that it follows a Markov process with persistence parameter $\rho > \frac{1}{2}$. That is, we have

$$\begin{aligned} \Pr(S^t &= 0) = \Pr(S^t = 1) = \frac{1}{2} \\ \Pr(S^{t+1} &= 0|S^t = 0) = \Pr(S^{t+1} = 1|S^t = 1) = \rho. \end{aligned}$$

There is also a deterministic investment cost, c, which is strictly between zero and one and common to all investors. Thus, the realized payoff to an investor in period t is $S^t - c$, and the realized payoff to an agent that does not invest is 0.

At the beginning of each period, each agent receives a binary private signal correlated with the investment state. These private signals are independent across agents, conditional on the investment state. Denoting the signal of an agent in period t as s, we have

$$\Pr(s = 0|S^t = 0) = \Pr(s = 1|S^t = 1) = \alpha,$$

where the parameter $\alpha \in (1/2, 1)$ captures the accuracy of signals. We will refer to agents as either type-0 or type-1, depending on whether they receive the low signal or the high signal. Also, we denote the number of agents who invest in period t as I^t, and the history of investments as $h^{t-1} = (I^1, ..., I^{t-1})$.

The timing of the No-Waiting Game is as follows. At the beginning of period t, the investment state is realized according the the Markov process described above. Each agent observes her signal and the history of past investments, h^{t-1}. Then the agents alive in period t simultaneously decide whether to invest or not, we proceed to period $t + 1$ with a new generation of agents, and so on.

Since all agents alive in period t observe the same investment history, they share the same initial belief about the investment state (before observing a private signal). Denote the probability that the investment state is high, conditional on the history h^{t-1}, as $\mu(h^{t-1})$. We will sometimes suppress the history, and simply refer to the beginning-of-period belief as μ. We denote the probability of the high investment state, conditional on an agent being type-1 or type-0, and conditional on the beginning-of-period belief μ, as

$$\mu_1 \equiv \Pr(S^t = 1|s = 1, \mu) = \frac{1}{1 + (\frac{1-\alpha}{\alpha})(\frac{1-\mu}{\mu})} \quad (1)$$

$$\mu_0 \equiv \Pr(S^t = 1|s = 0, \mu) = \frac{1}{1 + (\frac{\alpha}{1-\alpha})(\frac{1-\mu}{\mu})}. \quad (2)$$

Clearly, $\alpha > 1/2$ implies $\mu_1 > \mu_0$. Moreover, both μ_0 and μ_1 are increasing in the initial belief μ.

The following proposition shows that the No-Waiting Game has a Bayesian Nash equilibrium, characterized by cutoffs,

$$\underline{\mu}^{NW} \equiv \frac{1}{1 + (\frac{\alpha}{1-\alpha})(\frac{1-c}{c})} \text{ and} \quad (3)$$

$$\overline{\mu}^{NW} \equiv \frac{1}{1 + (\frac{1-\alpha}{\alpha})(\frac{1-c}{c})}, \quad (4)$$

determining whether no one invests, just type-1 agents invest, or all agents invest.

Proposition 2.1: *The No-Waiting Game has a Bayesian Nash equilibrium, characterized as follows.*

(i) Within-period behavior: If $\mu(h^{t-1}) < \underline{\mu}^{NW}$ *holds, then no one invests in period t (Regime 0). If* $\underline{\mu}^{NW} \leq \mu(h^{t-1}) < \overline{\mu}^{NW}$ *holds, then all of the type-1 agents and none of the type-0 agents invest in period t (Regime 1). If* $\overline{\mu}^{NW} \leq \mu$ *holds, then all agents invest in period t (Regime 2).*

(ii) Updating of beliefs: If we are in Regime 0 or Regime 2 in period t, beliefs in period $t+1$ *are given by*

$$\mu(h^t) = \rho\mu(h^{t-1}) + (1-\rho)(1-\mu(h^{t-1})). \tag{5}$$

If we are in Regime 1 in period t, beliefs at the end of period t are given by

$$\mu^{I^t} = \frac{1}{1 + \frac{1-\mu(h^{t-1})}{\mu(h^{t-1})}(\frac{1-\alpha}{\alpha})^{2I^t-n}}, \tag{6}$$

and beliefs in period $t+1$ *are given by*

$$\mu(h^t) = \rho\mu^{I^t} + (1-\rho)(1-\mu^{I^t}). \tag{7}$$

Proof. Given any history, a type-1 agent strictly prefers to invest if and only if we have $\mu_1 - c > 0$, which is equivalent to $\mu > \underline{\mu}^{NW}$. Similarly, a type-0 agent strictly prefers to invest if and only if we have $\mu_0 - c > 0$, which is equivalent to $\mu > \overline{\mu}^{NW}$. It follows that no agent has an incentive to deviate from their behavior as specified in part (i). When we are in Regime 0 or Regime 2, then nothing is learned from activity in period t, so beliefs are updated by considering the probability of a change in the investment state, using to Bayes' rule, according to (5). When we are in Regime 1, the number of type-1 agents is revealed to be I^t. The probability of the high investment state, conditional on there being I^t type-1 agents, is given by

$$\frac{\mu(h^{t-1})\binom{n}{I^t}\alpha^{I^t}(1-\alpha)^{n-I^t}}{\mu(h^{t-1})\binom{n}{I^t}\alpha^{I^t}(1-\alpha)^{n-I^t} + (1-\mu(h^{t-1}))\binom{n}{I^t}\alpha^{n-I^t}(1-\alpha)^{I^t}},$$

which can be simplified to (6). The probability of the high investment state at the beginning of period $t+1$ is updated, using Bayes' rule and the probability of a switch in the investment state, according to (7). □

The proof makes clear that the equilibrium characterized in Proposition (2.1) is essentially the unique Bayesian Nash equilibrium of the No-Waiting Game. The only possibility for multiplicity occurs when beliefs happen to lie exactly on the boundary between regimes.[4] It is worth emphasizing that different regimes give rise to different degrees of information aggregation. Specifically, no information about the current underlying state is revealed in either Regime 0 or Regime 2, where either no one invests or everyone invests. However, in Regime 1, all available information about the investment state is aggregated.

If not for the possibility of a shift in the investment state from period to period, Regime 0 and Regime 2 would lead to traditional information cascades, where subsequent generations do not pay attention to their signals. However, the possiblity of a shift in the investment state causes subsequent generations to update their beliefs, so that the expected investment return rises towards 1/2 in Regime 0, and falls towards 1/2 in Regime 2. Thus, for reasonable parameter values, the equilibrium results in self-correcting information cascades.

[4] The characterization of behavior and beliefs off the equilibrium path does not affect the incentive to deviate and is therefore unimportant. For completeness, we specify that, if an agent invests when no one is supposed to invest, that agent is believed to be type-1, and if an agent does not invest when everyone is supposed to invest, that agent is believed to be type-0. Continuation strategies and belief updating are defined in the obvious way.

Proposition 2.2: *For the No-Waiting Game, if $\alpha < c$ holds, the economy starts and remains in Regime 0. If $1-\alpha < c < \alpha$ holds, the economy starts in Regime 1. If $c < 1-\alpha$ holds, the economy starts and remains in Regime 2. For the interesting case, $1-\alpha < c < \alpha$, all recessions and booms are self-correcting. That is, whenever the economy reaches either Regime 0 or Regime 2, it will leave that regime with probability one. Regime 0 and Regime 2 can be reached in equilibrium, for sufficiently large ρ. A necessary condition is $\rho > \max[\overline{\mu}^{NW}, 1-\underline{\mu}^{NW}]$, which is also sufficient as $n \to \infty$.*

Proof. The first part follows from the fact that the expected investment return in period 1 is $1-\alpha$ for a type-0 agent and α for a type-1 agent. If $\alpha < c$ holds, neither type of agent will invest in period 1, so expected investments returns in period 2 are the same as they were in period 1. Therefore, no one invests in round 2, and so on. If $c < 1-\alpha$ holds, both type-0 and type-1 agents will invest in period 1, expected returns in period 2 are the same as they were in period 1, everyone invests in period 2, and so on.

Now suppose that $1-\alpha < c < \alpha$ holds, and suppose that the economy reaches Regime 0 after some history, with expected investment return $\mu(h^{t-1})$. We will show that the economy moves out of Regime 0 deterministically, in a finite number of periods. Since no information is revealed as long as we are in Regime 0, beliefs evolve according to (5). Since $\mu(h^{t-1}) < \underline{\mu}^{NW}$ and $\alpha > c$ hold, it follows that we have $\mu(h^{t-1}) < \underline{\mu}^{NW} < 1/2$. Therefore, (5) defines a sequence of increasing beliefs that converges to 1/2. After a finite number of periods, the expected investment return must exceed $\underline{\mu}^{NW}$, at which point the economy moves out of Regime 0. The same reasoning implies that, if the economy is in Regime 2 after some history, the economy moves out of Regime 2 deterministically, in a finite number of periods.

To show that Regime 0 is consistent with the equilibrium characterized in Proposition (2.1), we must show that, after a finite number of periods with no investment in Regime 1, the economy moves to Regime 0. Substituting $I^t = 0$ into (6) and (7) yields the difference equation for the evolution of beliefs:

$$\mu(h^t) = \frac{2\rho - 1}{1 + (\frac{1-\mu(h^{t-1})}{\mu(h^{t-1})})(\frac{\alpha}{1-\alpha})^n} + 1 - \rho. \tag{8}$$

If we have $\rho = 1$, we can solve (8) by iterative substitution, yielding

$$\mu(h^{t+\tau}) = \frac{1}{1 + (\frac{1-\mu(h^{t-1})}{\mu(h^{t-1})})(\frac{\alpha}{1-\alpha})^{(\tau+1)n}}. \tag{9}$$

From (9), we see that, after a finite number of periods, τ, beliefs will be below $\underline{\mu}^{NW}$, moving the economy into Regime 0. By continuity, we must move into Regime 0 for sufficiently large ρ. A similar argument establishes that, after a finite number of periods in Regime 1 with everyone investing (revealing themselves to be type-1), the economy moves to Regime 2.

A necessary condition for both Regime 0 and Regime 2 to be possible is $\rho > \max[\overline{\mu}^{NW}, 1-\underline{\mu}^{NW}]$. From (7), it is clear that $\mu(h^t)$ cannot be greater than or equal to ρ, which would occur if the investment state were known to be high in period t. Therefore, if $\rho \leq \overline{\mu}^{NW}$ holds, beliefs cannot be high enough to reach the threshold for Regime 2. Similarly, from (7), $\mu(h^t)$ cannot be less than or equal to $1-\rho$, which would occur if the investment state were known to be low in period t. Therefore, if $\rho \leq 1-\underline{\mu}^{NW}$ holds, beliefs cannot be low enough to reach the threshold for Regime 0. As $n \to \infty$, since type-1 agents invest and type-0 agents do not invest in Regime 1, the law of large numbers implies that activity will reveal the investment state. With probability arbitrarily close to one, beliefs are arbitrarily close to ρ when the investment state is high, and arbitrarily close to $1-\rho$ when the investment state is low. Thus, when the number of agents approaches infinity, $\rho > \max[\overline{\mu}^{NW}, 1-\underline{\mu}^{NW}]$ is sufficient to guarantee that the economy moves from Regime 1 to Regime 2 when the investment state is high, and from Regime 1 to Regime 0 when the investment state is low. □

Proposition (2.2) allows us to describe the cyclical investment patterns generated by the equilibrium. Consider the case in which all three regimes are possible, which occurs when $1-\alpha < c < \alpha$ and

$\rho > \max[\overline{\mu}^{NW}, 1-\underline{\mu}^{NW}]$ are satisfied. The economy starts in Regime 1, in which all private information is revealed by investment activity, since all the type-1 agents invest and the type-0 agents do not. In Regime 1, the number of realized investments, I^t, determines the Regime for period $t+1$. In general, there are two cutoffs, such that the economy switches to Regime 2 if I^t exceeds the larger cutoff, and the economy switches to Regime 0 if I^t is below the smaller cutoff.[5] For intermediate values of I^t, the economy will remain in Regime 1. When the economy switches to Regime 0 or Regime 2, the number of periods that the economy will remain in one of these regimes depends on I^t, because I^t affects how optimistic or pessimistic the beliefs are. Also, because $\underline{\mu}^{NW} < 1/2 < \overline{\mu}^{NW}$ holds, all transitions out of Regime 0 or Regime 2 must go through Regime 1.

While Proposition (2.2) allows the cyclical patterns to be characterized, it does not tell us anything about the long run frequencies of boom or recession. Long run frequencies are difficult to characterize analytically, because the entire history affects the current beliefs about the investment return, which affects the transition. Section 4 provides simulation results. In section 5, we solve the important special case of the large, patient, persistent economy.

3 The Waiting Game

In the Waiting Game, agents have an opportunity to postpone investment, in order to make a decision after observing market activity. For tractability, we assume that every period is divided into two rounds. An agent can either invest in the first round (and remain invested in second round) or postpone the decision until the second round. By waiting until round 2, the payoff will be discounted by a factor, $\delta < 1$. However, the investment state and the process generating signals is exactly as given in the No-Waiting Game. In particular, the realized investment return and signals remain constant during the two rounds of a period. Thus, the realized payoff to an investor in period t, round 1, is $S^t - c$, and the realized payoff to an investor in period t, round 2, is $\delta(S^t - c)$.

Let k_1^t and k_2^t denote the number of investments in round 1 and round 2 of period t, respectively. Thus, we have $I^t = k_1^t + k_2^t$. Now the history at the beginning of period t is given by $h^{t-1} = (k_1^1, k_2^1, ..., k_1^{t-1}, k_2^{t-1})$. We assume that k_1^t is observed before round 2 begins. By observing k_1^t, an agent is able to make more informative decisions, which is balanced against the cost of delay, as in Chamley and Gale (1994) and Levin and Peck (2006).

Since the equilibrium may involve mixing by the type-1 agents, we define the round-1 expected payoff of a type-1 agent, who plans to delay the investment decision until round 2. This expected payoff depends on the probability that other type-1 agents invest, q, and the beginning of period belief of a type-1 agent, μ_1, and is given by

$$V_1(\mu_1, q) = \delta \sum_{k=0}^{n-1} \max\{\mu_1^k - c, 0\} \Pr(k_1^t = k | s = 1, q, \mu_1)$$

where μ_1^k is a type-1 agent's probability assessment of the high investment return, given that other type-1 agents invest in round 1 with probability q, and conditional on observing k investments in round 1. More explicitly, we have

$$\mu_1^k = \frac{1}{1 + \frac{1-\mu_1}{\mu_1}(\frac{1-\alpha}{\alpha})^k [\frac{1-(1-\alpha)q}{(1-\alpha q)}]^{n-1-k}}. \tag{10}$$

[5] Depending on where beliefs lie within the range for regime 1, it is possible that the economy remains in regime 1, even if $I^t = 0$ or $I^t = n$. Our condition on ρ guarantees that the economy can eventually move out of regime 1, but not necessarily after one period. If n is large, then a lot of information will be revealed, making the possibility of moving out of regime 1 more likely.

