

A Model in which Outside and Inside Money are Essential

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Abstract

Cavalcanti-Wallace (1999) study implementable allocations in a random matching model of money in which some people have publicly known histories (bankers) and others have private histories (non-bankers), and in which bankers have access to a printing press that enables them to issue bank-notes. They compare the use of outside money and inside money and show that implementable outside-money allocations are a strict subset of implementable inside-money allocations. I generalize their model by introducing a lag in the updating of the public record of the bankers. Such a lag is one way to model imperfect information about banker histories. I show that for some parameters in the generalized model, the class of symmetric inside-money allocations studied by Cavalcanti-Wallace is not a superset of outside-money allocations. Indeed, for those parameters, there exist allocations that are implementable only via the combined use of both outside money and inside money.

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1 Introduction

Cavalcanti-Wallace (1999) study implementable allocations in a random matching model of money in which some people have publicly known histories (bankers) and others have private histories (nonbankers), and in which bankers have access to a printing press that enables them to issue bank-notes. They compare the use of outside money and inside money and show that implementable outside-money allocations are a strict subset of implementable inside-money allocations. I generalize their model by introducing a lag in the updating of the public record of the bankers. Such a lag is one way to model imperfect information about banker histories. I show that for some parameters in the generalized model, the class of symmetric inside-money allocations studied by Cavalcanti-Wallace is not a superset of outside-money allocations. Indeed, for those parameters, there exist allocations that are implementable only via the combined use of both outside money and inside money.

Cavalcanti-Wallace use a background environment that is similar to the settings of Trejos-Wright (1995) and Shi (1995). However, there are two main differences. Those papers rule out any sort of credit by assuming that people cannot commit to future actions and that histories are private information. Cavalcanti-Wallace and this paper weaken the privacy-of-histories assumption. Some weakening is necessary to get inside money to work; some knowledge of histories is necessary to induce bankers to redeem banknotes. A second way in which Cavalcanti-Wallace and this paper differ from previous work is that both use mechanism design rather than imposing a given bargaining rule. That permits us to examine the entire set of implementable allocations, rather than those that depend on a particular way trade is conducted. Such an analysis allows us to describe in terms of the environment when different types of monetary arrangements are essential. Specifically, I say that the simultaneous use of both outside money and inside money is essential if it implements allocations that cannot be implemented via the exclusive use of outside money, or the exclusive use of inside money.

The Cavalcanti-Wallace strict subset result arises due to the ability of bankers to consume independent of their recent trades whenever inside money is used; a banker's ability to offer something tangible to a nonbanker depends on the banker's recent trades. This potential advantage of inside money is also present in my version. However, a new consideration arises. When inside money is used and there is a lag in updating banker histories, there is added

inducement of bankers to issue notes when the mechanism says they should not and there is added inducement for them not to redeem notes when the mechanism says they should. These inducements are strongest using inside money because the states in which bankers are not to issue or are to redeem are private information for a while—until histories are updated. In other words, with a lag, inside money has a potential disadvantage; namely, tighter and additional incentive constraints. The implied trade-off is what gives rise to the possibility that the use of both outside money and inside money are essential.

I compare three sets of implementable allocations: those that arise via the use of two outside monies, those that arise via the use of two inside monies, and those that use one outside money and one inside money. (In contrast to Cavalcanti-Wallace, I need to consider the simultaneous usage of both outside money and inside money. Given the assumed indivisibility of money, I need to compare their simultaneous use to the use of either two outside monies or two inside monies¹.) I find that for a given environment, one in which the lag in updating histories is neither too short nor too long, there is no subset ordering of implementable allocations. That is, given that environment, I find three allocations, each of which is implementable in only one monetary arrangement. In particular, one of the allocations is implementable only using both outside money and inside money, thereby demonstrating the essentiality of both kinds of money.

The rest of the paper is organized as follows. Section 2 presents the environment. Section 3 describes the restricted class of mechanisms I study and the conditions for implementability. Section 4 relates the model to that in Cavalcanti-Wallace (1999). Section 5 presents the result that there is no subset ordering of implementable allocations, while section 6 offers concluding remarks.

2 The Economic Environment

The environment follows closely that of Cavalcanti-Wallace (1999), which is based on the random-matching model of Trejos-Wright (1995).

Time is discrete and the horizon is infinite. There are S distinct, divisible, and perishable types of goods at each date and there is a $[0; 1]$ continuum

¹Two distinct assets are helpful when money is indivisible. See Aiyagari-Wallace-Wright (1996).

of each of S specialization types of agents, where $S > 2$. An agent whose specialization type is s consumes only good s and produces only good $s + 1$ (modulo S), for $s = 1; 2; \dots; S$. Each agent maximizes expected discounted utility with a discount factor $\beta \in (0; 1)$. Utility in a period is given by $u(x; y)$ where x is the amount consumed and y is the amount produced. The function u is defined on $[0; 1)$, is increasing, twice differentiable, and satisfies $u(0) = 0$, $u'' < 0$, and $u'(0) = 1$. Moreover, there exists a $y^0 > 0$ such that $u(y^0) = y^0$:

In each period, agents are randomly matched in pairs. There are two kinds of meetings. A single-coincidence meeting is a meeting that contains a type s agent (the producer) and a type $s + 1$ agent (the consumer) for some s . A no-coincidence meeting is a meeting in which neither agent produces what the other consumes. Because $S > 2$; there are no meetings in which there is a double-coincidence of wants.

It is assumed that agents cannot commit to future actions. This implies that those agents who produce must receive a future reward for doing so.

As in Cavalcanti and Wallace (1999), the society is able to keep a public record of the actions of a fraction B of each specialization type of agent, where $B \in [0; 1]$: Agents whose histories are a part of the public record are called bankers. Society has no public record for the remaining fraction $1 - B$ of each specialization type, the nonbankers. We can think of B as society's capacity for keeping track of individual trading histories.

In order to extend the model to allow for imperfect information about bankers, I borrow from Kocherlakota-Wallace (1998) and assume that the public record is not updated immediately after every action. Specifically, there is a deterministic lag of T periods, where $T \geq 0$.² At the beginning of each date t , the bankers' trading histories are known up through what they did until the beginning of date $t - T$. Thus, if $T = 0$, then banker histories are completely publicly known and if $T = 1$, then their histories are effectively unknown. We can think of T as society's ability to update records.

The above model embodies the aforementioned random-matching models of money as special cases. Trejos-Wright (1995) is essentially a special case of the above model where all agents have private histories, $B = 0$. Kocherlakota-Wallace (1998) is essentially the special case where all agents

²In their model, everyone's history ($B = 1$) is updated with probability $\frac{1}{2}$ at each date, which produces an average updating lag of $\frac{1}{2}$. There, that led to a slightly simpler formulation of the payoff from defecting than a deterministic lag. Here, the more straightforward deterministic lag is simpler.

have publicly observable histories, $B = 1$. Cavalcanti-Wallace (1999) is the special case where there is no lag in updating the histories of bankers, $B \geq 0; 1$ and $T = 0$.

There are two distinct assets in the model. These assets are indivisible and agents can carry at most one unit of some asset across dates³. These assets could be two outside monies, two inside monies, or one outside money and one inside money. In fact, three separate monetary arrangements or "regimes" will be compared: an outside-money regime (where two outside monies are used), an inside-money regime (where two inside monies are used), and a mixed regime (where one outside money and one inside money are used). I study the use of two outside monies and the use of two inside monies to provide comparable alternatives to an economy's use of one outside money and one inside money. If I, for example, compared an economy that uses one outside money and one inside money to one that uses only one inside money, then the former may be advantageous solely because there are two distinct assets.

Outside money is neither produced nor consumed. It is indivisible and perfectly durable. If both assets in the environment are outside monies, then the two are distinguishable in some way.