The probability of $k_1^t = k$, given that an agent is type-1 and the other type-1 agents invest in round 1 with probability q, can be computed as

$$\Pr(k_1^t = k|s = 1, q, \mu_1) = \mu_1 \binom{n-1}{k} (\alpha q)^k (1-\alpha q)^{n-1-k} + (1-\mu_1) \binom{n-1}{k} [(1-\alpha)q]^k [1-(1-\alpha)q]^{n-1-k}.$$

After some manipulation, $V_1(\mu_1, q)$ can be expressed as

$$V_1(\mu_1, q) = \delta \sum_{k=0}^{n-1} \max [0, T_1(k, \mu_1, q)] \tag{11}$$

where each term $T_1(k, \mu_1, q)$ represents the option value (as of round 1) of investing in round 2 after observing k investors in round 1, and is given by

$$\begin{aligned} T_1(k, \mu_1, q) \quad &= \quad \mu_1(1-c) \binom{n-1}{k} (\alpha q)^k (1-\alpha q)^{n-1-k} \\ &\quad -(1-\mu_1)c \binom{n-1}{k} [(1-\alpha)q]^k [1-(1-\alpha)q]^{n-1-k}. \end{aligned} \tag{12}$$

A similar computation yields the the round-1 expected payoff of a type-0 agent, who plans to delay the investment decision until round 2. This expected payoff depends on the probability that type-1 agents invest, q, and the beginning of period belief of a type-0 agent, μ_0, and is given by

$$V_0(\mu_0, q) = \delta \sum_{k=0}^{n-1} \max\{\mu_0^k - c, 0\} \Pr(k_1^t = k|s = 0, q, \mu_0)$$

where μ_0^k is a type-0 agent's probability assessment of the high investment return, given that type-1 agents invest in round 1 with probability q, and conditional on observing k investments in round 1. Thus, we have

$$\mu_0^k = \frac{1}{1 + \frac{1-\mu_0}{\mu_0} (\frac{1-\alpha}{\alpha})^k [\frac{1-(1-\alpha)q}{(1-\alpha q)}]^{n-1-k}}. \tag{13}$$

After some manipulation, $V_0(\mu_0, q)$ can be written as

$$V_0(\mu_0, q) = \delta \sum_{k=0}^{n-1} \max [0, T_0(k, \mu_0, q)], \tag{14}$$

where we have

$$\begin{aligned} T_0(k, \mu_0, q) \quad &= \quad \mu_0(1-c) \binom{n-1}{k} (\alpha q)^k (1-\alpha q)^{n-1-k} \\ &\quad -(1-\mu_0)c \binom{n-1}{k} [(1-\alpha)q]^k [1-(1-\alpha)q]^{n-1-k}. \end{aligned} \tag{15}$$

A comparison of (11) and (12) with (14) and (15) indicates that $V_1(\cdot, \cdot)$ and $V_0(\cdot, \cdot)$ are really the same function, reflecting the fact that an agent's type affects her expected payoff only through her belief about the investment return. Henceforth, we will drop the subscript, and use the notation, $V(\mu_s, q)$ to indicate the round-1 expected payoff from waiting, given that the belief contingent on history and type is μ_s, and given that other agents are investing in round 1 with probability q if they are of type 1. We also adopt the corresponding notation, $T(k, \mu_s, q)$. The following lemma will be useful in characterizing the equilibrium. Lemma (3.1) establishes that there is a cutoff level of round-1 investment, which could be degenerate, above which expected returns exceed the cost. Also, the expected payoff from waiting, $V(\mu_s, q)$, is weakly increasing in an agent's belief, and strictly increasing unless the agent will never invest in round 2. The expected payoff from waiting is weakly increasing in the probability with which type-1 agents invest in round 1, and strictly increasing unless the agent would either never invest or always invest in round 2. Finally, the advantage, of investing in round 1 over waiting until round 2, is strictly increasing in an agent's belief. The intuition for this last result is that, since the agent is trading off the cost of delay with the option not to invest if unfavorable information is revealed, the option not to invest is more valuable for an agent who has more pessimistic beliefs.

Lemma 3.1: *The function, $V(\mu_s, q)$, is continuous in μ_s and q. We have either (i) $T(k, \mu_s, q) \geq 0$ for all k, (ii) $T(k, \mu_s, q) \leq 0$* for all k, or (iii) t*here is a cutoff, $k^* > 0$, such that $T(k, \mu_s, q) \geq 0$* for *$k \geq k^*$ and $T(k, \mu_s, q) < 0$* for *$k < k^*$. $V(\mu_s, q)$ is weakly increasing in μ_s and strictly increasing if* $T(n-1, \mu_s, q) > 0$ holds. *$V(\mu_s, q)$ is weakly increasing in q, and strictly increasing if $T(n-1, \mu_s, q) > 0$ and $T(0, \mu_s, q) < 0$ holds. The difference between the payoff from investing in round 1 and the payoff from waiting, $\mu_s - c - V(\mu_s, q)$, is strictly increasing in μ_s.*

Proof. Continuity of $V(\mu_s, q)$ follows immediately from (11) and (12). The cutoff property follows from the fact that μ_1^k and μ_0^k are strictly increasing in k, so that expected investment returns are increasing in the amount of round-1 investment. From (12) it is easy to see that $T(k, \mu_s, q)$ is strictly increasing in μ_s. Thus, whenever $T(n-1, \mu_s, q) > 0$ holds, then $V(\mu_s, q)$ is a sum containing at least one positive term, each of which are strictly increasing in μ_s.

The fact that $V(\mu_s, q)$ is weakly increasing in q, and strictly increasing when $T(0, \mu_s, q) < 0$ holds, is a special case of a result in Chamley and Gale (1994). Their argument is that, comparing q and a smaller mixing probability q', without loss of generality we can think of q' as being generated by type-1 agents first mixing with probability q and then having the "successful" agents mix again with probability q'/q. Then the second, garbled information structure is less informative than the first information structure in the sense of Blackwell, and Blackwell's theorem implies that the expected payoff must be weakly higher. When $T(n-1, \mu_s, q) > 0$ and $T(0, \mu_s, q) < 0$ holds, then investment is unprofitable when we have $k_1^t = 0$ and profitable when we have $k_1^t = n-1$. With a slightly lower mixing probability, q', there is a positive probability that $n-1$ agents are "successful" at the first randomization, but no investment survives the second randomization. Since $T(0, \mu_s, q') < 0$ holds for q' close to q, a profitable investment opportunity is lost with the second information structure, and there is no counteracting benefit.

For any q, investment in round 1 must yield that same payoff as always investing in round 2, but without discounting. We can therefore express $\mu_s - c$ as follows

$$\mu_s - c = \sum_{k=0}^{n-1} T(k, \mu_s, q). \tag{16}$$

Let us consider the possibilities. First, suppose that $T(k, \mu_s, q) \geq 0$ holds for all k. Then we have

$$\mu_s - c - V(\mu_s, q) = (1-\delta)V(\mu_s, q),$$

which is strictly increasing in μ_s, because $T(n-1, \mu_s, q) > 0$ must hold. Second, suppose that $T(k, \mu_s, q) \leq 0$ holds for all k. Then we have $\mu_s - c - V(\mu_s, q) = \mu_s - c$, which is obviously strictly increasing in μ_s. Finally, suppose that there is an interior cutoff, k^*. Then we have

$$\begin{aligned} \mu_s - c - V(\mu_s, q) \quad &= \quad \sum_{k=0}^{n-1} T(k, \mu_s, q) - \delta \sum_{k=0}^{n-1} \max\left[0, T(k, \mu_s, q)\right] = \\ & \quad\quad \sum_{k=0}^{k^*-1} T(k, \mu_s, q) + (1-\delta) \sum_{k=k^*}^{n-1} T(k, \mu_s, q). \end{aligned} \tag{17}$$

We have already shown that $T(k, \mu_s, q)$ is strictly increasing in μ_s. Thus, k^* remains constant as a function of μ_s, except for a finite number of values at which increasing μ_s causes k^* to decrease by one. Away from the jump points, (17) is strictly increasing in μ_s, because each term is strictly increasing. At one of the jump points, as k^* decreases from, say, $\kappa + 1$ to κ, the term $T(\kappa, \mu_s, q)$ moves from the left summation to the right summation in (17). However, since this movement occurs exactly when we have $T(\kappa, \mu_s, q) = 0$, changing the weight on $T(\kappa, \mu_s, q)$ in (17) from 1 to $1 - \delta$ has no effect on the overall expression. □

The following proposition shows that the Waiting Game has a Bayesian Nash equilibrium, characterized by cutoffs, $\underline{\mu}^W$, $\widehat{\mu}^W$, and $\overline{\mu}^W$, determining whether no one invests, type-0 agents do not invest and type-1 agents mix, type-0 agents do not invest and all type-1 agents invest, or all agents invest. Let μ^* be the unique value of μ_s solving

$$\mu_s - c - V(\mu_s, 1) = 0. \tag{18}$$

We know that a solution to (18) exists, because the expression is continuous, takes the value $-c$ at $\mu_s = 0$, and takes the value $(1-c)(1-\delta)$ at $\mu_s = 1$. Uniqueness follows from monotonicity. The interpretation of μ^* is the belief (conditional on learning one's type) such that an agent is indifferent between investing in round 1 and making a decision in round 2 after learning the types of all agents. The cutoffs are defined as follows

$$\underline{\mu}^W = \frac{1}{1 + (\frac{\alpha}{1-\alpha})(\frac{1-c}{c})} \tag{19}$$

$$\widehat{\mu}^W = \frac{1}{1 + (\frac{\alpha}{1-\alpha})(\frac{1-\mu^*}{\mu^*})} \tag{20}$$

$$\overline{\mu}^W = \frac{1}{1 + (\frac{1-\alpha}{\alpha})(\frac{1-\mu^*}{\mu^*})} \tag{21}$$

Before presenting our characterization theorem, we must develop notation for the end of period beliefs. Given the history, h^{t-1}, the mixing probability for type-1 agents in round 1, q, and the number of investors in round 1 is k, $k_1^t = k$, the beliefs of an outside observer, a type-1 agent that has not invested, and a type-0 agent are denoted as follows.

$$\mu^{k,q} = \frac{1}{1 + \frac{1-\mu(h^{t-1})}{\mu(h^{t-1})}(\frac{1-\alpha}{\alpha})^k(\frac{1-(1-\alpha)q}{1-\alpha q})^{n-1-k}} \tag{22}$$

$$\mu_1^{k,q} = \frac{1}{1 + \frac{1-\mu(h^{t-1})}{\mu(h^{t-1})}(\frac{1-\alpha}{\alpha})^{k+1}(\frac{1-(1-\alpha)q}{1-\alpha q})^{n-1-k}} \tag{23}$$

$$\mu_0^{k,q} = \frac{1}{1 + \frac{1-\mu(h^{t-1})}{\mu(h^{t-1})}(\frac{1-\alpha}{\alpha})^{k-1}(\frac{1-(1-\alpha)q}{1-\alpha q})^{n-1-k}} \tag{24}$$

Proposition 3.2: *The Waiting Game has a Bayesian Nash equilibrium, characterized as follows.*

(i) Regime 0: If $\mu(h^{t-1}) < \underline{\mu}^W$ holds, then no one invests in period t. Beliefs in period $t+1$ are given by

$$\mu(h^t) = \rho\mu(h^{t-1}) + (1-\rho)(1-\mu(h^{t-1})).$$

(ii) Regime M: If $\underline{\mu}^W \leq \mu(h^{t-1}) < \widehat{\mu}^W$ holds, then the type-0 agents wait and the type-1 agents invest in round 1 with probability $q(h^{t-1})$, where $q(h^{t-1})$ is the unique solution to $\mu_1 - c - V(\mu_1, q) = 0$. Based

on round-1 investment, if $\mu_1^{k_1^t,q(h^{t-1})} < c$ *holds, then no one invests in round 2, and beliefs in period* $t+1$ *are given by*

$$\mu(h^t) = \rho\mu^{k_1^t,q(h^{t-1})} + (1-\rho)(1-\mu^{k_1^t,q(h^{t-1})}).$$

If $\mu_0^{k_1^t,q(h^{t-1})} \leq c < \mu_1^{k_1^t,q(h^{t-1})}$ *holds, then the remaining type-1 agents (but no type-0 agents) invest in round 2, and beliefs in period* $t+1$ *are given by*

$$\mu(h^t) = \rho\mu^{I^t} + (1-\rho)(1-\mu^{I^t}).$$

If $c \leq \mu_0^{k_1^t,q(h^{t-1})}$ *holds, then all remaining agents invest in round 2, and beliefs in period* $t+1$ *are given by*

$$\mu(h^t) = \rho\mu^{k_1^t,q(h^{t-1})} + (1-\rho)(1-\mu^{k_1^t,q(h^{t-1})}).$$

(iii) Regime 1: If $\widehat{\mu}^W \leq \mu(h^{t-1}) < \overline{\mu}^W$ *holds, then the type-0 agents wait and the type-1 agents invest in round 1. Based on round-1 investment, if* $\mu_0^{k_1^t,1} < c$ *holds, then no one invests in round 2. If* $c \leq \mu_0^{k_1^t,1}$ *holds, then all type-0 agents invest in round 2. Beliefs in period* $t+1$ *are given by*

$$\mu(h^t) = \rho\mu^{k_1^t,1} + (1-\rho)(1-\mu^{k_1^t,1}).$$

(iv) Regime 2: If $\overline{\mu}^W \leq \mu(h^{t-1})$ *holds, then all agents invest in round 1. Beliefs in period* $t+1$ *are given by*

$$\mu(h^t) = \rho\mu(h^{t-1}) + (1-\rho)(1-\mu(h^{t-1})).$$

Proof. If $\mu(h^{t-1}) < \underline{\mu}^W$ holds, then investment in round 1 is not profitable for either type of agent, and nothing is learned from observing no investment in round 1. Therefore, no one has an incentive to deviate from the prescribed behavior in part (i). Beliefs in period $t+1$ are updated, according to Bayes' rule, based on the probability of a change in the investment state.

If $\underline{\mu}^W \leq \mu(h^{t-1}) < \widehat{\mu}^W$ holds, then investment in round 1 is profitable for a type-1 agent, so that $\mu_1 - c - V(\mu_1, 0) \geq 0$ holds. Also, from the definition of $\widehat{\mu}^W$ in (20), a type-1 agent would prefer to wait if waiting would allow her to learn the signals of all agents, so $\mu_1 - c - V(\mu_1, 1) < 0$ holds. From Lemma (3.1) there must be a unique q such that $\mu_1 - c - V(\mu_1, q) = 0$ holds, which we denote by $q(h^{t-1})$. For agents that do not invest in round 1, the specified behavior is to invest in round 2 if and only if investment is profitable given their beliefs at that point. Clearly, no beneficial deviation in round 2 is possible. We constructed $q(h^{t-1})$ to make type-1 agents indifferent between investing in round 1 and waiting, followed by investing in round 2 if and only if investment is profitable at that point. Therefore, type-1 agents cannot do better than to mix according to the specified probability. Since type-0 agents have more pessimistic beliefs than type-1 agents, Lemma (3.1) implies $\mu_0 - c - V(\mu_0, q(h^{t-1})) < 0$, so type-0 agents strictly prefer to wait. The specified beliefs in period $t+1$ are determined from Bayes' rule. If round-1 investment is either small or large, then either no one invests or everyone invests in round 2, so nothing additional is learned. If round-1 investment is intermediate, then only type-1 agents invest in round 2, and the signals of all agents are revealed.

If $\widehat{\mu}^W \leq \mu(h^{t-1}) < \overline{\mu}^W$ holds, then a type-1 agent would prefer to invest in round 1 rather than wait, if waiting would allow her to learn the signals of all agents. A type-0 agent would rather wait than invest, if waiting would allow her to learn the signals of all agents. Since type-1 agents invest in round 1 and type-0 agents do not, then round-1 behavior reveals everyone's signals. Therefore, no agent can benefit by deviating from the specified behavior. Beliefs in period $t+1$ are determined from Bayes' rule, and the fact that the number of type-1 agents is revealed to be k_1^t.