Each agent has a technology that permits that agent to create indivisible and perfectly durable objects, called notes. These notes are inside money. If both assets in the environment are inside monies, then each agent can create two distinguishable kinds of notes⁴. The notes issued by a single agent are distinguishable from those issued by another agent so that counterfeiting is not a problem.

When two agents meet, I assume that they have the following common information. They know each other's specialization type, asset holdings, and whether or not their trading partner is a banker or nonbanker. They also know the past actions of the bankers in the meeting that occurred up to $t_j - T$ periods ago.

³It is not possible for an agent to simultaneously hold a unit of both assets.

⁴There are other ways to introduce two inside monies. For example, we could say a subset of each specialization type has a printing press that can print inside money 1 and another subset of each specialization type has a printing press that can print inside money 2. I choose to think of each agent as having two printing presses, one that can print inside money 1 and one that can print inside money 2 for its ease of exposition.

3 Mechanisms

I look at a class of symmetric, stationary and deterministic mechanisms. A mechanism consists of a distribution of agents over states, state transitions for the agents, and, in single-coincidence meetings, output. The state transitions and output are functions of both the identity (banker or nonbanker) and announced states of the agents in a meeting. A mechanism is symmetric in the sense that state transitions and output are independent of an individual's specialization type and his relative position within that specialization type. Further, notes issued by bankers are treated symmetrically by the agents in the economy and those issued by nonbankers are ignored⁵. In the case of the inside-money regime, inside money 1 issued by bankers is treated symmetrically and inside money 2 is treated symmetrically, although the treatment of inside money 1 and inside money 2 is not. The mechanism is stationary because I require the distribution of agents over states to be a steady-state distribution, given the state transitions. A mechanism is deterministic in the sense that given the identities and announced states of the agents in a meeting, the suggested output and state transitions are known with certainty.

Let the set of individual identities be $fb; ng$; where b indicates that an agent is a banker and n indicates that an agent is a nonbanker. The state of an agent is a member of the set $A = \{0; 1; 2\}$. For nonbankers, the state represents money holdings, where state 0 indicates no asset holdings, state 1 indicates that a nonbanker holds a unit of asset 1; and state 2 indicates that a nonbanker holds a unit of asset 2. The state of a nonbanker is always observable in meetings because asset holdings are common information.

The state interpretation for bankers, however, is contingent on the type of regime being considered. In the outside-money regime, a banker's state describes a banker's asset holdings. Thus, banker states are completely observable in the outside-money regime.

Due to the symmetry imposed on mechanisms, the states cannot repre-

⁵Focusing on mechanisms that have a symmetric treatment of bank-notes is a limitation of the analysis. One could think of alternative ways in which inside money could be used that would permit bankers to hold inside money. An example of such a scheme can be found in Cavlacanti-Erosa-Temzelides (1999). They model inside money in a random matching model where banker histories are summarized by balance sheets. Whenever a banker's notes are redeemed too frequently, she is no longer a banker. It is not clear whether such a scheme is preferable in their model because they did not use a mechanism design approach.

sent asset holdings for bankers in the inside-money regime. Bankers do not hold inside money because they can costlessly produce their own. In the inside-money regime, therefore, a banker's state is extraneous. There is an implicit mapping of banker histories into A . It is important to note that this mapping is from nondefecting banker histories and does not represent banker defections. Bankers must be assigned extraneous states in the inside-money regime to provide an adequate comparison of allocations implementable under different regimes. In the outside-money regime, banker money-holdings may determine whether a banker produces or consumes. For example, a banker with a unit of outside money (in state 1 or 2) may be able to consume in certain meetings while a banker without a unit of outside money (in state 0) may be induced to produce in order to acquire money in certain meetings. In order to talk about duplication in the inside money world, we must be able to distinguish the meetings in which bankers should produce and those in which they should consume. This requires a set of states at least as large as A .⁶ In the case of the inside-money regime, a banker's state is private information whenever $T > 0$.

For the mixed regime, without loss of generality, refer to asset 1 as the inside money and asset 2 as the outside money. If a banker is in state 2, that implies that she has a unit of outside money. If the banker is in either state 0 or state 1, then that implies she does not have a unit of outside money. States 0 and 1 are extraneous in the sense that only one is needed to represent a banker without a unit of money. As with the inside-money regime, we need the set of states to be of size A for comparison with other regimes. Whether a banker is in state 0 or state 1 is private information whenever $T > 0$. If a banker is in state 2, her state is observable because money-holdings are observable.

The state interpretations above enables us to use the same notation for each of the three regimes considered. I need to define four functions. The first maps the announced states of two trading partners in a single-coincidence meeting into output. Let $y_{ij}^{kl} \in \mathbb{R}^+$ be output when a producer with identity k announces state i and a consumer with identity l announces state j . Formally, for all $(k; l) \in \mathbb{R}^+ \times \mathbb{R}^+$, let

$$y_{ij}^{kl} : A \times A \rightarrow \mathbb{R}^+$$

⁶Of course, we could have richer descriptions of banker histories in the inside-money regime. But they could also be richer in both the outside-money regime and the mixed regime and so do not add much in terms of the comparisons of the regimes.

Similarly, let

$$p_{c;c}^{kl} : A \in A \rightarrow A$$

be the state transition for the producer in the meeting and

$$q_{c;c}^{kl} : A \in A \rightarrow A$$

be that for the consumer in the meeting.

Let r_{ij}^{kl} be the state-transition function for the agent with identity k who announces state i in a no-coincidence meeting with an agent with identity l who announces state j . Formally, for all $(k; l) \in \mathbb{F}_b; \mathbb{N}_g \in \mathbb{F}_b; \mathbb{N}_g$, let

$$r_{c;c}^{kl} : A \in A \rightarrow A:$$

I describe a mechanism in the following way. There are two stages within a meeting. At the first stage agents simultaneously announce states. Such an announcement is superfluous for nonbankers because their states always represent asset-holdings which are common knowledge in a meeting. The announcement for bankers is contingent on regime. Formally, let $A_p \subseteq A$ be the set of states over which a banker can misrepresent. In the outside-money regime $A_p = \emptyset$; since states represent money-holdings and are common knowledge in a meeting. In the inside-money regime, however, $A_p = A$; bankers have complete freedom to misrepresent their state because it is private information. In the mixed regime, $A_p = \{0; 1\}$. A banker can report a state in A_p only when her true state is in A_p : A banker cannot misrepresent at all when her true state is 2, because it represents outside money-holdings.

Given the identities of the agents, and their announced states from the first stage, a mechanism suggests state transitions and, for single-coincidence meetings, output. At the second stage, agents simultaneously decide whether to agree or disagree to the suggestions. If both agree, the suggestions are carried out. If at least one agent disagrees, then there are no state transitions and, for single-coincidence meetings, no output.

A mechanism is implementable if it induces a) truth-telling by bankers in the first stage and b) participation by both bankers and nonbankers in the second stage. Notice that participation requires agreement in the second stage to be ex-post sequentially individually rational for agents; the announcements in the first stage eliminate any uncertainty about what the suggested state transitions and output are at the time an agent must decide to agree or disagree.

If a banker either misrepresents her state in the first stage, or chooses not to participate in the suggested state transitions and output in the second stage, then that banker is a defector. A nonbanker does not receive such a label since a defection by him would never be discovered. A defection by a banker is discovered T periods from the date it occurred (say date t); a banker's true state and announced state at date t are publicly revealed at date $t + T$, as well as whether or not a banker agreed to the state transitions and output in the second stage. For the $T - 1$ periods that follow an initial defection, a banker is an undiscovered defector. An undiscovered defector can costlessly choose to either truthfully participate in trade or to defect again. From period $t + T$ on, a defecting banker is a discovered defector and is punished with autarky. What is relevant for implementability is the expected payoff to a banker from defecting for the first time. This expected payoff incorporates the potential for positive gains from trade for $T - 1$ periods after the initial defection, and autarky from period T on.

I study three subsets of allocations that are implementable via the above mechanism: outside-money allocations, inside-money allocations, and mixed allocations (allocations that arise from the use of both outside money and inside money). These subsets differ in two respects: the number of feasibility constraints that are imposed on state transitions, and the number of incentive constraints imposed on agents. As we shall see, the feasibility constraints are most restrictive whenever outside money is used and the incentive constraints are most restrictive whenever inside money is used. In fact, there are no truth-telling constraints whenever outside money is used because money-holdings are assumed to be common knowledge in a meeting⁷.

I compare the implementability of three examples of allocations given a set of parameters. The parameter of interest is the lag of updating banker histories, T . I show that there are allocations that are implementable in each regime that are not implementable in the other two. This includes an example that provides an essential role for the simultaneous use of both outside money and inside money as defined above. The fact that there are alloca-

⁷It may be possible that individuals who have a unit of money can hide it from their trading partner. If the assumption that money-holdings are common knowledge is dropped, then truth-telling constraints would be necessary when outside money is used in the above situation. The use of outside money would still have fewer truth-telling constraints than the use of inside money because it is not possible for individuals to claim to have a unit of money when they do not. Thus, the assumption that money-holdings are common knowledge in a meeting should not significantly change the results.

tions implementable only with both outside and inside money arises from the trade-off between outside money having the most feasibility restrictions and inside money having the most incentive constraints.

3.1 Steady-State and Feasibility Requirements

As mentioned above, I look at stationary mechanisms in the sense that the distribution of agents over states is a steady-state distribution, given the state transitions. Let us now describe the requirements imposed on state transitions so that they follow a steady-state distribution.

Let x_i^k denote the fraction of each specialization type who are of identity k in state i : Because each person must be in one of the states, the fractions of each specialization type in each state must satisfy:

$$\sum_i x_i^b = B \text{ and } \sum_i x_i^n = 1 - B \quad (1)$$

A steady-state distribution of agents over states requires that the fraction of bankers in each state and the fraction of nonbankers in each state be constant. This can be expressed by equating the inflow and outflow of each state for nonbankers and bankers. For nonbankers, an inflow into a state i occurs whenever a nonbanker in state $h \in i$ in a meeting with a banker leaves the meeting in state i . An outflow from state i occurs whenever a nonbanker in state i in a meeting with a banker leaves the meeting in state $h \in i$. For bankers, the inflows and outflows have a similar interpretation in meetings with nonbankers. Formally, the inflow-equal-outflow conditions are

$$\sum_{h \in i} x_h^n \sum_j x_j^b [p_{hj}^{nb}(i) + q_{jh}^{bn}(i) + (S_i - 2)r_{hj}^{nb}(i)] = x_i^n \sum_j x_j^b \sum_{h \in i} [p_{ij}^{nb}(h) + q_{ji}^{bn}(h) + (S_i - 2)r_{ij}^{nb}(h)] \quad (2)$$

for nonbankers and

$$\sum_{h \in i} x_h^b \sum_j x_j^n [p_{hj}^{bn}(i) + q_{jh}^{nb}(i) + (S_i - 2)r_{hj}^{bn}(i)] = x_i^b \sum_j x_j^n \sum_{h \in i} [p_{ij}^{bn}(h) + q_{ji}^{nb}(h) + (S_i - 2)r_{ij}^{bn}(h)] \quad (3)$$

for bankers, where the indicator variable $p_{ij}^{kl}(h) = 1$ if $p_{ij}^{kl} = h$ and $p_{ij}^{kl}(h) = 0$ otherwise. The indicator variables $q_{ij}^{kl}(h)$ and $r_{ij}^{kl}(h)$ are defined similarly.

In addition to the steady-state restrictions imposed on state transitions, there are also feasibility constraints implied by the preservation of total money holdings in meetings. This applies to outside-money holdings in all

meetings and to inside-money holdings in meetings between nonbankers. The outside-money regime has the largest number of feasibility restrictions because outside-money holdings must be preserved in all meetings, regardless of agent identities. The inside-money regime has the fewest number of feasibility constraints because inside-money holdings must be preserved only in meetings between nonbankers. Finally, the mixed regime has an intermediate number of feasibility constraints. The preservation of outside-money holdings must be satisfied in meetings regardless of agent identities, but this pertains only to asset 2. Asset 1, the inside money, must be preserved only in meetings between nonbankers. The following definitions formally express the feasibility constraints as they pertain to each regime.

Definition 1 A stationary and feasible outside-money allocation is $(x_i^k; y_{ij}^k; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i; j \in A$ and $k; l \in fb; ng$ that satisfies (1) - (3) and

$$p_{ii}^{kl} = q_{ii}^{kl} = r_{ii}^{kl} = i \quad (4)$$

$$\begin{aligned} p_{ij}^{kl} = j & \text{ if and only if } q_{ij}^{kl} = i \\ r_{ij}^{kl} = j & \text{ if and only if } r_{ji}^{kl} = i \end{aligned} \quad (5)$$

for $i; j \in A$ and $k; l \in fb; ng$:

Condition (4) says that if both agents have the same money-holdings, then they leave with the same money-holdings. Condition (5) says that agents swap money-holdings.

Definition 2 A stationary and feasible inside-money allocation is $(x_i^k; y_{ij}^k; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i; j \in A$ and $k; l \in fb; ng$ that satisfies (1) - (3), and

$$p_{ii}^{nn} = q_{ii}^{nn} = r_{ii}^{nn} = i \quad (6)$$

$$\begin{aligned} p_{ij}^{nn} = j & \text{ if and only if } q_{ij}^{nn} = i \\ r_{ij}^{nn} = j & \text{ if and only if } r_{ji}^{nn} = i: \end{aligned} \quad (7)$$

In the inside-money regime, conditions (6) and (7) are weaker than (4) and (5) in the sense that they only pertain to meetings between nonbankers.

In order to define a stationary mixed allocation, we need to be explicit about which asset is the outside money and which asset is the inside money. Recall that asset 1 refers to the inside money and asset 2 refers to the outside money.

Definition 3 A stationary and feasible mixed allocation is $(x_i^k; y_{ij}^{kl}; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i, j \in A$ and $k, l \in fb; ng$ that (i) satisfies (1) - (3) for $i \in A$ and $k \in fb; ng$,

$$\begin{aligned} p_{ii}^{nn} &= q_{ii}^{nn} = r_{ii}^{nn} = i \\ p_{22}^{bl} &= q_{22}^{bl} = r_{22}^{bl} = 2 \\ p_{22}^{lb} &= q_{22}^{lb} = r_{22}^{lb} = 2 \end{aligned} \tag{8}$$

$$\begin{aligned} p_{ij}^{nn} &= j \text{ if and only if } q_{ij}^{nn} = i \\ r_{ij}^{nn} &= j \text{ if and only if } r_{ji}^{nn} = i: \end{aligned} \tag{9}$$

$$\begin{aligned} p_{i2}^{lb} &= 2 \text{ if and only if } q_{i2}^{lb} \notin 2 \\ p_{i2}^{bl} &= 2 \text{ if and only if } q_{i2}^{bl} \notin 2 \\ q_{2i}^{bl} &= 2 \text{ if and only if } p_{2i}^{bl} \notin 2 \\ q_{2i}^{lb} &= 2 \text{ if and only if } p_{2i}^{lb} \notin 2 \\ r_{i2}^{lb} &= 2 \text{ if and only if } r_{2i}^{bl} \notin 2 \\ r_{i2}^{bl} &= 2 \text{ if and only if } r_{i2}^{bl} \notin 2 \end{aligned} \tag{10}$$

for $i, j \in A$ and $l \in fb; ng$:

Condition (10) states that whenever someone acquires a unit of outside money, the trading partner must give it up. This is weaker than requiring the agents to swap states. It is easy to see that condition (8) is less restrictive than (4) (it applies to strictly fewer meetings) but is more restrictive than (7) (it pertains to some meetings involving bankers). For similar reasons, conditions (9) and (10) are less restrictive than (5) but more restrictive than (7) for similar reasons.