If $\overline{\mu}^W \leq \mu(h^{t-1})$ holds, then all agents would prefer to invest in round 1 rather than wait, if waiting would allow her to learn the signals of all agents. Therefore, no agent can benefit by deviating not to invest in round 1. Beliefs in period $t+1$ are determined from Bayes' rule, and the fact that nothing is learned from behavior in period t. $\square$

For the equilibrium characterized in Proposition (3.2), the amount of information revealed by market activity depends on the regime. In Regimes 0 and 2, nothing is revealed, because either everyone invests or no one invests. In Regime 1, all of the private information is revealed. In Regime M, the amount of information revealed depends on $q(h^{t-1})$ and the realized investment in round 1. From the proof of Lemma (3.1), we see that higher $q(h^{t-1})$ leads to more informative market activity, in the sense that, ex ante, agents prefer a higher mixing probability. It is easy to show that, within region M, $q(h^{t-1})$ is increasing in $\mu(h^{t-1})$, so the informativeness of market activity is increasing in initial beliefs. The realization of k_1^t also determines the amount of information revealed, but the connection is complicated. If k_1^t is small, then higher values leave agents less sure that the investment return is low, so more investment is less informative. If k_1^t is large, then higher values leave agents more sure that the investment return is high, so more investment is more informative. If k_1^t is intermediate, then this induces full information revelation in round 2.

In general, the Waiting Game will have multiple equilibria, for the same reasons as that discussed in Chamley (2004). The construction in Proposition (3.2) specifies that type-0 agents will delay investment in Regime M and Regime 1. It is possible that investment is profitable for a type-0 agent, but waiting is more profitable based on the amount of information that will be revealed. However, in such circumstances, one could construct a different equilibrium in which all agents invest in round 1, supported by the self-fulfilling expectation that no information will be revealed. Proposition (3.2) characterizes the most informationally efficient equilibrium.[6]

The cutoffs for the various regimes in the Waiting Game depend on α, c, δ, and n, while the cutoffs in the No-Waiting Game depend only on α and c. The analysis is therefore somewhat complicated, but several results can be demonstrated. In particular, the set of (beginning of period) beliefs for which the economy is in Regime 0 is identical for the two games, and the set of beliefs for which the economy is in Regime 2 is strictly smaller when waiting is allowed.

Proposition 3.3: *We have $\underline{\mu}^W = \underline{\mu}^{NW}$ and $\overline{\mu}^W > \overline{\mu}^{NW}$, so the range of beliefs corresponding to Regime 0 is identical for the two games, and the range of beliefs corresponding to regime 2 is strictly smaller for the Waiting Game than the No-Waiting Game. For the Waiting Game, if $\alpha < c$ holds, the economy starts and remains in Regime 0. If the economy starts in Regime M or Regime 1 (a sufficient condition is $1-\alpha < c < \alpha$), then all recessions and booms are self-correcting, in the sense that whenever the economy reaches either Regime 0 or Regime 2, it will leave that regime with probability one.*

Proof. That $\underline{\mu}^W = \underline{\mu}^{NW}$ holds follows immediately from (3) and (19). From (4) and (21), showing $\overline{\mu}^W > \overline{\mu}^{NW}$ is equivalent to showing $c < \mu^*$. We know that μ^* solves (18). Also, $\mu_s - c - V(\mu_s, 1)$, evaluated at $\mu_s = c$, is strictly negative, because the initial beliefs yield an expected payoff of zero, so the observation that $n-1$ other agents are type 1 makes the expected payoff positive, and thus, $-V(\mu_s, 1)$ is negative. From the monotonicity of $\mu_s - c - V(\mu_s, 1)$, $c < \mu^*$ follows.

If $\alpha < c$ holds in the Waiting Game, the economy starts and remains in Regime 0, by the same argument as given in the proof of Proposition (2.2). Clearly, $1-\alpha < c < \alpha$ is sufficient for the economy to start in Regime M or Regime 1, because investment in round 1 of period 1 is profitable for a type-1 agent, so we cannot be in Regime 0, and investment in round 1 of period 1 is unprofitable for a type-0 agent, so we cannot be in Regime 2. The argument that the economy deterministically moves out of Regime 0 or Regime 2 in a finite number of periods, assuming that we start in Regime M or Regime 1,

[6] There is also another type of equilibrium, in which type-1 agents invest and type-0 agents mix. However, this equilibrium is unstable, in the sense of fictitious play dynamics or other adjustment procedures.

is the same as in the proof of Proposition (2.2). In particular, $\overline{\mu}^W > \overline{\mu}^{NW} > 1/2$ holds, so the economy cannot remain in Regime 2 forever. □

Proposition 3.4: *In the limit, as $n \to \infty$, the regime cutoffs for the Waiting Game are as follows*

$$\begin{aligned}
\underline{\mu}^W &= \frac{1}{1+(\frac{\alpha}{1-\alpha})(\frac{1-c}{c})}, \\
\widehat{\mu}^W &= \frac{1}{1+(\frac{\alpha}{1-\alpha})(\frac{1-c}{c})(1-\delta)}, \\
\overline{\mu}^W &= \frac{1}{1+(\frac{1-\alpha}{\alpha})(\frac{1-c}{c})(1-\delta)}.
\end{aligned}$$

The regime in period 1 is given by

$$\begin{aligned}
\text{Regime 0} &: \quad \alpha < c \\
\text{Regime M} &: \quad \frac{\alpha(1-\delta)}{1-\alpha\delta} < c < \alpha \\
\text{Regime 1} &: \quad \frac{(1-\alpha)(1-\delta)}{1-\delta+\alpha\delta} < c < \frac{\alpha(1-\delta)}{1-\alpha\delta} \\
\text{Regime 2} &: \quad c < \frac{(1-\alpha)(1-\delta)}{1-\delta+\alpha\delta}
\end{aligned}$$

Proof. The expression for $\underline{\mu}^W$ follows by definition. As $n \to \infty$, when all type-1 agents invest in round 1, the investment state is revealed, so an agent who waits can invest if and only if the investment state is high. Therefore, we have

$$V(\mu_s, 1) = \delta\mu_s(1-c). \tag{25}$$

From (18) and (25), we have

$$\lim_{n\to\infty} \mu^* = \frac{c}{1-\delta(1-c)},$$

which, when substituted into (20) and (21), allows us to derive the expressions for $\widehat{\mu}^W$ and $\overline{\mu}^W$. The intervals characterizing the regime in period 1 are computed by comparing the initial belief, 1/2, with the regime cutoffs. □

For large economies in Regime M, it is easy to show that the mixing probability must converge to zero as $n \to \infty$. If the probability were bounded away from zero, then an agent could learn the investment state by waiting, but being in Regime M requires that all agents would prefer to wait if they could learn the other agents' signals. This is the same phenomenon discovered by Chamley and Gale (1994) and discussed in Levin and Peck (2006). What is required for the indifference condition is that something between no information and full information be revealed by round 1 activity. Thus, although the probability that any single agent invests is converging to zero, the probability that no agent invests in the good and bad investment states, $(1-\alpha q(h^{t-1}))^n$ and $(1-(1-\alpha)q(h^{t-1}))^n$, converge to a positive limit.

For the No-Waiting Game, if we have $1-\alpha < c < \alpha$, then we start in Regime 1 and it would take an unusual lack of persistence not to cycle between Regimes 0, 1, and 2. For the Waiting Game, $1-\alpha < c < \alpha$ does not guarantee that we start in Regime 1. From Proposition (3.4), we see that if δ is close to one, then the economy starts in Regime M. Indeed, moving into Regime 1 or Regime 2 would require near certainty that the investment state is high. Thus, it would not be unusual to have an economy that cycles only between Regimes 0 and M if agents are fairly patient. For the *large, persistent, patient economy* (where n is large, δ is near one, and ρ is near one), whether Regimes 1 or 2 can be reached depends on the order of limits. The long-run properties of this economy are studied in Section 5.

4 Long-Run Patterns of Boom and Recession

The patterns of boom and recession in the Waiting Game differ substantially from the No-Waiting Game. First, we see that the economy is in Regime 2 for a narrower range of beliefs in the Waiting Game than in the No-Waiting Game ($\overline{\mu}^W > \overline{\mu}^{NW}$). The threshold to fall into Regime 1 occurs when investment is unprofitable for type-0 agents in the No-Waiting Game, while it occurs when investment is *less profitable* than waiting and learning all agents' signals in the Waiting Game. Thus, beliefs do not have to fall all the way to the zero-profit point for type-0 agents. A smaller region of Regime 2 means that the length of a Regime 2 cascade is shorter in the Waiting Game. A second difference is that the economy typically enters Regime M when it emerges from Regime 0 in the Waiting Game, but it enters Regime 1 in the No-Waiting Game. Since information aggregation is less efficient in Regime M than in Regime 1, it will take agents longer, in the Waiting Game than in the No-Waiting Game, to learn that the investment state has switched from low to high.

We are now ready to study the long-run cyclical patterns of boom and recession. We define a boom to be a period in which either all agents invest, or "almost all" agents invest in the sense that the investment state is high and all type-1 agents invest. Thus, to be in a boom period, we are either in Regime 2, Regime 1 when the investment state is high, Regime M when the state is high and all type-1 agents invest by the end of round 2 (in the case of the Waiting Game), or Regime M when the state is low but all agents invest by the end of round 2 (in the case of the Waiting Game). We define a recession to be a period in which we are not in a boom.[7]

In general, it is not possible to derive analytical expressions for the long-run probability of being in a recession, the expected length of recessions and booms, and the expected amount of underinvestment (lack of investment in the good state) or overinvestment (investment in the bad state). The dynamical systems characterized by Propositions (2.1) and (3.2) cannot be represented as finite-state Markov processes, because beliefs depend on the entire history (the transition matrix is not time-homogenous). Fortunately, Propositions (2.1) and (3.2) suggest a procedure to compute equilibrium trajectories numerically, for both the No-Waiting Game and the Waiting Game. Basically, the computation is feasible because expectations of what future agents might do have no impact on current choices. Equilibrium can be computed, history by history, by updating beliefs, determining the regime, computing the mixing probability if necessary, and so on. Although the number of histories to compute grows exponentially as the number of periods increases, it is relatively easy to compute an equilibrium trajectory, by drawing a realization of the investment state in each period, then drawing signals, drawing the outcome of each type-1 agent's mixing (if necessary), and so on. Figures 1-4 demonstrate equilibrium trajectories for the two games, with parameters $\alpha = 0.7$, $c = 0.5$, $n = 100$, $\rho = 0.99$, $\delta = 0.9$, and a time horizon of 1000 periods.[8]

For the No-Waiting Game, the symmetry of parameters leads to symmetric cycles. The economy starts in Regime 1, but the large number of agents yields almost perfect information about the investment state. Beliefs jump into either Regime 2 or Regime 0, with $\mu(h^t) \simeq 1$ if about 70 agents invest in period t, and $\mu(h^t) \simeq 0$ if about 30 agents invest in period t. However, Figure 2 shows the self-correcting nature of the cascades, as type-1 agents occasionally invest, or type-0 agents occasionally choose not to invest, as the probability of an unobserved change in the investment return builds up. Average investment per period is 50.6, reflecting the symmetry of cycles. For the Waiting Game, parameters are symmetric, but

[7] Suppose we have a large economy and the investment return remains high for many consecutive periods. It seems reasonable to consider the No-Waiting Game to remain in a boom throughout if the economy is in Regime 2, then moves to Regime 1 for one period, observes αn agents invest, which reveals the high investment state and propels the economy back to Regime 2. Similarly, if the investment state remains low for many consecutive periods, it seems reasonable to consider the economy to remain in a recession if we periodically leave Regime 0 for one period, observe only $(1-\alpha)n$ agents invest, which reveals the investment state to be low and propels the economy back to Regime 0. However, other definitions of boom and recession are possible, and could be studied by our methods.

[8] Programming was done using SAS version 9.1. Source code is available upon request. The same "seed" was used in both games to draw random numbers, so the investment state and signal realizations are the same across the two games. We are grateful to Hammad Qureshi for doing an excellent job programming the algorithms.

the pattern of cycles is not. The Regime 0 cascade lasts for the same number of periods in the Waiting Game as in the No-Waiting Game, but Regime 2 cascades are much shorter, as seen in Figure 3. Average investment per period is 24.6.

Several features emerge from this example. First, the average length of a recession is longer and the average length of a boom is shorter under the Waiting Game than under the No-Waiting Game. Second, under the Waiting Game the economy spends more time in recession than in a boom. Third, there is less overinvestment and more underinvestment under the Waiting Game than under the No-Waiting Game.[9] These features generalize across all parameter values for which we have performed simulations.

Booms are shorter in the Waiting Game, because the shorter Regime 2 cascade reduces the chance that the investment return changes undetected, from high to low and back again to high, all within the same boom. Also, when the investment state has changed from high to low, the economy will learn about the change more quickly, on average. Recessions are longer in the Waiting Game, because when the economy moves out of Regime 0, we move into Regime M. Thus, while the No-Waiting Game moves into Regime 1 and activity almost reveals the state, the Waiting Game moves into Regime M and activity reveals little, on average. Figure 3 shows that the economy must wait several periods until a single type-1 agent decides to invest, and having two or more agents invest in round 1 of any one period is very unlikely. This pattern is established analytically, for the large, patient, persistent economy, in the next section.

5 The Large, Patient, Persistent Economy

All of our simulations indicate that investment cycles have longer recessions and shorter booms in the Waiting Game than in the No-Waiting Game. While we conjecture that this is a general result, the nonstationary nature of the dynamics makes proving this result impossible (at least, for us). However, in this section we derive analytical results for the important case of the *large, patient, persistent economy.* The example simulated in Figures 1-4 approximates this limit and illustrates the nature of equilibrium. Basically, Regime 0 and Regime 2 are entered only after the previous period's investment state is revealed, which pins down beliefs and the lengths of these regimes. Thus, the No-Waiting Game can be characterized by a finite-state Markov process. For the Waiting Game, when the economy falls out of Regime 2, it enters Regime 1, which reveals the investment state. When the economy falls out of Regime 0, it enters Regime M. Because type-1 agents are very patient, almost nothing can be learned from round 1 activity. That is, the mixing probability must be so low that, conditional on no one investing in round 1, then investment for a type-1 agent in round 2 must yield zero expected profits. The probability of $k_1^t = 1$ is nearly zero, and conditional on positive investment, the probability of $k_1^t > 1$ is nearly zero. If one agent does invest in round 1, then it follows that the remaining type-1 agents will invest in round 2, but type-0 agents will not invest. Thus, the Waiting Game can be characterized by a finite-state Markov process.

5.1 The No-Waiting Game

For the No-Waiting Game, the approximation of the dynamics to the first-order Markov process described above follows from the law of large numbers. Thus, there is a large number, r^{NW}, of Markov states corresponding to Regime 0, a large number, b^{NW}, of Markov states corresponding to Regime 2, and two Markov states corresponding to Regime 1. Our strategy is to derive expressions for transition probabilities and other variables in terms of ρ, and take limits as ρ approaches one. Certain variables, like the expected

[9] The average overinvestment (probability that the state is low and a random agent invests) in the Waiting Game is 0.0103 and in the No-Waiting Game is 0.06326. The average underinvestment in the Waiting Game is 0.275 and in the No-Waiting Game is 0.06821.

length of booms and recessions, will become unbounded, but other variables, like the long run probablities of boom and recession, will converge. The transition probabilities are determined as follows.

Within Regime 0, Markov state 1 transitions to Markov state 2 with probability one, which transitions to Markov state 3 with probability one, and so on until Markov state r^{NW}. If the economy is in Markov state r^{NW}, the transition is to either Markov state $r^{NW}+1$ (the state corresponding to Regime 1 with a low investment return) or Markov state $r^{NW}+2$ (the state corresponding to Regime 1 with a high investment return). The probability of transitioning to Markov state $r^{NW}+1$ is the probability of the investment state being low after r^{NW} periods, given that it was low initially. Denoting this probability as $1-\underline{p}^{NW}$, and denoting the probability of transitioning to Markov state $r^{NW}+2$ as $\underline{p}^{NW}$, we have

$$\underline{p}^{NW} = \frac{1}{2} - \frac{1}{2}(2\rho-1)^{r^{NW}+1}. \tag{26}$$

In Markov state $r^{NW}+1$, all type-1 agents invest, and by the law of large numbers, the investment state is revealed to be low. The transition is to Markov state 1 with probability one. Similarly, Markov state $r^{NW}+2$ transitions to the first Markov state in Regime 2, $r^{NW}+3$, with probability one.

Within Regime 2, Markov state $r^{NW}+3$ transitions to Markov state $r^{NW}+4$ with probability one, which transitions to Markov state $r^{NW}+5$ with probability one, and so on until Markov state $r^{NW}+2+b^{NW}$. If the economy is in Markov state $r^{NW}+2+b^{NW}$, the transition is to either Markov state $r^{NW}+1$ or Markov state $r^{NW}+2$. The probability of transitioning to Markov state $r^{NW}+1$ is the probability of the investment state being low after b^{NW} periods, given that it was high initially. Denoting this probability as $(1-\overline{p}^{NW})$, and denoting the probability of transitioning to Markov state $r^{NW}+2$ as $\overline{p}^{NW}$, we have

$$(1-\overline{p}^{NW}) = \frac{1}{2} - \frac{1}{2}(2\rho-1)^{b^{NW}+1}. \tag{27}$$

These transitions are captured in the transition probability matrix, P^{NW}, given by

$$P^{NW} = \begin{bmatrix} & 1 & & & & & & & & \\ & & 1\ldots & & & & & & & \\ & & & 1 & & & & & & \\ & & & & (1-\underline{p}^{NW}) & \underline{p}^{NW} & & & & \\ 1 & & & & & & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1\ldots & \\ & & & & & & & & & 1 \\ & & & & (1-\overline{p}^{NW}) & \overline{p}^{NW} & & & & \end{bmatrix}$$

Because ρ is close to 1, beliefs adjust very gradually in Regime 0 and Regime 2, so that the transition to Regime 1 occurs exactly when beliefs first cross the boundaries, $\underline{\mu}^{NW}$ and $\overline{\mu}^{NW}$. Thus, we have

$$\underline{p}^{NW} = \rho\underline{\mu}^{NW} + (1-\rho)(1-\underline{\mu}^{NW}) \tag{28}$$

$$(1-\overline{p}^{NW}) = \rho\overline{\mu}^{NW} + (1-\rho)(1-\overline{\mu}^{NW}) \tag{29}$$

From (26)-(29), we can derive expressions for r^{NW} and b^{NW}.

$$r^{NW} = \frac{\log(1-2\underline{\mu}^{NW})}{\log(2\rho-1)} \tag{30}$$

$$b^{NW} = \frac{\log(2\overline{\mu}^{NW}-1)}{\log(2\rho-1)} \tag{31}$$

Equations (30) and (31) indicate that the number of periods in Regime 0 and Regime 2 grow without bound as $\rho \to 1$, but the ratio converges to a well defined limit.

Since all of the states are ergodic, we can calculate the stationary distribution of states π^{NW} (a row vector), which is defined as $\pi^{NW} P^{NW} = \pi^{NW}$. Notice that every time we reach Markov state $r^{NW}+1$, we transition to Markov state 1, and conversely, every time we reach Markov state 1, we have transitioned from Markov state $r^{NW}+1$. Thus, the long run probability of each Markov state in Regime 0, and the probability of Markov state $r^{NW}+1$, must all be equal. By the same reasoning, the long run probability of each Markov state in Regime 2, and the probability of Markov state $r^{NW}+2$, must all be equal. Denoting the j^{th} component of π^{NW} as $\pi^{NW}(j)$, we have

$$\pi^{NW}(1) = \pi^{NW}(2) = ... = \pi^{NW}(r^{NW}) = \pi^{NW}(r^{NW}+1) \equiv \pi_0^{NW} \quad \text{and}$$
$$\pi^{NW}(r^{NW}+2) = \pi^{NW}(r^{NW}+3) = ... = \pi^{NW}(r^{NW}+2+b^{NW}) \equiv \pi_2^{NW}.$$

We also know that: (i) from Markov state r^{NW}, we reach Markov state $r^{NW}+1$ with probability $(1-\underline{p}^{NW})$, (ii) from Markov state $r^{NW}+2+b^{NW}$, we reach Markov state $r^{NW}+1$ with probability $(1-\overline{p}^{NW})$, and (iii) these are the only two ways of reaching Markov state $r^{NW}+1$. This yields the condition,

$$\pi_0^{NW}(1-\underline{p}^{NW}) + \pi_2^{NW}(1-\overline{p}^{NW}) = \pi_0^{NW}$$

Combining the previous condition with the fact that all long run probabilities must sum to one,

$$(r^{NW}+1)\pi_0^{NW} + (b^{NW}+1)\pi_2^{NW} = 1,$$

yields

$$\pi_0^{NW} = \frac{(1-\overline{p}^{NW})}{(r^{NW}+1)(1-\overline{p}^{NW}) + (b^{NW}+1)\underline{p}^{NW}}$$
$$\pi_2^{NW} = \frac{\underline{p}^{NW}}{(r^{NW}+1)(1-\overline{p}^{NW}) + (b^{NW}+1)\underline{p}^{NW}}.$$

Therefore, the fraction of time the economy spends in booms, π_B^{NW}, is

$$\pi_B^{NW} = (b^{NW}+1)\pi_2^{NW} = \frac{(b^{NW}+1)\underline{p}^{NW}}{(r^{NW}+1)(1-\overline{p}^{NW}) + (b^{NW}+1)\underline{p}^{NW}}, \tag{32}$$

with the probability of recession equal to $1-\pi_B^{NW}$. Using (28)-(31) and the result that b^{NW} and r^{NW} are large, (32) can be expressed in terms of our underlying parameters as

$$\pi_B^{NW} = \frac{1}{1 + \frac{\log(1-2\underline{\mu}^{NW})}{\log(2\overline{\mu}^{NW}-1)}\left[\frac{1-\overline{\mu}^{NW}}{\underline{\mu}^{NW}}\right]}. \tag{33}$$

Clearly, if we have $c = 1/2$, then by (3) and (4), $\underline{\mu}^{NW} = 1-\overline{\mu}^{NW}$ holds. It follows from (30) and (31) that $b^{NW} = r^{NW}$ and $\underline{p}^{NW} = (1-\overline{p}^{NW})$ hold. By (32), we have $\pi_B^{NW} = \pi_R^{NW} = 1/2$. As we will see, this symmetry of investment cycles for the symmetric model with $c = 1/2$ does not carry over to the Waiting Game.

5.2 The Waiting Game

It is not obvious that we can use a first-order Markov process to approximate the dynamics of the Waiting Game for the large, patient, persistent economy. To justify this approximation, we will use the equilibrium characterization in Proposition (3.2) to argue that (i) Regimes 0 and 2 are only entered following the previous period's investment state being revealed, so that the beliefs do not depend on the entire history, (ii) Regime 1 is only entered after b^W periods in Regime 2, so beliefs do not depend on the entire history, (iii) the beginning of period beliefs in Regime M do not depend on the history, (iv) in Regime M, if one agent invests in round 1, $k_1^t = 1$, then the remaining type-1 agents and none of the type-0 agents will invest in round 2, thereby revealing the investment state, and (v) in Regime M, the probability of more than one agent investing in round 1, $k_1^t > 1$, is negligible, even as compared to the already small probability that exactly one agent invests.

In the large, patient, persistent No-Waiting Game, observing a lot of investment in a situation where only type-1 agents invest always pushes the economy into Regime 2 during the following period. However, the Waiting Game is different. Suppose that, in the Waiting Game, we observe a lot of investment in a situation in which only type-1 agents invest. The following period can be in Regime 2, Regime 1, or a "high mixing probability" Regime M, depending on the rates of convergence of n, ρ and δ. Intuitively, even if agents are very sure that the investment state was high in period t (because n is large), and that the investment state is high in period $t+1$ (because ρ is close to one), it does not hurt an agent in period $t+1$ to wait until round 2, because δ is close to one. We will assume that, as ρ and δ approach one, the ratio $(1-\delta)/(1-\rho)$ is equal to a constant, denoted by D. Thus, we have

$$\delta = 1 - (1-\rho)D.$$

High values of D would arise if agents are impatient relative to the persistence of the investment return, where impatience and persistence are both measured with respect to a given unit of real time. As the number of periods per unit of real time goes to infinity, both ρ and δ approach one, but D does not change. Also, we will assume that n becomes large "faster" than ρ and δ approach one, so that learning the number of type-1 agents reveals the investment state during that period.

We want to find a condition on D to guarantee that the economy moves into Regime 2 in period t after learning that the investment state was high in period $t-1$. We have $\mu(h^{t-1}) = \rho$, which implies

$$\mu_0 = \frac{1}{1 + \left(\frac{1-\rho}{\rho}\right)\left(\frac{\alpha}{1-\alpha}\right)} \quad \text{and} \tag{34}$$

$$\mu_1 = \frac{1}{1 + \left(\frac{1-\rho}{\rho}\right)\left(\frac{1-\alpha}{\alpha}\right)}. \tag{35}$$

Since Regime 2 occurs if and only if beliefs are such that $\mu_0 - c \geq \delta[\mu_0(1-c)]$ holds, substituting (34) and manipulating the expression yields the condition,

$$D = \frac{1-\delta}{1-\rho} \geq \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{c}{1-c}\right). \tag{36}$$

By a similar calculation, learning that the investment state was high in period $t-1$ pushes the economy into Regime 1 in period t if we have

$$\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{c}{1-c}\right) < D < \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{c}{1-c}\right),$$

and into Regime M in period t if we have

$$D < \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{c}{1-c}\right).$$

For the remainder of this section, we will assume that (36) holds.[10] It will be convenient to adopt the shorthand notation,

$$z = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-c}{c}\right).$$

Now let us turn our attention to Regime M. Because agents are infinitely patient in the limit, the value of any information received by waiting must approach zero. Thus, whenever $\mu(h^{t-1}) < 1$ holds, a type-1 agent that does not invest in round 1 must not benefit from the option not to invest in round 2 when $k_1^t = 0$. Based on this observation, we can characterize the equilibrium choices made in Regime M. First we define notation for the probabilities of round-1 investment, given the investment state:

$$prk0 \equiv pr(k_1^t = k | S^t = 0) \quad \text{and} \quad prk1 \equiv pr(k_1^t = k | S^t = 1).$$

Also, based on the mixing probability, q, we define Q to be the ratio of the probability that no one invests when the investment state is low, to the probability that no one invests when the investment state is high:

$$Q \equiv \left(\frac{1-(1-\alpha)q}{1-\alpha q}\right)^n \tag{37}$$

Lemma 5.1: *For the large, patient, persistent Waiting Game in Regime M, let the beginning of period belief be given by $\mu(h^{t-1}) = \mu < 1$. Then in the limit, as $n \to \infty$, $\delta \to 1$, and $\rho \to 1$ hold, Q and round-1 investment probabilities are characterized as follows.*

$$\begin{aligned}
Q &= \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1-c}{c}\right)\left(\frac{\mu}{1-\mu}\right) \qquad (38)\\
pr00 &= Q^{-(1-\alpha)/(2\alpha-1)} \\
pr10 &= \left(\frac{1-\alpha}{2\alpha-1}\right) Q^{-(1-\alpha)/(2\alpha-1)} \log(Q) \\
pr01 &= Q^{-\alpha/(2\alpha-1)} \\
pr11 &= \left(\frac{\alpha}{2\alpha-1}\right) Q^{-\alpha/(2\alpha-1)} \log(Q)
\end{aligned}$$

Also, suppose $\mu = \underline{\mu}^W$ holds. Then Q, $pr00$, and $pr01$ converge to one. Although the probability of $k_1^t = 1$ is converging to zero, the probability of $k_1^t > 1$, relative to the probability of $k_1^t = 1$, also converges to zero.

Proof. See the Appendix.

Lemma (5.1) characterizes the probabilities of round 1 investment being zero or one, when the economy is in Regime M. In order for a nearly infinitely patient type-1 agent to be indifferent between investing in round 1 and waiting, there cannot be a positive probability of learning that profits are strictly negative, or else the agent strictly prefers to wait; profits cannot be positive for all realizations of k_1^t, or else the agent strictly prefers to invest in round 1. Therefore, a type-1 agent's expected investment return, conditional on $k_1^t = 0$, must approximately equal the cost, c. Then following no investment in round 1, beliefs at the end of the current period must be at the boundary between Regime 0 and Regime M, $\mu(h^t) = \underline{\mu}^W$.

[10] Our assumptions regarding rates of convergence will ensure that Regime 2 exists. If, say, δ approached one much faster than ρ, the characterization in Proposition (3.2) would involve type-1 agents mixing during boom periods. The probability of investment by a type-1 agent in round 1 would approach zero in the limit, but the expected aggregate investment would be large enough to reveal the investment state with near certainty. Determining long run boom frequencies for such an equilibrium poses problems, but we conjecture that they are unaffected by relative rates of convergence.

By Lemma (5.1), we remain in Regime M in period $t+1$, and observe no investment with very high probability. The probability of $k_1^{t+1} = 1$ approaches zero, but cannot be ignored when we consider long-run boom and recession frequencies, because the number of periods we must wait to observe the next investment could be of the same order of magnitude as the length of a Regime 0 cascade or a Regime 2 cascade. However, Lemma (5.1) allows us to ignore the possibility that round 1 investment is two or more in this circumstance.

Proposition (5.2) below shows that the large, patient, persistent Waiting Game long run equilibrium dynamics can be characterized by a first-order Markov process. Markov state 1 through Markov state r^W correspond to Regime 0, where r^W is a large number. Markov state $r^W + 1$ corresponds to Regime M and a low investment return, and Markov state $r^W + 2$ corresponds to Regime M and a high investment return. Markov state $r^W + 3$ corresponds to Regime 1 and a low investment return, and Markov state $r^W + 4$ corresponds to Regime 1 and a high investment return. Markov state $r^W + 5$ through Markov state $r^W + 4 + b^W$ correspond to Regime 2, where b^W is not necessarily a large number. Our strategy is to derive expressions for transition probabilities and other variables in terms of ρ, and take limits as ρ approaches one.