3.2 Incentive Constraints

This section describes a general set of participation and truth-telling constraints. The participation constraints reflect the fact that agents can always choose to reject the suggested state transitions and output in the second stage of a meeting. The truth-telling constraints are necessary for bankers whenever their states are private information.

I begin by describing the no-defection, expected discounted utility for bankers and nonbankers, and for a banker, the initial-defector expected discounted utility, all given that no one else defects.

Let v_i^k denote the no-defection, expected discounted utility of an agent with identity k who is in state i at the start of a period. Also, for a single-coincidence meeting in which the producer is of identity k in state i and the consumer is of identity l in state j , let P_{ij}^{kl} and Q_{ij}^{kl} be producer and consumer payoffs, respectively, from following the suggested output and state transitions in the second stage of a meeting when everyone else follows the suggested output and state transitions. Then

$$P_{ij}^{kl} = y_{ij}^{kl} + \sum_h p_{ij}^{kl}(h) v_h^k \quad (11)$$

and

$$Q_{ij}^{kl} = u(y_{ij}^{kl}) + \sum_h q_{ij}^{kl}(h) v_h^k \quad (12)$$

For a no-coincidence meeting in which an agent is of identity k in state i and his partner in the meeting is of identity l in state j , let R_{ij}^{kl} be the payoff to the agent of identity k in state i from following the suggested state transitions in the second stage of a meeting when everyone else follows the suggested state transitions. Then

$$R_{ij}^{kl} = \sum_h r_{ij}^{kl}(h) v_h^k \quad (13)$$

Given these definitions, we can express v_i^k as

$$v_i^k = \sum_j \frac{x_j^n}{S} [P_{ij}^{kn} + Q_{ji}^{nk} + (S-2)R_{ij}^{kn}] + \sum_j \frac{x_j^b}{S} [P_{ij}^{kb} + Q_{ji}^{bk} + (S-2)R_{ij}^{kb}] \quad (14)$$

It is easy to show, via a contraction mapping argument, that there are unique solutions to the v_i^k given an allocation.

I calculate recursively the initial-defector expected discounted utility. The defecting banker's payoff must include the option of misrepresenting herself whenever her state is private information and she must also have the option of disagreeing to the suggested output and state transitions in a meeting. It must also reflect the fact that she knows with certainty that her first defection will be discovered T periods after it occurs and she will be punished with autarky from that date on. Let $v_{i_2}^b$ denote the expected discounted utility of

a defecting banker who enters the period in state i and who ...rst defected ζ periods ago.

I now describe producer and consumer payoffs in single-coincidence meetings and payoffs from no-coincidence meetings for undiscovered defecting bankers. Let $\mathbb{P}_{ij\zeta}^{bl}$ and $\mathbb{Q}_{ji\zeta}^{lb}$ be producer and consumer payoffs from following the suggested output and state transitions in the second stage of a meeting when everyone else does too in a single-coincidence meeting with a trading partner of identity l in state j , respectively, for undiscovered defecting bankers who announce state m and ...rst defected ζ periods ago. Then

$$\mathbb{P}_{mj\zeta}^{bl} = i y_{mj}^{bl} + \sum_h p_{mj}^{bl}(h) v_{h;\zeta+1}^b \quad (15)$$

and

$$\mathbb{Q}_{jm\zeta}^{lb} = u(y_{jm}^{lb}) + \sum_h q_{jm}^{lb}(h) v_{h;\zeta+1}^b \quad (16)$$

Similarly, let $\mathbb{R}_{ij\zeta}^{bl}$ be the payoff for an undiscovered defecting banker in state i who ...rst defected ζ periods ago in a no-coincidence meeting with an agent of identity l who is in state j from following the suggested state transitions in the second stage of a meeting when everyone else follows the suggested output and state transitions. Then

$$\mathbb{R}_{mj\zeta}^{bl} = \sum_h r_{mj}^{bl}(h) v_{h;\zeta+1}^b \quad (17)$$

The difference between (11)-(13) and (15)-(17) is that v_h^b is replaced with $v_{h;\zeta+1}^b$. The need for ζ reflects the fact that the continuation payoff of a defecting banker is dependent on the time left before discovery.

Now consider a banker in the ...rst stage of a meeting whose true state is i . The announced state of an undiscovered defecting banker depends on whether the banker is a producer or a consumer in a single-coincidence meeting or in a no-coincidence meeting, the ability of that banker to misrepresent her state, and what is known about the state of her trading partner. Recall that $A_p \subseteq A$ is the space over which a banker can misrepresent. Let I_i be an indicator variable that equals 1 if $i \in A_p$ and is 0 otherwise. Thus, if $I_i = 1$, a banker can announce a state in A_p , and when $I_i = 0$, she is constrained to report truthfully. Similarly, let J_j^l be an indicator variable that equals 1 if $j \in A_p$ and is 0 otherwise, where j is the true state of the banker's trading

partner. J_j^l not only depends on the true state of the trading partner, but also his identity⁸.

Let $m_i^p(I_i; J_j^l)$ be the optimal message of a defecting banker who is a producer in a single-coincidence meeting, whose true state is i and who knows whether or not her trading partner can misrepresent. Similarly, define $m_i^q(I_i; J_j^l)$ to be the optimal message of a defecting banker who is a producer in a single-coincidence meeting, and $m_i^r(I_i; J_j^l)$ to be that of a defecting banker in a no-coincidence meeting.

We can distinguish between four types of optimal messages for producers: $m_i^p(0; 0)$; $m_i^p(0; 1)$; $m_i^p(1; 0)$; $m_i^p(1; 1)$.

$$\begin{aligned}
 m_i^p(0; 0) &= i \\
 m_i^p(0; 1) &= i \\
 m_i^p(1; 0) &= \arg \max_m \mathbb{P}_{mj\ell}^{bl} \\
 m_i^p(1; 1) &= \arg \max_m \prod_{j \in A_p} \frac{x_j^l}{x_j^l} \mathbb{P}_{mj\ell}^{bl} \tag{18}
 \end{aligned}$$

The first two messages reflect the fact that if a banker cannot misrepresent her type, i.e. $I_i = 0$, then she reports truthfully. The final two messages say that when given the freedom to misrepresent, a defecting banker chooses the state that gives the highest expected discounted utility; $m_i^p(1; 0)$ indicates that the state of the trading partner is known with certainty while $m_i^p(1; 1)$ indicates that what is known is that the state of the trading partner is an element of A_p . Similar interpretations hold for $m_i^q(I_i; J_j^l)$ where $\mathbb{P}_{mj\ell}^{bl}$ is replaced with $\mathbb{Q}_{mj\ell}^{lb}$ and for $m_i^r(I_i; J_j^l)$ with $\mathbb{R}_{mj\ell}^{bl}$:

It is useful to state here that in the outside-money regime, the only possible message is $m_i^p(0; 0)$, because states are always observable. In the inside-money regime, however, the only possible messages are $m_i^p(1; 0)$ and $m_i^p(1; 1)$ because banker states are completely unobservable. When in a meeting with a nonbanker, $m_i^p(1; 0)$ is relevant, and in a meeting with banker $m_i^p(1; 1)$ is relevant. In the mixed regime, all messages are possible; $m_i^p(0; 0)$ and $m_i^p(0; 1)$ are relevant when a defecting banker has a unit of outside money ($i = 2$), and $m_i^p(1; 0)$ and $m_i^p(1; 1)$ are relevant when she does not have a unit of outside money.