Proposition 5.2: *Assume that $c < \alpha$ holds, and that the economy reaches Regime 0. From that point, the large, patient, persistent Waiting Game equilibrium can be approximated by the Markov process defined by the transition matrix, P^W, where we have*

$$\underline{p}^W = \frac{\rho + (1-\rho)z}{1+z} \quad , \quad \underline{Q} = \frac{z\underline{p}^W}{1-\underline{p}^W} \tag{39}$$

$$\lambda_0 \equiv pr(k_1^t = 1 | S^t = 0; \mu = \underline{p}^W) = \left(\frac{1-\alpha}{2\alpha - 1}\right) \underline{Q}^{-(1-\alpha)/(2\alpha-1)} \log(\underline{Q}) \tag{40}$$

$$\lambda_1 \equiv pr(k_1^t = 1 | S^t = 1; \mu = \underline{p}^W) = \left(\frac{\alpha}{2\alpha - 1}\right) \underline{Q}^{-\alpha/(2\alpha-1)} \log(\underline{Q}) \tag{41}$$

$$\overline{\mu}^W = \frac{1}{1 + (\frac{1-\alpha}{\alpha})(\frac{1-c}{c})(1-\rho)D} \tag{42}$$

$$\overline{p}^W = \rho\overline{\mu}^W + (1-\rho)(1-\overline{\mu}^W) \tag{43}$$

$$P^W = \begin{bmatrix}
 & 1 & & & & & & & & & & \\
 & & 1... & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
 & & & & (1-\underline{p}^W) & \underline{p}^W & & & & & & \\
\lambda_0 & & & & (1-\lambda_0)\rho & (1-\lambda_0)(1-\rho) & & & & & & \\
 & & & & (1-\lambda_1)(1-\rho) & (1-\lambda_1)\rho & & & \lambda_1 & & & \\
1 & & & & & & & & & & & \\
 & & & & & & & & 1 & & & \\
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & 1... & \\
 & & & & & & & & & & & 1 \\
 & & & & & & (1-\overline{p}^W) & \overline{p}^W & & & &
\end{bmatrix}$$

Remark: *Within Regime 0, (Markov) state 1 transitions to state 2 w.p. 1, which transitions to state 3 w.p. 1, and so on until state r^W. State r^W transitions to state $r^W + 1$ w.p. $(1 - \underline{p}^W)$, and to state $r^W + 2$ w.p. $\underline{p}^W$.*

Within Regime M, state r^W+1 transitions to state r^W+1 w.p. $(1-\lambda_0)\rho$, transitions to state r^W+2 w.p. $(1-\lambda_0)(1-\rho)$, and transitions to state 1 w.p. λ_0. State r^W+2 transitions to state r^W+1 w.p. $(1-\lambda_1)(1-\rho)$, transitions to state r^W+2 w.p. $(1-\lambda_1)\rho$, and transitions to state r^W+5 w.p. λ_1.

Within Regime 1, state r^W+3 transitions to state 1 w.p. 1, and state r^W+4 transitions to state r^W+5 w.p. 1. Within Regime 2, state r^W+5 transitions to state r^W+6 w.p. 1, and so on until state r^W+4+b^W. State r^W+4+b^W transitions to state r^W+3 w.p. $1-\overline{p}^W$ and transitions to state r^W+4 w.p. $\overline{p}^W$.

Proof of Proposition 5.2. The beliefs during the period in which the economy first enters Regime 0 will correspond to one of the Markov states between 1 and r^W. Because investment is unprofitable for type-1 agents, no one invests, beliefs adjust deterministically based on ρ, and we transition to the next Markov state. From Markov state r^W, the probability of transitioning to Markov state r^W+1 is the probability of the investment state being low after r^W periods, given that it was low initially. As we have already shown for the No-Waiting Game, this transition occurs when beliefs first cross the boundary, $\underline{\mu}^W$, where expected profits are zero for a type-1 agent. Thus, the probability of transitioning to Markov state r^W+1 is

$$\rho(1-\underline{\mu}^W)+(1-\rho)\underline{\mu}^W=\rho(1-\frac{1}{1+z})+(1-\rho)(\frac{1}{1+z})=1-\underline{p}^W,$$

and the probability of transitioning to Markov state r^W+2 is $\underline{p}^W$.

In Markov state r^W+1, the expected investment return is $\underline{p}^W$. We are (barely) in Regime M, and from Lemma (5.1) it follows that the probability of zero round 1 investment, $k_1^t=0$, is $(1-\lambda_0)$, the probability of one investor in round 1, $k_1^t=1$, is λ_0, and the probability of $k_1^t>1$ is negligible.[11] The mixing probability is such that, if no one invests in round 1, then a type-1 agent's conditional probability of the high investment return continues to be c, so no one invests in round 2, and next period's expected investment return is $\underline{p}^W$. If we have $k_1^t=1$, then from (23) and (24), a type-1 agent finds it profitable to invest and a type-0 agent does not; thus, only the remaining type-1 agents invest in round 2, thereby revealing the investment state to be low. The specified transition probabilities follow immediately.[12] Similarly, in Markov state r^W+2, the expected investment return is $\underline{p}^W$. From Lemma (5.1), the probability of zero round 1 investment, $k_1^t=0$, is $(1-\lambda_1)$, the probability of one investor in round 1, $k_1^t=1$, is λ_1, and the probability of $k_1^t>1$ is negligible. If no one invests in round 1, then a type-1 agent's conditional probability of the high investment return continues to be c, no one invests in round 2, and next period's expected investment return is $\underline{p}^W$. If we have $k_1^t=1$, then only the remaining type-1 agents invest in round 2, thereby revealing the investment state to be high. The specified transition probabilities follow immediately.

Markov state r^W+3 corresponds to Regime 1 with a low investment return. Therefore, the transition is to Markov state 1 with probability one. Similarly, Markov state r^W+4 corresponds to Regime 1 with a high investment return. Therefore, all type-1 agents invest in round 1, and by the law of large numbers, the investment state is revealed to be high, so all of the type-0 agents invest in round 2. We claim that the transition is to the first Markov state in Regime 2, r^W+5, with probability one. This follows from condition (36), which guarantees that, after the investment state S^{t-1} is revealed to be high, type-0 agents in period t prefer to invest in round 1 rather than wait and learn S^t.

Within Regime 2, Markov state r^W+5 transitions to Markov state r^W+6 with probability one, which transitions to Markov state r^W+7 with probability one, and so on until Markov state r^W+4+b^W. At

[11] There is a subtlety here, because we are considering ρ close to, but not equal to, one. Thus, the actual mixing condition for Q, $(1-\delta)(\mu_1-c)/\delta=Pr(k_1^t=0|s=1,q,\mu_1)(\mu_1^{0,q}-c)$, is slightly different from the condition, $\mu_1^{0,q}=c$, which is used to derive (39). Although the actual condition does not have a closed form solution, we can differentiate and solve for $\partial Q/\partial\rho$, and verify that $\lim_{\rho\to1}\partial Q/\partial\rho=\lim_{\rho\to1}\partial\underline{Q}/\partial\rho$. Therefore, λ_0 and λ_1 are correctly specified for ρ close to one.

[12] Indeed, the probability of $k_1^t=1$ approaches zero as $\rho\to1$ (and therefore, $\delta\to1$), but this rare event is the only way to escape Regime M, and may occur several times within a cycle. Long run frequencies of boom and recession depend, in part, on the relative likelihoods of $k_1^t=1$ in the low and high investment states.

this point, beliefs are given by $\overline{p}^W$. To derive (42), we substitute $(1-\rho)D$ for $(1-\delta)$ into the expression for $\overline{\mu}^W$ in Proposition (3.4), and derive (43) by computing beliefs when the economy leaves Regime 2 in the following period. From Markov state r^W+4+b^W, the economy transitions to either Markov state r^W+3 (the state corresponding to Regime 1 with a low investment return) or Markov state r^W+4 (the state corresponding to Regime 1 with a high investment return). Transition probabilities are based on the probability of the investment state being low or high after b^W periods, given that it was high initially. □

As long as the economy eventually reaches Regime 0, investment patterns before that point do not affect the long run frequencies. We conjecture this assumption is always satisfied, that for any $\varepsilon > 0$, there exists $T < \infty$, such that the probability of reaching Regime 0 before period T is greater than $1-\varepsilon$.[13]

The lengths of Regime 0 cascades and Regime 2 cascades, r^W and b^W, are determined as follows. The probability of the investment state changing from low to high after r^W periods must equal the probability of transition from Markov state r^W to Markov state r^W+2, and this limiting probability is $\underline{p}^W$. Thus, we have

$$\underline{p}^W = \frac{1}{2} - \frac{1}{2}(2\rho-1)^{r^W+1},$$

from which we can derive

$$r^W = \frac{\log(1-2\underline{\mu}^W)}{\log(2\rho-1)} = \frac{\log(\frac{z-1}{1+z})}{\log(2\rho-1)}. \tag{44}$$

Comparing (30) and (44), we see that $r^W = r^{NW}$ holds, and that the length of a Regime 0 cascade approaches infinity for the large, patient, persistent economy. The probability of the investment state changing from high to low after b^W periods must equal the probability of transition from Markov state r^W+4+b^W to Markov state r^W+3, which is $1-\overline{p}^W$. Thus, we have

$$1-\overline{p}^W = \frac{1}{2} - \frac{1}{2}(2\rho-1)^{b^W+1}. \tag{45}$$

From (45), (42), and (43), we have

$$b^W = \frac{\log(2\overline{\mu}^W-1)}{\log(2\rho-1)} = \frac{\log(\frac{2}{1+(\frac{1-\alpha}{\alpha})(\frac{1-c}{c})(1-\rho)D}-1)}{\log(2\rho-1)}. \tag{46}$$

Taking the limit of (46) as ρ approaches one, we have the length of a Regime 2 cascade for the large, patient, persistent economy, given by[14]

$$b^W = D(\frac{1-\alpha}{\alpha})(\frac{1-c}{c}). \tag{47}$$

From (47), we see that the limiting length of a Regime 2 cascade in the Waiting Game is finite, which is obviously shorter than in the No-Waiting Game, which grows without bound as ρ approaches one.

Given the Markov chain specified by the transition matrix P^W, we can compute the steady state distribution of each Markov state, π^W, which is defined as $\pi^W P^W = \pi^W$. We denote the j^{th} component of π^W as $\pi^W(j)$. Because some Markov states are visited in combination with other Markov states along each realization of the process, the corresponding long run probabilities must be equal, yielding

$$\pi^W(1) = \pi^W(2) = ... = \pi^W(r^W) \equiv \pi_0^W \quad \text{and}$$
$$\pi^W(r^W+5) = \pi^W(r^W+6) = ... = \pi^W(r^W+4+b^W) \equiv \pi_2^W.$$

[13] Of course, T will depend on ρ, so that as $\rho \to 1$ we also have $T \to \infty$.

[14] We are ignoring integer issues here. Also, we implicitly assume that α is not so close to 1/2 that, when we drop out of Regime 2, we drop into Regime 1 rather than Regime M.

Let us adopt some shorthand notation for the probabilities of Markov states r^W+1 through r^W+4. Regime M, low investment return is denoted as π^W_{M0}; Regime M, high investment return is denoted as π^W_{M1}; Regime 1, low investment return is denoted as π^W_{10}; and Regime 1, high investment return is denoted as π^W_{11}.

We know that: (i) from Markov state r^W+4, we reach Markov state r^W+5 with probability one, (ii) from Markov state r^W+2, we reach Markov state r^W+5 with probability λ_1, and (iii) these are the only two ways of reaching Markov state r^W+5. This yields the condition,

$$\pi^W_{11} + \lambda_1 \pi^W_{M1} = \pi^W_2. \tag{48}$$

Similarly, (i) from Markov state r^W+3, we reach Markov state 1 with probability one, (ii) from Markov state r^W+1, we reach Markov state 1 with probability λ_0, and (iii) these are the only two ways of reaching Markov state 1. This yields the condition,

$$\pi^W_{10} + \lambda_0 \pi^W_{M0} = \pi^W_0. \tag{49}$$

We also know that we reach Markov state r^W+3 if and only if last period's Markov state was r^W+4+b^W and the investment state is low this period, and we reach Markov state r^W+4 if and only if last period's Markov state was r^W+4+b^W and the investment state is high this period. This yields the conditions,

$$(1-\overline{p}^W)\pi^W_2 = \pi^W_{10}, \tag{50}$$

$$\overline{p}^W \pi^W_2 = \pi^W_{11}. \tag{51}$$

Finally, consider the ways of reaching the two Markov states corresponding to Regime M. We reach Markov state r^W+1 if and only if (i) last period's Markov state was r^W and the investment state is low this period, (ii) last period's Markov state was r^W+1, no one invested in round 1, and the investment state is low this period, or (iii) last period's Markov state was r^W+2, no one invested in round 1, and the investment state is low this period. This yields the condition,[15]

$$(1-\lambda_1)(1-\rho)\pi^W_{M1} + (1-\lambda_0)\rho\pi^W_{M0} + (1-\underline{p}^W)\pi^W_0 = \pi^W_{M0}. \tag{52}$$

Of course, we must also impose the condition that probabilities sum to one,

$$b^W\pi^W_2 + \pi^W_{11} + \pi^W_{10} + \pi^W_{M1} + \pi^W_{M0} + r^W\pi^W_0 = 1. \tag{53}$$

Equations (48)-(53) are linear equations that can be solved for π^W_2, π^W_{11}, π^W_{10}, π^W_{M1}, π^W_{M0}, and π^W_0. The following intermediate calculations, which express long run probabilities relative to π^W_{M0}, will be useful in later proofs.

$$\pi^W_2 = \frac{\lambda_1\pi^W_{M1}}{\overline{p}^W}; \quad \pi^W_{11} = \frac{1-\overline{p}^W}{\overline{p}^W}\lambda_1\pi^W_{M1} \tag{54}$$

$$\pi^W_{10} = \lambda_1\pi^W_{M1}; \quad \pi^W_0 = \lambda_1\pi^W_{M1} + \lambda_0\pi^W_{M0} \tag{55}$$

$$\pi^W_{M1} = \frac{1-\rho+\lambda_0[\rho-(1-\underline{p}^W)]}{1-\rho+\lambda_1(\rho-\underline{p}^W)}\pi^W_{M0} \tag{56}$$

By (56), it can be easily shown that $\pi^W_{M1} < \pi^W_{M0}$ holds, since we have $\lambda_0 < \lambda_1$ and $(1-\underline{p}^W) > \underline{p}^W$.

After solving (48)-(53), substituting the derived values of λ_0, λ_1, b^W, etc., and taking the limit as $\rho \to 1$, we can derive an expression for the limiting boom probability, π^W_B, which is[16]

[15] A similar condition, for π^W_{M1}, is redundant.
[16] Computations were performed using Maple 10.

$$\pi_B^W = b^W\pi_2^W + \pi_{11}^W + \lambda_1\pi_{M1}^W = \frac{b^W+1}{\overline{p}^W}\lambda_1\pi_{M1}^W = \frac{2c(\alpha-c)}{6c\alpha - 2c - 4c^2\alpha - (\alpha-c)\log(\frac{\alpha-c}{c+\alpha-2c\alpha})}. \quad (57)$$

The probability of recession is $1-\pi_B^W$, which is also given by

$$\begin{aligned}\pi_R^W &= \pi_{10}^W + (1-\lambda_1)\pi_{M1}^W + \pi_{M0}^W + r^W\pi_0^W \\ &= (r\lambda_1+1)\pi_{M1}^W + (1+r\lambda_0)\pi_{M0}^W.\end{aligned} \quad (58)$$

5.3 Comparing the Long Run Dynamics Across Games

In this subsection, we compare the long-run dynamics of the No-Waiting Game and the Waiting Game, for the large, patient, persistent economy. We demonstrate that the expected length of a boom is shorter and the expected length of a recession is longer in the Waiting Game. The probability of being in a recession is greater in the Waiting Game than in the No-Waiting Game. Somewhat surprisingly, the Waiting Game always spends more time in recession than in boom. We also show that overinvestment (investing in the low state) is more prevalent in the No-Waiting Game, and underinvestment (not investing in the high state) is more prevalent in the Waiting Game.

Let L_B^{NW} be the expected length of a boom for the No-Waiting Game. The actual length of a boom is a random variable that can take one of the values, $b^{NW}+1$, $2(b^{NW}+1)$, $3(b^{NW}+1)$, and so on. The probability that a boom lasts for $k(b^{NW}+1)$ periods is $(1-\overline{p}^{NW})(\overline{p}^{NW})^{k-1}$. Thus, we have

$$L_B^{NW} = (b^{NW}+1)\sum_{k=1}^{\infty} k(1-\overline{p}^{NW})(\overline{p}^{NW})^{k-1} = (b^{NW}+1)\frac{1}{(1-\overline{p}^{NW})} = (b^{NW}+1)\frac{2}{1-(2\rho-1)^{b^{NW}+1}} \quad (59)$$

Similarly, the expected length of recessions L_R^{NW} is

$$L_R^{NW} = (r^{NW}+1)\frac{1}{\underline{p}^{NW}} = (r^{NW}+1)\frac{2}{1-(2\rho-1)^{r^{NW}+1}}. \quad (60)$$

Although the expected lengths of booms and recessions grow without bound as $\rho \to 1$, the ratios have well defined limits.

Now consider the Waiting Game. The facts that a Regime 2 cascade is finite, and a Regime 0 cascade grows without bound, do not necessarily tell us anything about the expected lengths of booms and recessions. For example, after a Regime 2 cascade ends, the economy is very likely to discover that the investment state remains high, leading to another Regime 2 cascade within the same boom. The expected length of booms in the Waiting Game, denoted by L_B^W, is given by

$$L_B^W = (b^W+1)\sum_{k=1}^{\infty} k(1-\overline{p}^W)(\overline{p}^W)^{k-1} = \frac{b^W+1}{1-\overline{p}^W}. \quad (61)$$

Denote the expected length of recessions in the Waiting Game by L_R^W. It will be convenient to keep track of the expected length of recessions starting from Regime M when the investment state is high, which we denote by ℓ_1, and starting from Regime M when the investment state is low, which we denote by ℓ_0. From P^W, we have the following equations:

$$L_R^W = (r^W + 1) + \underline{p}^W \ell_1 + (1 - \underline{p}^W)\ell_0 \quad (62)$$

$$\ell_1 = (1 - \lambda_1)[1 + \rho\ell_1 + (1 - \rho)\ell_0] \quad (63)$$

$$\ell_0 = (1 - \lambda_0)[1 + \rho\ell_0 + (1 - \rho)\ell_1] + \lambda_0 L_R^W \quad (64)$$

Solving the above equations simultaneously, we can compute L_R^W.

Proposition 5.3: *For the large, patient, persistent economy with $c < \alpha$, the expected length of a boom is shorter, and the expected length of a recession is longer, in the Waiting Game than in the No-Waiting Game. That is, we have $L_B^W < L_B^{NW}$ and $L_R^W > L_R^{NW}$.*

Proof. See the Appendix.

The average length of a boom is shorter in the Waiting Game than in the No-Waiting Game, because Regime 2 cascades are shorter with the possibility of waiting. The shorter Regime 2 cascades reduce the chance that the investment return switches from high to low without being detected. The average length of a recession is longer in the Waiting Game, because of the presence of Regime M. Suppose the investment state switches to high during a Regime 0 cascade. In the No-Waiting Game, the economy moves to Regime 1, and the high investment state is revealed, ending the recession. However, in the Waiting Game, the economy moves to Regime M, and is likely to stay there for many periods. This directly prolongs the recession, and also allows for the possibility that the investment state switches back to the low state before the high state is detected.

Next we consider long-run probabilities of boom and recession.

Proposition 5.4: *Under the non-trivial case in which $c < \alpha$ holds for the large, patient, persistent economy, (i) the long-run probability of being in a recession in the Waiting Game is greater than in the No-Waiting Game, $\pi_R^W > \pi_R^{NW}$ and $\pi_B^W < \pi_B^{NW}$, and (ii) in the Waiting Game, the long-run probability of being in a recession is greater than the long-run probability of being in a boom, $\pi_R^W > \frac{1}{2} > \pi_B^W$.*

Proof. See the Appendix.

The intuition for Proposition (5.4) is the following. The economy is oscillating between boom and recession. Given that the average length of a boom is shorter and the average length of a recession is longer in the Waiting Game than in the No-Waiting Game (Proposition (5.3)), the economy must spend relatively more time in a recession in the Waiting Game. Part (ii) of the proposition follows from the fact that, in the Waiting Game, Regime 0 cascades are longer than Regime 2 cascades,[17] which would imply that the average length of a recession is longer than that of a boom under the hypothetical assumption that the economy learned the investment state after emerging from Regime 0. The fact that the economy enters Regime M after emerging from Regime 0 only adds to the relative likelihood of recession.

Finally, we show that the possibility of waiting reduces expected overinvestment and increases expected underinvestment. Let O be the overinvestment index, measuring the per capita investment when the state is low, divided by the number of periods. More specifically, we have

$$O = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} [\frac{I^t}{n} | S^t = 0]$$

[17] Recall that, for the No-Waiting Game, Regime 0 cascades are longer than Regime 2 cascades if and only if we have $c > 1/2$. However, for the Waiting Game, Regime 0 cascades approach infinity, while Regime 2 cascades are finite.

Similarly, define U as the underinvestment index:

$$U = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} [\frac{n - I^t}{n} | S^t = 1]$$

For the large, patient, persistent No-Waiting Game, the overinvestment index can be expressed as

$$O^{NW} = \{\frac{b^{NW}}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{b^{NW}}]\}\pi_2 + (1-\alpha)\pi_0 \tag{65}$$

where the term in braces is the expected number of periods that the investment state is low during b^{NW} consecutive periods of Regime 2. The second term is the probability that the investment state is low during Regime 1, multiplied by the fraction of agents that invest.

Similarly, U^{NW} can be calculated as

$$U^{NW} = \{\frac{r^{NW}}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{r^{NW}}]\}\pi_0 + (1-\alpha)\pi_{11} \tag{66}$$

For the large, patient, persistent Waiting Game, O^W can be computed as:

$$\begin{aligned} O^W &= \pi_{10}^W(1-\alpha) + \pi_{M0}^W(1-\alpha)\lambda_0 + \pi_2^W\{\frac{b^W}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{b^W}]\} \\ &= (1-\alpha)(\lambda_1\pi_{M1}^W + \lambda_0\pi_{M0}^W) + \frac{\lambda_1\pi_{M1}^W}{1 - \overline{p}^W}\{\frac{b^W}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{b^W}]\} \end{aligned} \tag{67}$$

Similarly, the underinvestment index for the Waiting Game, U^W, can be computed as:

$$U^W = \pi_{M1}^W[(1-\lambda_1) + \lambda_1(1-\alpha)] + \pi_0^W\{\frac{r^W}{2} - \frac{2\rho - 1}{4(1-\rho)}[1 - (2\rho - 1)^{r^W}]\} \tag{68}$$

Proposition 5.5: *Under the non-trivial case in which $c < \alpha$ holds for the large, patient, persistent economy, (i) the overinvestment index is higher for the No-Waiting Game than for the Waiting Game, $O^{NW} > O^W = 0$, and (ii) the underinvestment index is higher for the Waiting Game than for the No-Waiting Game, $U^{NW} < U^W$.*
Proof. See the Appendix.

The intuition for part (i) of Proposition (5.5) is that a Regime 2 cascade is a significant fraction of the investment cycle in the No-Waiting Game, but it is a negligible fraction of the investment cycle in the Waiting Game. The only possibilities for overinvestment in the Waiting Game are during a Regime 2 cascade in which the investment state has switched from high to low, during Regime 1 when the investment state is low, and during Regime M when the investment state is low and one agent invests in round 1. All of these situations form a negligible fraction of an investment cycle. The intuition for part (ii) is that the presence of Regime M increases the probability that no agent invests, even though the investment state has switched from low to high.

6 Concluding Remarks

The features of the equilibrium would not change much if we embedded an infinite number of rounds into each period, because the optimal choice of whether to invest or not involves a comparison between the expected profits of investing and the expected profits of delaying the decision for one round.[18] Because there are no payoff externalities, an agent that has invested does not care about future choices made by other agents. This feature makes computation of equilibrium trajectories feasible, even in large economies. Similarly, it would not be difficult to model investment payoffs as stretching into future periods. However, it would change the analysis substantially to allow agents to live, receive signals, and delay investment choices across periods.

Our equilibrium investment cycles do not look like the empirical patterns of investment or GDP for actual economies. Any attempt to calibrate the model would require the addition of new features. The fluctuations could be made less sharp by introducing asymmetries in investment cost, as in Levin and Peck (2006). That way, some type-1 agents (and perhaps some type-0 agents with very low cost) would be investing while others would be waiting. In large economies, fluctuations could be made less sharp by dropping the assumption of conditional independence. Also, more realistic looking cycles could arise if some information flowed directly from the investment activity, rather than equilibrium inferences based on the fact that an agent has invested.[19]

We hope that those interested in business cycles take the following messages from this paper. First, we have formally demonstrated that the option to delay investment will tend to lengthen recessions and shorten booms, as compared to the pattern that would have arisen without the option to wait. Second, asymmetric information gets aggregated by market activity, but imperfectly and in chunks. Eventually the economy finds itself in a regime in which different types choose different actions, and a lot of information gets revealed. Third, even large economies can find themselves in a situation where it takes just one agent to act, thereby releasing the information that either prolongs the recession further or else starts a boom.

7 Appendix

Proof of Lemma 5.1: For a type-1 agent that does not invest in round 1 and observes $k_1^t = 0$, the probability of the high investment state, $\mu_1^{0,q}$, is

$$\mu_1^{0,q} = \frac{1}{1 + \frac{1-\mu}{\mu}(\frac{1-\alpha}{\alpha})(\frac{1-(1-\alpha)q}{1-\alpha q})^{n-1}}.$$

Clearly q must be near zero in order to be in Regime M, so as n approaches infinity $\mu_1^{0,q}$ can be simplified to

$$\mu_1^{0,q} = \frac{1}{1 + \frac{1-\mu}{\mu}(\frac{1-\alpha}{\alpha})Q}. \tag{69}$$

Since $\mu_1^{0,q}$ must make a type-1 agent indifferent between investing and not investing in round 2, we must have $\mu_1^{0,q} = c$. From (69), we can solve for Q, yielding (38). From (37), the mixing probability q can be written as

$$q = \frac{Q^{\frac{1}{n}} - 1}{\alpha Q^{\frac{1}{n}} - (1-\alpha)}. \tag{70}$$

The probability of $k_1^t = 0$ in the high investment state is given by

$$\begin{aligned} pr01 &= (1-\alpha q)^n = [1 - \frac{\alpha(Q^{\frac{1}{n}} - 1)}{\alpha Q^{\frac{1}{n}} - (1-\alpha)}]^n \\ &= [\frac{\alpha Q^{\frac{1}{n}} - (1-\alpha)}{2\alpha - 1}]^{-n}. \end{aligned}$$

[18] Chamley and Gale (1994) refer to this as the one-step property.
[19] Veldkamp (2005) has this feature, without the asymmetric information.

Taking the limit of $\log(pr01)$, as n approaches infinity, yields

$$\lim_{n\to\infty} \log(pr01) = -\lim_{n\to\infty} \frac{\log[\frac{\alpha Q^{\frac{1}{n}}-(1-\alpha)}{2\alpha-1}]}{1/n}$$
$$= -\frac{\alpha}{2\alpha-1}\log Q.$$

Thus, we have

$$\lim_{n\to\infty} pr01 = Q^{-\alpha/(2\alpha-1)}.$$

By a similar computation, one can show that the probability of $k_1^t = 0$ in the low investment state is given by

$$\lim_{n\to\infty} pr00 = Q^{-(1-\alpha)/(2\alpha-1)}.$$

The probability of $k_1^t = 1$ in the high investment state is given by

$$n\alpha q(1-\alpha q)^{n-1}.$$

Therefore, we have

$$\lim_{n\to\infty} pr11 = \lim_{n\to\infty}(n\alpha q(pr01))$$
$$= \lim_{n\to\infty}(n\alpha\left[\frac{Q^{\frac{1}{n}}-1}{\alpha Q^{\frac{1}{n}}-(1-\alpha)}\right]pr01)$$
$$= \left(\frac{\alpha}{2\alpha-1}\right)Q^{-\alpha/(2\alpha-1)}\log(Q).$$

By a similar computation, one can show that the probability of $k_1^t = 1$ in the low investment state is given by

$$\lim_{n\to\infty} pr10 = \left(\frac{1-\alpha}{2\alpha-1}\right)Q^{-(1-\alpha)/(2\alpha-1)}\log(Q).$$

Now suppose that $\mu = \underline{\mu}^W$ holds. Equation (38) implies that we have $\lim_{n\to\infty} Q = 1$, so $\lim_{n\to\infty} pr01 = 1$ and $\lim_{n\to\infty} pr00 = 1$ hold. Although we have $\lim_{n\to\infty} pr11 = 0$ and $\lim_{n\to\infty} pr10 = 0$, having one agent invest is infinitely more likely than having more than one agent invest. To see this, note that

$$\lim_{n\to\infty}\left(\sum_{k=2}^{n} prk1\right) = 1 - \lim_{n\to\infty} pr11 - \lim_{n\to\infty} pr01 = 1 - \lim_{n\to\infty} pr11 - Q^{-\frac{\alpha}{2\alpha-1}}.$$

Thus, we have

$$\lim_{n\to\infty}\left(\frac{\sum_{k=2}^{n} prk1}{pr11}\right) = \lim_{Q\to1}[\frac{1-Q^{-\frac{\alpha}{2\alpha-1}}}{\frac{\alpha}{2\alpha-1}Q^{-\frac{\alpha}{2\alpha-1}}\log Q}] - 1 = \lim_{Q\to1}[\frac{Q^{\frac{\alpha}{2\alpha-1}}-1}{\frac{\alpha}{2\alpha-1}\log Q}] - 1$$
$$= \lim_{Q\to1}\frac{\frac{\alpha}{2\alpha-1}Q^{\frac{\alpha}{2\alpha-1}-1}}{\frac{\alpha}{2\alpha-1}\frac{1}{Q}} - 1 = 0.$$

A similar calculation yields

$$\lim_{n\to\infty}\left(\frac{\sum_{k=2}^{n} prk0}{pr10}\right) = 0,$$

which completes the proof. □

Proof of Proposition 5.3. Since $r^W = r^{NW}$ and $\underline{p}^W = \underline{p}^{NW}$, we will drop the superscripts without causing confusion. From (44), we have

$$\lim_{\rho\to 1}(r+1)(1-\rho) = \lim_{\rho\to 1} r(1-\rho) = \lim_{\rho\to 1}\frac{(1-\rho)\log[(2\rho-1)\frac{z-1}{z+1}]}{\log(2\rho-1)} = \frac{1}{2}\log\frac{1+z}{z-1}. \tag{71}$$

By (39), (40) and (41), we have

$$\lim_{\rho\to 1}\frac{\lambda_0}{\lambda_1} = \lim_{\rho\to 1}\frac{1-[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]^{\frac{1-\alpha}{2\alpha-1}}}{1-[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]^{\frac{\alpha}{2\alpha-1}}} = \lim_{\rho\to 1}\frac{\frac{1-\alpha}{2\alpha-1}[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]'}{\frac{\alpha}{2\alpha-1}[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]'} = \frac{1-\alpha}{\alpha}, \tag{72}$$

$$\lim_{\rho\to 1}\frac{(1-\rho)}{\lambda_1} = \lim_{\rho\to 1}\frac{1-\rho}{1-[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]^{\frac{\alpha}{2\alpha-1}}} = \lim_{\rho\to 1}\frac{-1}{-\frac{\alpha}{2\alpha-1}[\frac{1-\rho+\rho z}{z[\rho+(1-\rho)z]}]'} = \frac{2\alpha-1}{\alpha}\frac{z}{z^2-1}. \tag{73}$$

Since λ_1 goes to 0 as ρ goes to 1, we have

$$\lim_{\rho\to 1} r\lambda_1 = \lim_{\rho\to 1}(r+1)\lambda_1 = \lim_{\rho\to 1}(r+1)(1-\rho)\frac{\lambda_1}{1-\rho} = \frac{1}{2}\frac{\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}. \tag{74}$$

where the last equality uses (71) and (73). By (72) and (74), we have

$$\lim_{\rho\to 1} r\lambda_0 = \lim_{\rho\to 1} r\lambda_1\frac{\lambda_0}{\lambda_1} = \frac{1}{2}\frac{1-\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}. \tag{75}$$

Finally, by (56) we have

$$\begin{aligned}\lim_{\rho\to 1}\frac{\pi^W_{M0}}{\pi^W_{M1}} &= \lim_{\rho\to 1}\frac{1-\rho+\lambda_1(\rho-\underline{p}^W)}{1-\rho+\lambda_0[\rho-(1-\underline{p}^W)]} = \lim_{\rho\to 1}\frac{\frac{(1-\rho)}{\lambda_1}+(1-\frac{1}{1+z})}{\frac{(1-\rho)}{\lambda_1}+\frac{\lambda_0}{\lambda_1}\frac{1}{1+z}} \\ &= \frac{1+\frac{\alpha}{2\alpha-1}(z-1)}{1+\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}},\end{aligned} \tag{76}$$

where the last inequality uses (72) and (73).