Finally, consider a defecting banker in the second stage. The announcements concerning states have been revealed and now a defecting banker

⁸For example, $J_j^n = 0$ for all regimes while J_j^b depends on regime.

chooses whether to agree to the suggested state transitions and output. Let $\mathbf{p}_{\ell}^{bl}(m_i^p(I_i; J_j^l); j) = \mathbf{p}_{mj_{\ell}}^{bl}$ where $m = m_i^p(I_i; J_j^l)$ ⁹. Similarly define $\mathbf{q}_{\ell}^{bb}(j; m_i^q(I_i; J_j^l))$ and $\mathbf{r}_{\ell}^{br}(m_i^r(I_i; J_j^l); j)$.

Then for $i = 0; 1; 2$ and $\ell \in \{1; 2; \dots; T - i - 1\}$:

$$\begin{aligned}
\mathbf{v}_{i\ell}^b &= \sum_j \frac{x_j^n}{S} \max[\mathbf{p}_{\ell}^{bn}(m_i^p(I_i; 0); j); -\mathbf{v}_{i;\ell+1}^b] \\
&+ \sum_j \frac{x_j^n}{S} \max[\mathbf{q}_{\ell}^{nb}(j; m_i^q(I_i; 0)); -\mathbf{v}_{i;\ell+1}^b] \\
&+ (S - i - 2) \sum_j \frac{x_j^n}{S} \max[\mathbf{r}_{\ell}^{bn}(m_i^r(I_i; 0); j); -\mathbf{v}_{i;\ell+1}^b] \\
&+ \sum_j \frac{x_j^b}{S} \max[\mathbf{p}_{\ell}^{bb}(m_i^p(I_i; J_j^b); j); -\mathbf{v}_{i;\ell+1}^b] \\
&+ \sum_j \frac{x_j^b}{S} \max[\mathbf{q}_{\ell}^{bb}(j; m_i^q(I_i; J_j^b)); -\mathbf{v}_{i;\ell+1}^b] \\
&+ (S - i - 2) \sum_j \frac{x_j^b}{S} \max[\mathbf{r}_{\ell}^{bb}(m_i^r(I_i; J_j^b); j); -\mathbf{v}_{i;\ell+1}^b]
\end{aligned} \tag{19}$$

Equation (19) takes into account the two decisions a defecting banker makes. The first three terms on the right-hand side pertain to meetings with nonbankers. Consider the first expression. The sum over $\frac{x_j^n}{S}$ pertains to the probability that a banker meets a nonbanker-consumer in state j . A defecting banker knows the state of a nonbanker and chooses $m_i^p(I_i; 0)$ in the first stage. Given that she chooses $m_i^p(I_i; 0)$, she then compares the payoff to agreeing at the second stage, given that her trading partner is agreeing, $\mathbf{p}_{\ell}^{bn}(m_i^p(I_i; 0); j)$, with the payoff to disagreeing at the second stage, $-\mathbf{v}_{i;\ell+1}^b$. She chooses so that she receives the higher payoff. The second and third expressions on the right-hand side of (19) reflect the similar decision when a defecting banker is the consumer in a single coincidence meeting with a nonbanker and is in a no-coincidence meeting with a nonbanker, respectively. All three incorporate the fact that $J_j^n = 0$ in all possible meetings.

The final three terms pertain to meetings with bankers. In this situation, J_j^b could be either 0 or 1, depending on the regime. That is the only significant difference between the first three terms and the final three terms.

⁹Recall that we are considering individual defection, so that the trading partner is announcing truthfully.

As mentioned above, we are interested in the expected discounted utility for a banker from an initial defection given that no one else defects, v_{i1}^b . This is obtained by solving $v_{i,t}^b$ recursively from the initial condition $v_{iT}^b = 0$. The initial condition incorporates the fact that once discovered, a defecting banker is punished with autarky forever. It is easy to establish that v_{i1}^b is finite and unique.

Now consider the constraints that are relevant for implementation. There are two broad sets of constraints: participation constraints and truth-telling constraints. As mentioned earlier, the participation constraints imply that agreeing to a suggested output and state transitions are ex-post sequentially individually rational. That is, there is no uncertainty about what those suggestions are. The truth-telling constraints are necessary for bankers only, because nonbanker states are observable in a meeting because they always represent a nonbanker's money-holdings regardless of the regime considered.

Participation constraints require that agents are ex-post sequentially rational. This is equivalent to them receiving non-negative gains from trade. For nonbankers, the participation constraints are

$$\min\{P_{ij}^{nl}; Q_{ji}^{ln}; R_{ij}^{ng}\} \geq v_i^n \quad (20)$$

for all $i, j \in A$ and $l \in fb; ng$. The right-hand side of (20) is due to the fact that defecting nonbankers can only be punished with no output and no state transition at that date because they will never be discovered.

For bankers, the participation constraints are

$$\min\{P_{ij}^{bl}; Q_{ji}^{lb}; R_{ij}^{lg}\} \geq v_{i1}^b \quad (21)$$

for all $i, j \in A$ and $l \in fb; ng$. Notice for bankers that the right-hand side of (21) reflects the fact that if a banker disagrees to the suggested output and state transition, she receives nothing and does not change states in the period, and becomes an initial defector.

Now consider truth-telling constraints on bankers whose states are not known. All that is required here is that bankers report their true state in the first stage of a meeting for all possible meetings:

$$m_i^p(I_i; J_j^l) = m_i^q(I_i; J_j^l) = m_i^r(I_i; J_j^l) = i \quad (22)$$

for all $i, j \in A$ and $l \in fb; ng$ where $m_i^k(I_i; J_j^l)$ for $k \in fp; q; rg$ is defined as in (18).

Notice from (18) that $m_i^p(0; J_j^1) = m_i^q(0; J_j^1) = m_i^r(0; J_j^1) = i$ so that truth-telling constraints are necessarily satisfied and are redundant. Conversely $m_i^p(1; J_j^1)$; $m_i^q(1; J_j^1)$; and $m_i^r(1; J_j^1)$ are the results of optimizing behavior and truth-telling constraints must be considered. Thus, in the outside-money regime, there are no truth-telling constraints because $A_p = \emptyset$; and for defecting bankers $I_i = 0$ for all i . For the inside-money regime, $A_p = A$ and $I_i = 1$ for all i so that there are truth-telling constraints for all bankers. In the mixed regime, $A_p = \{0, 1\}$ and so $I_i = 1$ when $i \in A_p$ and truth-telling constraints must be satisfied for those meetings. When $i \notin A_p$, $I_i = 0$, and no truth-telling constraints are needed. This is summarized by the definitions that follow, which list the incentive constraints that must be satisfied for each regime.

Definition 4 An implementable outside-money allocation $(x_i^k; y_{ij}^{kl}; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i \in A$ and $k \in \{f, g\}$ is an allocation that satisfies Definition 1 and (11)-(21).

Definition 5 An implementable inside-money allocation $(x_i^k; y_{ij}^{kl}; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i \in A$ and $k \in \{f, g\}$ is an allocation that satisfies Definition 2 and (11)-(22).

Before describing what is required for implementation in the mixed regime, recall that without loss of generality, asset 1 represents inside money and asset 2 represents outside money.

Definition 6 An implementable mixed allocation $(x_i^k; y_{ij}^{kl}; p_{ij}^{kl}; q_{ij}^{kl}; r_{ij}^{kl})$ for $i \in A$ and $k \in \{f, g\}$ is an allocation that satisfies Definition 3 and (11)-(21), and for $i \in \{0, 1\}$; (22).

The incentive constraints are the strictest for the inside-money regime, and the most relaxed for the outside-money regime. This is true for two reasons. The first is that the outside-money regime has no truth-telling constraints, while the mixed regime has truth-telling constraints for some bankers and the inside-money regime has truth-telling constraints for every banker.