By (59) and (61), we have

$$L_B^{NW} - L_B^W = \frac{b^{NW}+1}{\overline{p}^{NW}} - \frac{b^W+1}{\overline{p}^W}. \tag{77}$$

To show that the right side of (77) is positive, we define the function, $p^{10}(b) \equiv \frac{1}{2} - \frac{1}{2}(2\rho-1)^{b+1}$, which is the probability of the investment state switching from high to low after b periods. Note that $p^{10}(b^{NW}) = \overline{p}^{NW}$ and $p^{10}(b^W) = \overline{p}^W$ hold. We will now show that $\frac{b+1}{p^{10}(b)}$ is increasing in b. We have

$$\begin{aligned}&\frac{b+1}{p^{10}(b)} - \frac{b}{p^{10}(b-1)} = (b+1)\frac{2}{1-(2\rho-1)^{b+1}} - b\frac{2}{1-(2\rho-1)^b} \\ =\;& \frac{2}{[1-(2\rho-1)^{b+1}][1-(2\rho-1)^b]}[1-(b+1)(2\rho-1)^b + b(2\rho-1)^{b+1}].\end{aligned}$$

The sign of the previous expression is the same as the sign of the last term in brackets, which can be rewritten as:

$$\begin{aligned}&[1-(2\rho-1)^{b+1}] - (b+1)(2\rho-1)^b[1-(2\rho-1)] \\ =\;& [1-(2\rho-1)]\{[1+(2\rho-1)+...+(2\rho-1)^b] - (b+1)(2\rho-1)^b]\} \\ >\;& [1-(2\rho-1)]\{(b+1)(2\rho-1)^b - (b+1)(2\rho-1)^b\} = 0.\end{aligned} \tag{78}$$

where the last inequality follows from the fact that $(2\rho-1)<1$ holds. Thus, $\frac{b+1}{p^{10}(b)}$ is increasing in b, so we have $L_B^{NW}>L_B^W$.

To show $L_R^W>L_R^{NW}$, first recall that $L_R^{NW}=\frac{r+1}{\underline{p}}$ holds. By (62), (63), and (64), L_R^W can be rewritten as

$$\begin{aligned}L_R^W &= \frac{1}{\lambda_0\lambda_1\underline{p}+(1-\rho)(1-\lambda_0)\lambda_1}\{(r+1)[\lambda_0(1-\rho+\rho\lambda_1)+(1-\rho)(1-\lambda_0)\lambda_1]...\\ &\quad ...+\underline{p}(1-\lambda_1)[\lambda_0+(1-\lambda_0)\frac{(1-\rho)\lambda_1}{1-\rho+\lambda_1}]+(1-\lambda_0)[1-\rho+\rho\lambda_1-\underline{p}\lambda_1](2-\frac{\lambda_1}{1-\rho+\rho\lambda_1})\}.\end{aligned}$$

Thus, we have

$$\begin{aligned}L_R^W-L_R^{NW} &\propto \underline{p}^2(1-\lambda_1)[\lambda_0+(1-\lambda_0)\frac{(1-\rho)\lambda_1}{1-\rho+\rho\lambda_1}]+(1-\lambda_0)\underline{p}[1-\rho+\rho\lambda_1-\underline{p}\lambda_1](2-\frac{\lambda_1}{1-\rho+\rho\lambda_1})\\ &\quad -(r+1)(1-\rho)[\lambda_1(1-\lambda_0)(1-\underline{p})-\lambda_0(1-\lambda_1)\underline{p}].\end{aligned}$$

Since $\frac{\lambda_1}{1-\rho+\rho\lambda_1}<1$ holds, to show $L_R^{NW}<L_R^W$, the following condition is sufficient:

$$\begin{aligned}&(r+1)(1-\rho)[\lambda_1(1-\lambda_0)(1-\underline{p})-\lambda_0(1-\lambda_1)\underline{p}]\\ \leq\ &\underline{p}^2(1-\lambda_1)[\lambda_0+(1-\lambda_0)\frac{(1-\rho)\lambda_1}{1-\rho+\rho\lambda_1}]+(1-\lambda_0)\underline{p}[1-\rho+\rho\lambda_1-\underline{p}\lambda_1]\end{aligned}$$

By eliminating a positive term (the term with $\frac{(1-\rho)\lambda_1}{1-\rho+\rho\lambda_1}$) on the right hand side, and dividing both sides by $\lambda_1(1-\lambda_0)(1-\underline{p})$, the above inequality is implied by the following condition:

$$(r+1)(1-\rho)[1-\frac{\underline{p}}{1-\underline{p}}\frac{1-\lambda_1}{1-\lambda_0}\frac{\lambda_0}{\lambda_1}]\leq\underline{p}+\frac{\underline{p}}{1-\underline{p}}\frac{1-\lambda_1}{\lambda_1}(1-\rho)+\frac{\underline{p}^2}{1-\underline{p}}\frac{1-\lambda_1}{1-\lambda_0}\frac{\lambda_0}{\lambda_1}. \tag{79}$$

Using the limits (72)-(76), when ρ converges to 1, (79) becomes

$$\begin{aligned}\frac{1}{2}(1-\frac{1-\alpha}{\alpha}\frac{1}{z})\log\frac{1+z}{z-1} &\leq \frac{1}{1+z}+\frac{1}{z^2-1}\frac{2\alpha-1}{\alpha}+\frac{1}{z(z+1)}\frac{1-\alpha}{\alpha}\\ &\Leftrightarrow\ \frac{1}{2}\log\frac{1+z}{z-1}\leq\frac{1}{z^2-1}\frac{z^2-\frac{1-\alpha}{\alpha}}{z-\frac{1-\alpha}{\alpha}}.\end{aligned}$$

since $z>1$ holds, which implies

$$\frac{z^2-\frac{1-\alpha}{\alpha}}{z-\frac{1-\alpha}{\alpha}}\geq\frac{z^2}{z}=z,$$

the following inequality is sufficient to show $L_R^{NW}<L_R^W$:

$$\frac{2z}{z^2-1}-\log\frac{1+z}{z-1}\geq 0. \tag{80}$$

Given z is bounded, (80) holds. To verify the condition, the derivative of the expression with respect to z is $\frac{-4}{(z^2-1)^2}$, which is negative, so the left side of (80) is decreasing in z. Moreover, we have

$$\lim_{z\to\infty}[\frac{2z}{z^2-1}-\log\frac{1+z}{z-1}]=0,$$

which implies (80). □

Proof of Proposition 5.4. To establish part (i), it is sufficient to show

$$\lim_{\rho\to 1}\frac{\pi_R^{NW}}{\pi_B^{NW}} < \lim_{\rho\to 1}\frac{\pi_R^{W}}{\pi_B^{W}}, \tag{81}$$

because $\pi_R^{NW}+\pi_B^{NW}=1$ and $\pi_R^{W}+\pi_B^{W}=1$ hold.

By (32), (57), and (58), we have

$$\begin{aligned}\lim_{\rho\to 1}\frac{\pi_R^{NW}}{\pi_B^{NW}} &= \lim_{\rho\to 1}\frac{\frac{r+1}{\underline{p}}}{\frac{b^{NW}+1}{\overline{p}^{NW}}} = \lim_{\rho\to 1}\frac{\frac{r+1}{\underline{p}}\lambda_1}{\frac{b^{NW}+1}{\overline{p}^{NW}}\lambda_1} \quad \text{and}\\ \lim_{\rho\to 1}\frac{\pi_R^{W}}{\pi_B^{W}} &= \lim_{\rho\to 1}\frac{(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W}}{\frac{b^W+1}{\overline{p}^W}\lambda_1} > \lim_{\rho\to 1}\frac{(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W}}{\frac{b^{NW}+1}{\overline{p}^{NW}}\lambda_1}.\end{aligned} \tag{82}$$

Inequality (82) comes from the fact that $\frac{b^W+1}{\overline{p}^W} < \frac{b^{NW}+1}{\overline{p}^{NW}}$ holds. To show (81), it is sufficient to show

$$\lim_{\rho\to 1}(r\lambda_1+1)+(1+r\lambda_0)\frac{\pi_{M0}^W}{\pi_{M1}^W} \geq \lim_{\rho\to 1}\frac{r+1}{\underline{p}}\lambda_1. \tag{83}$$

Using the previous limiting results (72)-(76), inequality (83) is equivalent to

$$\begin{aligned}&\frac{1}{2}\frac{\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}+1+[1+\frac{1}{2}\frac{1-\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}]\frac{1+\frac{\alpha}{2\alpha-1}(z-1)}{1+\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}}\\ \geq\; &(1+z)\frac{1}{2}\frac{\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1},\end{aligned}$$

which can be simplified as

$$4+2\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}+2\frac{\alpha}{2\alpha-1}(z-1)+\frac{1-\alpha}{2\alpha-1}\frac{z^2-1}{z}\log\frac{z+1}{z-1}-\frac{\alpha}{2\alpha-1}(z^2-1)\log\frac{z+1}{z-1}\geq 0. \tag{84}$$

Note that

$$\frac{1}{2}\log\frac{z+1}{z-1}-\frac{1}{z-1}<0 \tag{85}$$

must hold. To see this, the derivative with respect to z is easily shown to be positive, which implies that the left side of (85) is increasing in z. Moreover, we have

$$\lim_{z\to\infty}[\frac{1}{2}\log\frac{z+1}{z-1}-\frac{1}{z-1}]=0,$$

so (85) holds. By (85), the left side of inequality (84) is strictly greater than

$$\begin{aligned}&4+2\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}+2\frac{\alpha}{2\alpha-1}(z-1)+2\frac{1-\alpha}{2\alpha-1}\frac{z+1}{z}-2\frac{\alpha}{2\alpha-1}(z+1)\\ =\; &4+4\frac{1-\alpha}{2\alpha-1}-4\frac{\alpha}{2\alpha-1}=0.\end{aligned}$$

Thus, inequality (84) holds, establishing part (i).

To show part (ii), note that

$$\lim_{\rho\to 1}\pi_B^W=\lim_{\rho\to 1}\frac{b^W+1}{\overline{p}^W}\lambda_1\pi_{M1}^W<\lim_{\rho\to 1}\frac{r+1}{\underline{p}}\lambda_1\pi_{M1}^W$$

holds. The inequality follows from the result, $r > b^W$. Now, (83) implies

$$\lim_{\rho\to 1} \frac{r+1}{\underline{p}}\lambda_1\pi_{M1}^W < \lim_{\rho\to 1}(r\lambda_1+1)\pi_{M1}^W + (1+r\lambda_0)\pi_{M0}^W.$$

From (58), we have

$$\lim_{\rho\to 1}\pi_B^W < \lim_{\rho\to 1}\pi_R^W.$$

□

Proof of Proposition 5.5. First, define the function, $A(b)$, by

$$A(b) \equiv \frac{1}{b}\{\frac{b}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^b]\} = \frac{1}{2} - \frac{2\rho-1}{4(1-\rho)}\frac{[1-(2\rho-1)^b]}{b}.$$

That is, $A(b)$ is the probability that the investment state is low during one of the b periods of Regime 2, chosen at random, and $A(r)$ is the probability that the investment state is high during one of the r periods of Regime 0, chosen at random. We show that $A(\cdot)$ is an increasing function. To see this, it is sufficient to show that

$$\frac{[1-(2\rho-1)^b]}{b} \text{ is decreasing in } b.$$

This condition is satisfied, since we have

$$\begin{aligned} & \frac{1-(2\rho-1)^b}{b} - \frac{1-(2\rho-1)^{b+1}}{b+1} \\ = & \frac{1}{b(b+1)}[1-(2\rho-1)^{b+1} - (b+1)(2\rho-1)^b(1-(2\rho-1))] > 0, \end{aligned}$$

where the last inequality follows from (78).

By (67), we have

$$\begin{aligned} \lim_{\rho\to 1} O^W &= \lim_{\rho\to 1}(1-\alpha)(\lambda_1\pi_{M1}^W + \lambda_0\pi_{M0}^W) + \frac{\lambda_1\pi_{M1}^W}{\overline{p}^W}\{\frac{b^W}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^{b^W}]\} \\ &= \lim_{\rho\to 1}\frac{b^w+1}{\overline{p}^W}\lambda_1\pi_{M1}^W A(b^W) = \lim_{\rho\to 1}\pi_B^W A(b^W). \end{aligned} \quad (86)$$

The second equality in (86) holds, since both λ_1 and λ_0 go to 0 as ρ goes to 1. On the other hand, by (65), we have

$$\begin{aligned} \lim_{\rho\to 1} O^{NW} &= \lim_{\rho\to 1}\{\frac{b^{NW}}{2} - \frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^{b^{NW}}]\}\frac{\underline{p}}{(r+1)\overline{p}^{NW} + (b^{NW}+1)\underline{p}} \\ & \quad +(1-\alpha)\frac{\overline{p}^{NW}}{(r+1)\overline{p}^{NW} + (b^{NW}+1)\underline{p}} \\ &= \lim_{\rho\to 1} A(b^{NW})\frac{(b^{NW}+1)\underline{p}}{(r+1)\overline{p}^{NW} + (b^{NW}+1)\underline{p}} = \lim_{\rho\to 1}\pi_B^{NW} A(b^{NW}). \end{aligned} \quad (87)$$

where the second inequality holds, since r and b^{NW} go to infinity as ρ goes to 1. Now we compare (86) and (87). From Proposition (5.4), we have $\lim_{\rho\to 1}\pi_B^{NW} > \lim_{\rho\to 1}\pi_B^W$. By the fact, $b^{NW} > b^W$, we have $A(b^{NW}) > A(b^W)$. Indeed, since b^W is finite, it is easy to see using l'Hopital's rule that $\lim_{\rho\to 1} A(b^W) = 0$ holds. Therefore, when ρ approches 1, we have $O^{NW} > O^W = 0$. This proves property (i).

Now we show property (ii). By (66), we have

$$\begin{aligned}\lim_{\rho\to1} U^{NW} &= \lim_{\rho\to1}\frac{\overline{p}^{NW}}{(r+1)\overline{p}^{NW}+(b^{NW}+1)\underline{p}}\{\frac{r}{2}-\frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^r]\} \\ &= \lim_{\rho\to1}\frac{r\overline{p}^{NW}}{(r+1)\overline{p}^{NW}+(b^{NW}+1)\underline{p}}A(r)=\lim_{\rho\to1}\pi_R^{NW}A(r). \end{aligned} \tag{88}$$

The second equality follows the fact that r goes to infinity as ρ goes to 1. On the other hand, by (68), we have

$$\begin{aligned}\lim_{\rho\to1} U^{W} &= \pi_{M1}^W[(1-\lambda_1)+\lambda_1(1-\alpha)]+\pi_0^W\{\frac{r}{2}-\frac{2\rho-1}{4(1-\rho)}[1-(2\rho-1)^r]\} \\ &= \lim_{\rho\to1}\pi_{M1}^W+(r\lambda_1\pi_{M1}^W+r\lambda_0\pi_{M0}^W)A(r) \\ &= \lim_{\rho\to1}[(r\lambda_1+1)\pi_{M1}^W+(r\lambda_0+1)\pi_{M0}^W]A(r)+[1-A(r)]\pi_{M1}^W-A(r)\pi_{M0}^W \\ &= \lim_{\rho\to1}\pi_R^W A(r)+\lim_{\rho\to1}[1-A(r)]\pi_{M1}^W-A(r)\pi_{M0}^W, \end{aligned} \tag{89}$$

where the second equality follows from the fact that λ_1 goes to 0 as ρ goes to 1, and the last equality follows from (58). Now we compare (88) and (89). Since by Proposition (5.3), $\lim_{\rho\to1}\pi_R^W > \lim_{\rho\to1}\pi_R^{NW}$ holds, the following condition is sufficient to show $\lim_{\rho\to1}(U^W-U^{NW})>0$:

$$\lim_{\rho\to1}[1-A(r)]\pi_{M1}-A(r)\pi_{M0}\geq 0, \tag{90}$$

which is equivalent to

$$\lim_{\rho\to1}[\frac{1}{2}+\frac{2\rho-1}{4(1-\rho)}\frac{1-(2\rho-1)^r}{r}]-[\frac{1}{2}-\frac{2\rho-1}{4(1-\rho)}\frac{1-(2\rho-1)^r}{r}]\frac{\pi_{M0}^W}{\pi_{M1}^W}\geq 0. \tag{91}$$

However, we have

$$\lim_{\rho\to1}\frac{2\rho-1}{4(1-\rho)}\frac{1-(2\rho-1)^r}{r}=\lim_{\rho\to1}\frac{\underline{p}}{2(r+1)(1-\rho)}=\frac{1}{(z+1)\log\frac{z+1}{z-1}}. \tag{92}$$

Substituting (92) into (91), we see that (90) is equivalent to

$$\frac{1}{2}+\frac{1}{(z+1)\log\frac{z+1}{z-1}}-(\frac{1}{2}-\frac{1}{(z+1)\log\frac{z+1}{z-1}})\frac{1+\frac{\alpha}{2\alpha-1}(z-1)}{1+\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}}\geq 0,$$

which is equivalent to

$$\frac{1}{2}\left(\frac{1-\alpha}{2\alpha-1}\right)\frac{z-1}{z}+\frac{1}{(z+1)\log\frac{z+1}{z-1}}[2+\frac{1-\alpha}{2\alpha-1}\frac{z-1}{z}+\frac{\alpha}{2\alpha-1}(z-1)]-\frac{1}{2}\frac{\alpha}{2\alpha-1}(z-1)\geq 0.$$

Since $\frac{1}{2}\log\frac{z+1}{z-1}<\frac{1}{z-1}$ holds, by (85), the left hand side of the above inequality is greater than

$$\begin{aligned}&\frac{1}{2}\left(\frac{1-\alpha}{2\alpha-1}\right)\frac{z-1}{z}-\frac{1}{2}\frac{\alpha}{2\alpha-1}(z-1)+\frac{z-1}{z+1}[1+\frac{1}{2}\left(\frac{1-\alpha}{2\alpha-1}\right)\frac{z-1}{z}+\frac{1}{2}\frac{\alpha}{2\alpha-1}(z-1)] \\ =&\left(\frac{1-\alpha}{2\alpha-1}\right)\frac{z-1}{z+1}+\frac{z-1}{z+1}-\frac{\alpha}{2\alpha-1}\frac{z-1}{z+1}=0.\end{aligned}$$

Therefore, inequality (90) holds. □

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Figure 1: State vs Time

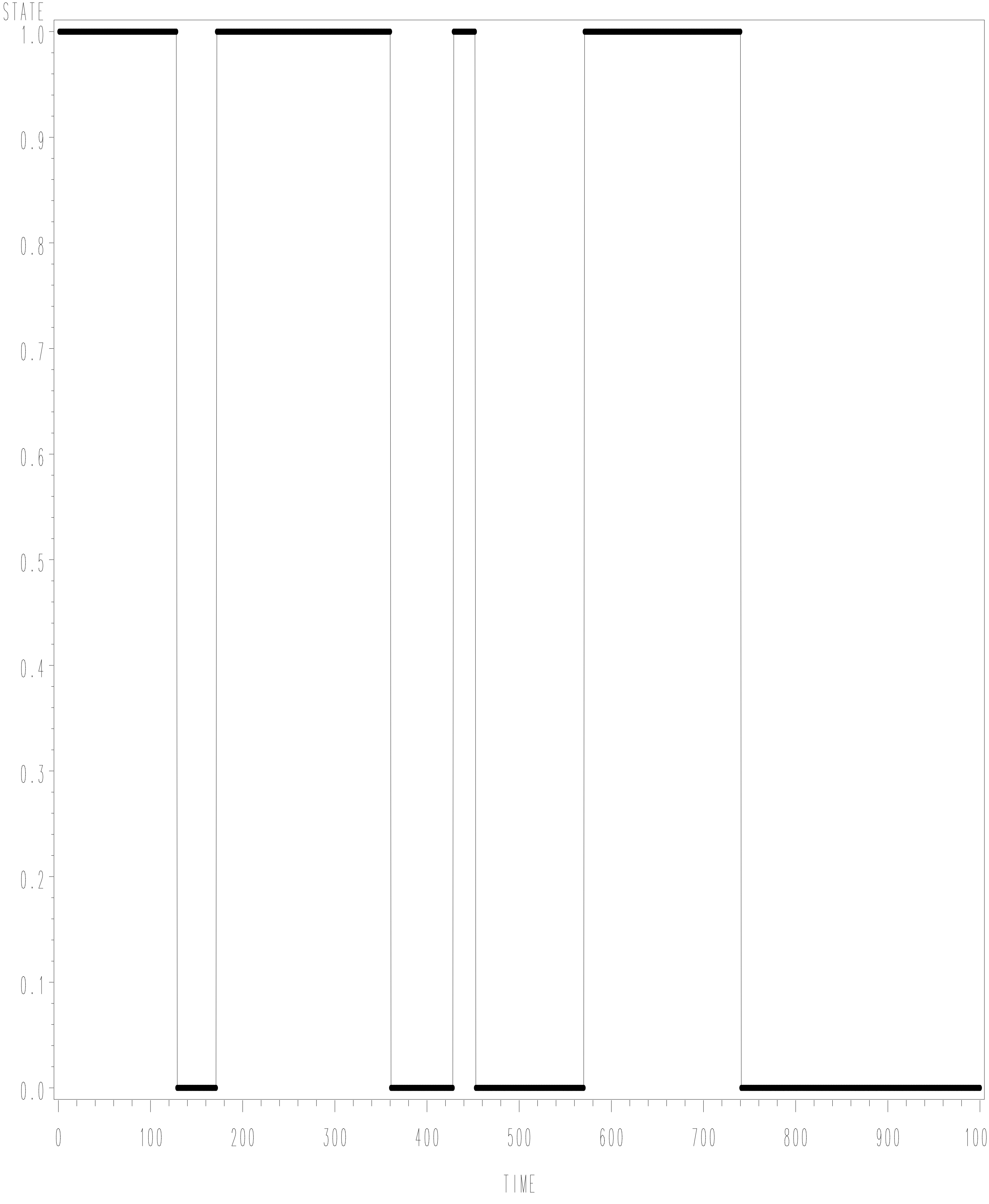

Figure 2: No–Waiting Game, Total Investment vs Time

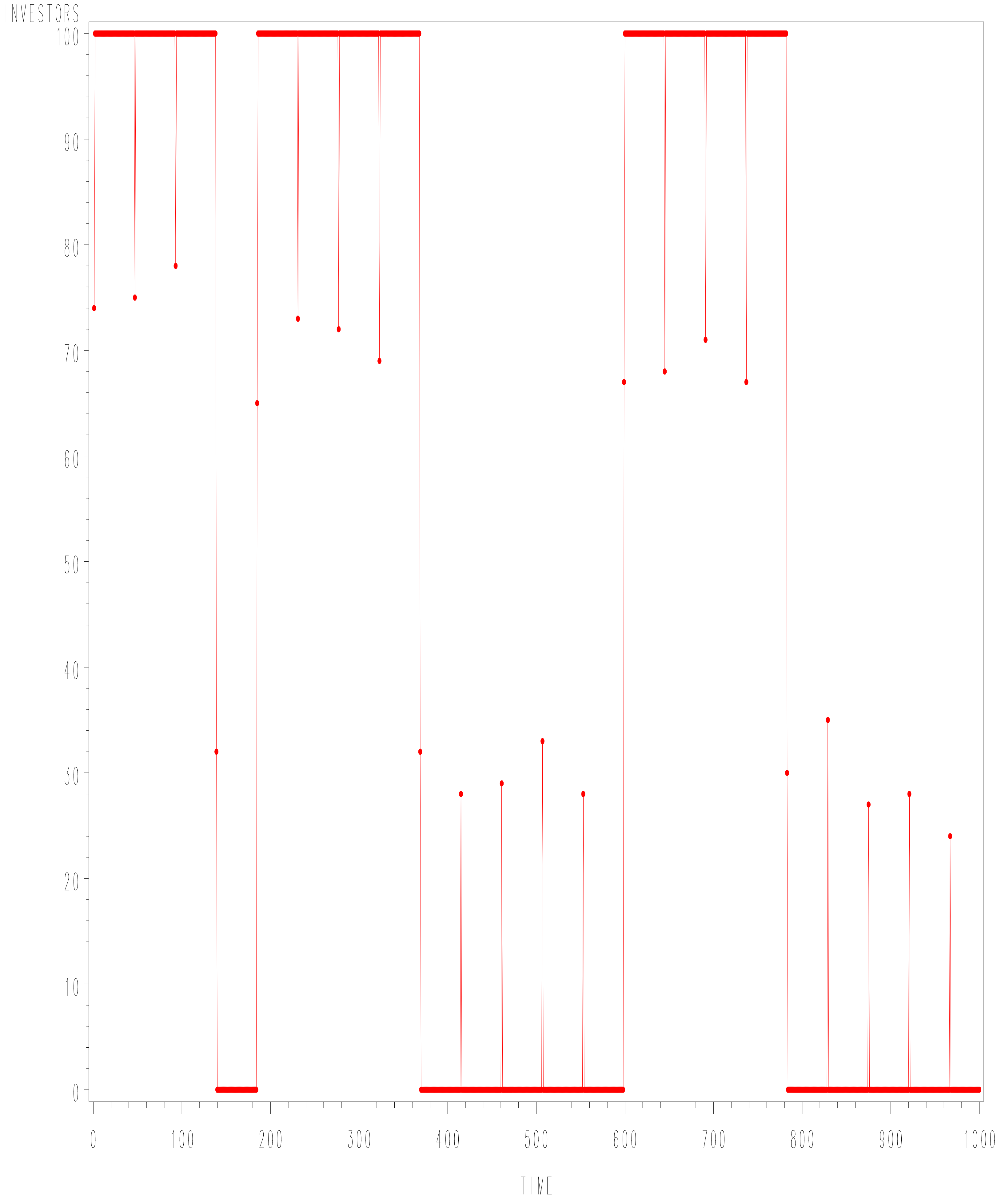

Figure 3: Waiting Game, Round 1 Investment vs Time

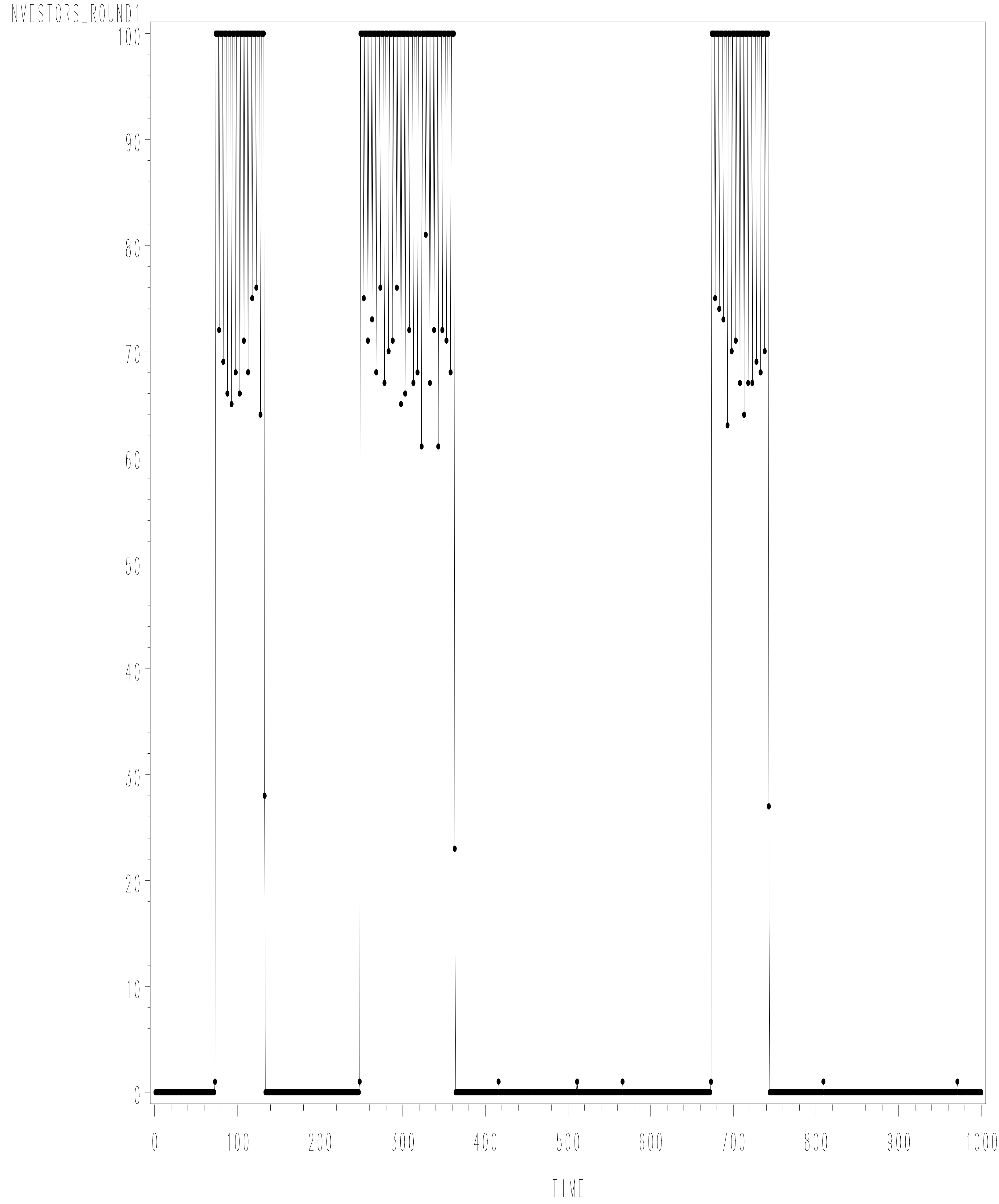

Figure 4: Waiting Game, Total Investment vs Time