The second reason is that, because the different regimes have different numbers of truth-telling constraints, the value to a banker from making an initial defection, v_{i1}^b , may also vary with regime. This leads to stricter participation constraints for bankers in regimes where v_{i1}^b is higher. In the

outside-money regime, there are no truth-telling constraints so future defections can only occur at the second stage of a meeting. In the mixed regime, future defections can occur at the first stage of meeting as well, but only if the defecting banker's true state and announced state are elements of $\{0, 1\}$. Thus, the value to a banker from an initial defection is potentially higher in the mixed regime than in the outside-money regime because there are more opportunities to defect in future periods. In the inside-money regime, a defecting banker can defect in the first stage regardless of her true state and she can announce any state in A . Thus, in the inside-money regime, a defecting banker has the greatest flexibility in defecting in future periods. Thus, the expected discounted utility for a banker from an initial defection is highest in the inside money-regime and lowest in the outside-money regime. Of course, it is possible that the payoffs to defecting are independent of regime. This is the case, for example, when allocations do not differ across banker states¹⁰.

4 A Special Case: $T=0$

Before presenting the comparisons of the ability of the different regimes to implement allocations, it is worth discussing how Cavalcanti-Wallace (1999) is a special case of the model. Their model is a special case in two distinct ways. First, an economy uses only one asset, be it inside money or outside money. This is equivalent to $x_2^k = 0$ and zero transitions into and out of state 2 in the above specification. The other is that there is no difficulty in updating banker histories. This is equivalent to setting $T = 0$.

Cavalcanti-Wallace compare two regimes: an outside-money regime and an inside-money regime. They show that implementable outside-money allocations are a strict subset of implementable inside-money allocations. The strictness comes directly from a comparison of the definitions concerning state transitions and the definitions concerning implementability for the two regimes. As in Definitions 1 and 2, the steady-state restrictions for outside money are more restrictive than those for inside money when $T = 0$. In contrast to Definitions 4 and 5, the incentive constraints that must be satisfied for implementation are identical under both regimes. When $T = 0$ banker states are observable in a meeting because what a banker did last period becomes part of the public record immediately. Thus, the announcement stage is superfluous in their specification of the model and no truth-telling constraints

¹⁰See Example 2 below.

are needed when inside money is used. So implementable outside-money allocations are a strict subset of inside-money allocations because outside money must satisfy more feasibility constraints while both outside money and inside money have identical incentive constraints.

Intuitively, the strictness arises because in meetings with nonbankers, bankers can always issue a unit of inside money, when inside money is used. A banker's ability to offer money to a nonbanker is not dependent on the banker's recent trades; bankers do not need to hold a unit of outside money to consume in single-coincidence meetings with nonbankers.

Cavalcanti-Wallace's strict subset result does not hold up for a general T . To see this, consider the opposite extreme when $T = 1$. Let an allocation suggest an output of $y > 0$ in single-coincidence meetings in which the consumer is in state 1 and the producer is in state 0, regardless of the identity of each. The state transition in such a meeting calls for the agents to swap states. In all other meetings there is no output and agents do not switch states. Such an allocation, which does not suggest gifts for any type of meeting, has been shown to be implementable in an outside-money regime if y is not too high.¹¹ Consumers with a unit of outside money use the outside money to consume in single-coincidence meetings when the producer does not have a unit of outside money.

When inside money is used, however, such an allocation is not implementable. To see why, consider a defecting banker. Such a defector is never discovered. Thus, given an initial defection, a defecting banker will always wish to issue new inside money for consumption and never redeem inside money and never produce. Thus, a defecting banker consumes with the same regularity as a nondefecting banker, but never produces in meetings in which a nondefecting banker does. Because production provides disutility, the value to defecting exceeds the value of not defecting and the example is not implementable in the inside-money regime when $T = 1$. Thus, the strict subset result of Cavalcanti-Wallace does not generalize.

Intuitively, when outside money is used, it carries with it a history that cannot be misrepresented. In order to be in state 1 a banker must have produced to get a unit of outside money. When inside money is used, bankers have an incentive to announce that they are in state 1 even if they are not. This is because they can costlessly issue inside money and consume. The use of inside money allows defecting bankers to misrepresent their recent

¹¹See Cavalcanti-Wallace (1999b).

histories, an opportunity not afforded them when outside money is used. Thus, when $T = 1$ we could say that inside money is a strict subset of outside money; only no-trade allocations are implementable with inside money.¹²

So what about updating lags between the two extremes? As Cavalcanti-Wallace demonstrate, inside money has an advantage over outside money in that banker consumption does not have to be tied down to recent histories; feasibility constraints are less stringent with its use. On the other hand, the use of inside money gives rise to truth-telling constraints whenever there is a lag in updating banker histories. The truth-telling constraints must not only be satisfied, but their presence raises the value to defecting and, thereby, makes it more difficult to satisfy participation constraints. As a result, outside money has an advantage over inside money in that it carries a recent trading history that cannot be misrepresented. It is this trade-off between the two types of money that leads to the surmise that there may exist allocations that can only be implemented via the use of both. The next section tests this surmise.

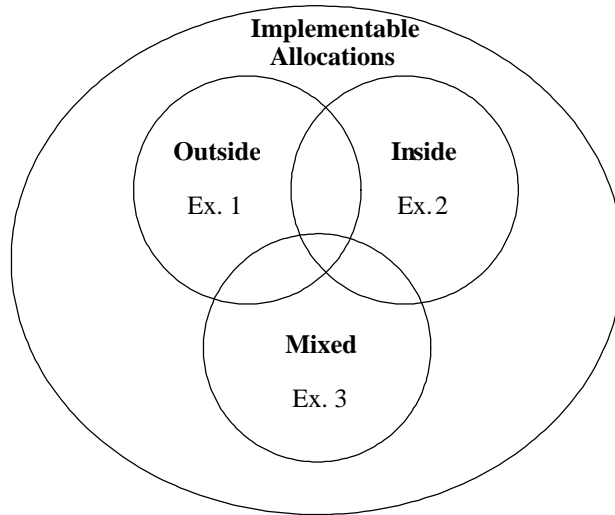
5 Main Result

This section presents three examples of allocations and compares their implementability in the three regimes. Each example takes as given the same set of parameters. Example 1 is implementable only in the outside-money regime, Example 2 is implementable only in the inside-money regime, and Example 3 is implementable only in the mixed regime (see Figure 1). Thus, for some parameters, there is no subset ordering.

To accomplish the task of formally proving that there is no subset ordering of implementable allocations, we fix a set of parameters $(T; S; B; \beta; u(x))$, where the parameters are the length of the updating lag, the number of specialization types in the economy, the measure of each specialization type that are bankers, the discount factor, and the specification of the period utility function. Of course, the parameter of most interest for this analysis is T , the length of the updating lag. When $T = 0$, the set of implementable outside-money allocations is a strict subset of the set of implementable inside-

¹²Actually, the agents in the economy could use the initial stock of banknotes and not accept notes newly issued by bankers. Such an allocation can have trade, but the inside money is just playing the role of outside money; no new notes are ever issued and no existing notes are ever redeemed.

Figure 1: Set of Implementable Allocations



money allocations. Conversely, when $T = 1$ the set of implementable inside-money allocations is a strict subset of the set of implementable outside-money allocations. Thus, in order to talk about allocations that are implementable only in the mixed regime, it must be the case that $T > 0$ and finite. It is for this reason that numerical examples must be used to establish the result.

For each example, the set of parameters is the following:

$$f; T; S; B; \beta; u(x) = f; 0.5; 3; 0.1; 0.99; x^{\frac{1}{2}}$$

Motivations for these choices are as follows. The choices of S ; B ; and β are somewhat arbitrary. The only importance given to the number of specialization types is that there are enough to eliminate the possibility of a double-coincidence of wants in a meeting. The minimum number of specialization types that accomplishes this is $S = 3$. The choice of a small fraction of each specialization type being bankers, B , is motivated by the use of a small B in Cavalcanti-Wallace (1999). The choice of a relatively high discount factor, β , while important in determining the values of the expected utilities, is somewhat arbitrary.

The explicit utility function for the calculations is $u(x) = x^{\frac{1}{2}}$. Such a

utility function has a very simple form that satisfies all of the assumptions made in Section 2. Note $u(x) = x^{\frac{1}{2}}$ implies that the solution to $\max u(y) \mid y$ is $y = 0.25$:

Most important is the choice of T , the length of the updating lag. To arrive at the choice of 65 I first used a version of an implementable inside-money allocation presented in Cavalcanti-Wallace (1999)¹³. Such an allocation provides a level of output consistent with $y = 0.25$. Then, I searched for the minimum lag such that banker-producer participation constraints would be violated. This lag was $T = 65$. This gives us an idea of how much output is too much to be traded for inside money.

In Example 1, both monies trade for more output in single-coincidence meetings than can be traded with inside money. This causes both truth-telling constraints and banker participation constraints to fail in the inside-money and mixed regimes. Example 2 cannot be duplicated by either the outside-money regime or the mixed regime because it does not satisfy the stricter feasibility constraints in Definitions 1 and 3. Example 3 cannot be duplicated due to a combination of the reasons given above. The outside-money regime cannot duplicate it because the example does not satisfy the stricter feasibility constraints in Definition 1. The inside-money regime cannot because the outside money in the mixed regime trades for more output than can be traded with inside money. This leads to the violation of several banker participation constraints.

Now let us consider the specific examples. Each example will stipulate that no gifts be given in no-coincidence meetings ($r_{ij}^{kl} = i$). Example 1 has the feature that money 2 is more valuable than money 1. It trades for a higher output, and in single-coincidence meetings between nonbankers, consumers who hold a unit of money 2 can swap assets with producers who hold a unit of money 1 and receive some positive consumption. In single-coincidence meetings between bankers, there is positive output. In most of those meetings, banker-producers give gifts to banker-consumers, but in certain meetings the bankers do switch states. In the outside-money regime, this corresponds to a banker-consumer with a unit of outside money surrendering the unit to the banker-producer who does not have a unit of money for consumption.

Example 1 Production in single-coincidence meetings is as follows:

$$y_{01}^{bb} = y_{01}^{nn} = y_{01}^{bn} = y_{01}^{nb} = 0.25$$

¹³See their Example 2 allocation.

$$\begin{aligned}
y_{02}^{bb} = y_{02}^{nn} = y_{02}^{bn} = y_{02}^{nb} &= 0:3 \\
y_{00}^{bb} = y_{1j}^{bb} = y_{2j}^{bb} &= 0:2 \\
y_{12}^{nn} &= 0:1
\end{aligned}$$

and $y_{ij}^{kl} = 0$ for all other meetings for $i \in A$. The following are the state transitions whenever they stipulate a new state for the agent:

$$\begin{aligned}
q_{01}^{bb} = q_{01}^{bn} = q_{01}^{nb} = q_{01}^{nn} = q_{02}^{bb} = q_{02}^{bn} = q_{02}^{nb} = q_{02}^{nn} &= 0 \\
p_{01}^{bb} = p_{01}^{bn} = p_{01}^{nb} = p_{01}^{nn} = q_{12}^{nn} &= 1 \\
p_{02}^{bb} = p_{02}^{bn} = p_{02}^{nb} = p_{02}^{nn} = p_{12}^{nn} &= 2
\end{aligned}$$

All other $p_{ij}^{kl} = i$ and $q_{ij}^{kl} = j$. For all no-coincidence meetings, $r_{ij}^{kl} = i$. The fractions of each specialization type in each state are

$$\begin{aligned}
x_0^n = 0:45; x_1^n = x_2^n &= 0:225 \\
x_0^b = 0:05; x_1^b = x_2^b &= 0:025:
\end{aligned}$$

It is easy to verify that Example 1 satisfies the feasibility restrictions in Definition 1, and, as a direct consequence, those in Definitions 2 and 3. To check the incentive constraints, we need to know the expected discounted utility for nonbankers, bankers, and defecting bankers who defect for the first time. Table 1 reproduces those values for the above example. Recall that the expected discounted utility to initial defectors is contingent on the type of regime being considered. Let $v_{i1}^b(O)$ be the expected discounted value to a banker of making an initial defection in the outside-money regime. Similarly define $v_{i1}^b(I)$ for the inside-money regime and $v_{i1}^b(M)$ for the mixed regime.

Table 1: Value Functions for Example 1

State	v_i^n	v_i^b	$v_{i1}^b(O)$	$v_{i1}^b(I)$	$v_{i1}^b(M)$
0	2.2715474	2.7475370	1.3613908	4.3590061	4.3069621
1	2.6264524	2.8490892	1.7083424	4.3590061	4.3069621
2	2.7475370	2.8945393	1.7537922	4.3980695	4.3524120

Given the expected discounted utilities in Table 1, we can verify that all of the participation constraints are satisfied for the outside-money regime and so Definition 4 is satisfied. Thus, Example 1 is implementable when two outside monies are used. Such is not the case for the inside-money regime and the mixed regime. Under both of those regimes, all banker participation

constraints and all relevant truth-telling constraints are violated. Table 1 hints at this because the initial value to defecting in both the inside-money and mixed regimes exceed the value to a banker who does not defect. It is obvious that (22) is violated whenever the mechanism calls for production by a banker because $v_{i1}^b(I)$ is independent of state, and a defecting banker can announce any state in A . It turns out that participation constraints for banker producers are violated as well. Banker truth-telling constraints are violated because banker consumers in different states receive significantly different allocations, and banker producers in different states receive significantly different allocations. This creates incentives to misrepresent their state to the state that is most favorable to them because they can defer punishment well enough into the future. Thus, Definitions 5 and 6 are violated for Example 1, and it is only implementable under the outside-money regime.

Example 2, like Example 1 has the second money more valuable than the first. In meetings between nonbankers, the second money trades for more output than the first, and consumers who hold a unit of money 2 can swap assets with producers who hold a unit of money 1 and receive some positive consumption. In contrast with Example 1, the states of the bankers never change in any type of meeting. Banker-producers give gifts to banker-consumers, and produce for nonbankers who have a unit of either money. In the inside-money regime, the latter is equivalent to bankers redeeming notes from nonbankers. Nonbanker-producers produce output to banker-consumers, regardless of state. In the inside-money regime, this is equivalent to bankers, regardless of state, issuing a note to a nonbanker for output.

Example 2 Production in single-coincidence meetings is as follows:

$$\begin{aligned}
 y_{ij}^{bb} = y_{01}^{nn} = y_{i2}^{bn} = y_{0i}^{nb} &= 0:2 \\
 y_{02}^{nn} &= 0:25 \\
 y_{i1}^{bn} &= 0:15 \\
 y_{12}^{nn} &= 0:1
 \end{aligned}$$

and $y_{ij}^{kl} = 0$ for all other meetings for $i \in A$. The following are the state transitions whenever they stipulate a new state for the agent:

$$\begin{aligned}
 q_{01}^{nn} = q_{02}^{nn} = q_{i1}^{bn} = q_{i2}^{bn} &= 0 \\
 p_{01}^{nn} = p_{00}^{nb} = p_{01}^{nb} = q_{12}^{nn} &= 1 \\
 p_{02}^{nn} = p_{02}^{nb} = p_{12}^{nn} &= 2
 \end{aligned}$$

All other $p_{ij}^{kl} = i$ and $q_{ij}^{kl} = j$. For all no-coincidence meetings, $r_{ij}^{kl} = i$. The fractions of each specialization type in each state are

$$x_0^n = 0.45; x_1^n = x_2^n = 0.225$$

$$x_0^b = x_1^b = 0.025; x_2^b = 0.05:$$

It is easy to verify that an Example 2 allocation satisfies Definition 2, but violates Definitions 1 and 3. The fact that bankers remain in the same state in which they entered in single-coincidence meetings with nonbankers that leave their original state violates the feasibility constraints whenever outside money is used. For example, $p_{02}^{nb} = 2$ and $q_{02}^{nb} = 2$ violates (5) for the outside-money regime and (10) for the inside-money regime when outside money is used.

Finally, we must demonstrate that an Example 2 allocation is implementable via the inside-money regime. Table 2 reports the expected discounted utilities for nonbankers, bankers, and initial banker defectors. From these values we can verify that all of the participation and truth-telling constraints are satisfied and therefore, Definition 5 is satisfied. We conclude that Example 2 is implementable only in the inside-money regime.

Table 2: Value Functions for Example 2

State	v_i^n	v_i^b	$v_{i1}^b(O)$	$v_{i1}^b(I)$	$v_{i1}^b(M)$
0	2.4531507	4.9072493	3.8895945	3.8895945	3.8895945
1	2.7505034	4.9072493	3.8895945	3.8895945	3.8895945
2	2.8714805	4.9072493	3.8895945	3.8895945	3.8895945

In Example 3, as in the other two, asset 2 is more valuable than asset 1 and banker-producers always produce for banker-consumers. Like Example 1, most banker-producers give gifts of output to banker-consumers, but in some meetings between bankers, they leave their states. In the mixed regime, this is a banker-consumer with a unit of outside money giving it to a banker-producer in exchange for output. Like Example 2, banker-producers produce for nonbankers who have a unit of either money and nonbanker-producers produce output for banker-consumers, regardless of state. In the mixed regime, the former is equivalent to bankers redeeming notes (if nonbanker state is 1) or acquiring outside money (if nonbanker state is 2). The latter is equivalent to bankers issuing a note (if banker is not in state 2) or surrendering a unit of outside money (if banker is in state 2). Thus, banker

states are important here in the sense that outside money exchanges for more output than inside money.

Example 3 Production in single-coincidence meetings is as follows:

$$\begin{aligned} y_{00}^{bb} = y_{01}^{bb} = y_{10}^{bb} = y_{11}^{bb} = y_{2j}^{bb} = y_{01}^{nn} = y_{i1}^{bn} = y_{00}^{nb} = y_{01}^{nb} = 0:2 \\ y_{02}^{bb} = y_{12}^{bb} = y_{02}^{nn} = y_{02}^{bn} = y_{12}^{bn} = y_{02}^{nb} = 0:25 \\ y_{12}^{nn} = 0:1 \end{aligned}$$

and $y_{ij}^{kl} = 0$ for all other meetings for $i \in A$. The following are the state transitions whenever they stipulate a new state for the agent:

$$\begin{aligned} q_{02}^{bb} = q_{12}^{bb} = q_{01}^{nn} = q_{02}^{nn} = q_{i1}^{bn} = q_{02}^{bn} = q_{12}^{bn} = q_{02}^{nb} = 0 \\ p_{01}^{nn} = p_{01}^{bn} = p_{00}^{nb} = p_{01}^{nb} = q_{12}^{nn} = 1 \\ p_{02}^{bb} = p_{12}^{bb} = p_{02}^{nn} = p_{12}^{nn} = p_{02}^{bn} = p_{12}^{bn} = p_{02}^{nb} = 2 \end{aligned}$$

All other $p_{ij}^{kl} = i$ and $q_{ij}^{kl} = j$. For all no-coincidence meetings $r_{ij}^{kl} = i$. The fractions of each specialization type in each state are

$$\begin{aligned} x_0^n = 0:36; x_1^n = 0:18; x_2^n = 0:36 \\ x_0^b = 0:0333; x_1^b = 0:0167; x_2^b = 0:05: \end{aligned}$$

Inspection of Example 3 shows that it satisfies the steady-state restrictions required in Definition 3, and therefore, Definition 2, since there are less feasibility constraints for the inside-money regime than for the mixed regime. By a similar argument, it is easy to verify that Definition 1 is not satisfied for the Example 3 allocation. There are more feasibility constraints for the outside-money regime than for the mixed regime. In the Example 3 allocation there exist single-coincidence meetings in which bankers and nonbankers do not swap states, even though nonbankers leave their original state. For example, $p_{01}^{nb} = 1$ and $q_{01}^{nb} = 1$ violates (5) for the outside-money regime. Such an allocation, therefore, cannot be implemented via the outside-money regime.

Table 3 provides the relevant expected discounted utilities for the Example 3 allocation. These values can be used to verify that the Example 3 allocation satisfies the participation constraints and relevant truth-telling constraints that are required by Definition 6, and so the allocation is implementable via the use of one outside money and one inside money. Finally,

we can use Table 3 to verify that the inside-money regime cannot implement Example 3. In fact, certain banker-producer participation constraints are violated. Specifically, in meetings with nonbanker-consumers who enter with a unit of money 1, bankers in states 0 and 1 are not willing to produce. Also, in meetings with banker-consumers, bankers in states 0 and 1 are not willing to produce for bankers in state 0 or 1.

Table 3: Value Functions for Example 3

State	v_i^b	v_i^b	$v_{i1}^b(O)$	$v_{i1}^b(I)$	$v_{i1}^b(M)$
0	2.4323032	3.7418523	3.2531154	3.5953568	3.2531154
1	2.7118472	3.7418523	3.2531154	3.5953568	3.2531154
2	2.8211107	3.8774455	3.3027632	3.5953568	3.3027632

The above three examples have proven the following proposition.

Proposition 1 There exists a set of parameters, $(T; S; B; \tau; u(x))$, such that there is no subset ordering of implementable allocations.

One implication of Proposition 1 is that for some parameters there exist allocations that can only be implemented via the combined use of one outside money and one inside money. Such allocations make use of the trade-off that inside money can increase the frequency of banker consumption, but, in the presence of a specified updating lag, give rise to truth-telling constraints that are not present when outside money is used.

6 Concluding Remarks

This paper has established a result linking the difficulty in updating public knowledge of individual histories with the kinds of money that are essential. It has been shown, given a set of parameters, that there does not exist a subset ordering of implementable allocations under different monetary arrangements or regimes.

An important consequence of this result is that for a given set of parameters, there exist allocations that can only be implemented via the combined use of one inside money and one outside money. Such allocations provide an essential role for their coexistence. They make use of a trade-off between inside money's ability to increase the frequency of banker consumption and

outside money's ability to signal a recent history that cannot be misrepresented.

The notion of essentiality used here is weak. It requires only that there exist allocations for which both outside and inside money must be used. A stronger result requires a welfare comparison of the three regimes. Such a comparison is the goal of future research. It involves defining welfare as the ex-ante expected discounted utility of a representative agent at the beginning of the world, before agents are given a specialization type, identity, and initial state. An example of an implementable mixed allocation then provides an ex-ante discounted utility. To satisfy this stronger notion of essentiality requires that such a utility not be beaten by any implementable inside-money allocation or implementable outside-money allocation given a set of parameters. The surmise is that there exists a set of parameters for which the trade-off between inside money and outside money makes the use of both desirable in terms of this welfare criterion.

What complicates deriving this result is the fact that bankers can give gifts to nonbankers. This can be in the form of production in single-coincidence meetings and in the form of an asset in any type of meeting. Such a complication led to a very limited analytical welfare result in Cavalcanti-Wallace when only one asset was considered. Thus, a test of this welfare criterion will involve a search over all possible implementable outside-money allocations and implementable inside-money allocations, given a set of parameters.

Finally, I conjecture that the role of the assumption that agents can hold at most one unit of money is not crucial. Such an assumption is made to keep the model tractable. As mentioned above, there is already great complication in calculating banker defector value functions so that specific parameters are needed for numerical calculation. Dropping that assumption only provides further complication. The assumption is not critical for the results, however, because they are driven by the trade-off between inside and outside money. This trade-off would still exist if the assumption about the unit upper bound on money-holdings were dropped.

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