Informational Efficiency, Trade And Incomplete Markets

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Abstract: In [Grossman and Stiglitz, 1980], it is argued that in a rational expectations setting the only way information is valuable to some traders is that it is not fully revealed through prices. In that case, if traders are allowed to purchase information before trade then some traders will acquire that information at a cost and non-zero information-based trade obtains. The intuition for this result is that agents can benefit from the costly acquisition of information only if they have an informational advantage over other agents.

Here, we argue that information-based trade obtains for reasons other than information differentials across agents. This is done by assuming that risk sharing is not optimal. Incomplete securities markets open twice, once before information is revealed, and once after information is revealed to all agents (directly or because of agents' rational expectations). We define market completeness in this setting and show that if markets are complete then the trade undertaken in the first period is still optimal in the second period, thereby recovering Grossman and Stiglitz's result that information cannot be valuable if it is fully revealed to all agents. In that case, there is no information-based trade. However, if markets are incomplete then retrade obtains in the second period even if information is fully revealed through prices and agents agree on the interpretation of signals. Our results shows that information-based trade and informational efficiency of prices can coexist. In the context of informationally efficient prices, we give necessary and sufficient conditions for information to be valuable to a trader and show that gains from information acquisition and trade volume can be made arbitrarily large through increased market incompleteness. This suggests that information-based volume of trade needn't be attributed to the presence of investor irrationality, or noise trading but instead may be attributed to continued information-based risk-sharing by rational agents in markets that do not offer optimal risk-sharing.

1 Introduction

It is well known ([Radner, 1979]) that in finite Walrasian economies, if agents maximize an expected utility, conditional on private and equilibrium information, prices reveal all private information to all agents in the economy, provided we assume agents are equipped with a forecast function relating prices and aggregate signals (this assumption is known as rational expectations). This theoretical result confirms the common wisdom that prices convey all relevant information needed for a decision maker in a large economy. On the other hand, it is thought that agents in ongoing securities markets profit from their private information. The intuition for this claim is that if agents have "monopoly" over their private information, they can strategically trade by selling shares of a stock if the private information indicates that the stock is overvalued and by buying shares if the private information indicates that the stock is undervalued. Over time, agents lose their information monopoly as their private information is revealed through the price adjustment process. As prices adjust, other agents can "judge quality by price" and extract (at least partially) private information from prices. This process of extracting information through prices is modelled in the celebrated paper of [Grossman and Stiglitz, 1980]. If information is costly to acquire, an equilibrium number of agents will acquire it while the other agents will extract any possible amount of this information through prices. However, if prices were fully revealing, uninformed agents would extract the whole signal, obviating the advantage gained by the costly acquisition of information. In this case, no one would acquire the costly signal. If, on the other hand, some noise is introduced in the economy, then prices convey the signal only partially. So, despite some of the signal being transmitted through prices from the informed to the uninformed, it is still worth for some agents to acquire information at a positive cost. This is what Grossman and Stiglitz call an "equilibrium degree of disequilibrium". These results, that prices convey all necessary information and that agents profit from private information, are at odds. Indeed, if information is thought to be valuable only if it is private, then prices should not convey information perfectly, otherwise there wouldn't be trading based on news.

The reason that information is not valuable in the Grossman and Stiglitz model without noise in the economy is that agents begin with a Pareto optimal allocation of the consumption good and so the arrival of "perfect" information cannot generate trade because expected normalized marginal utilities of all agents adjust to keep the original allocation Pareto optimal. In the absence of noise, both informed and uninformed agents interpret the arrival of a particular signal identically. This makes re-adjusting their portfolios unnecessary since their original portfolios are Pareto optimal. Noisy information changes the beliefs of the uninformed in a way that makes rebalancing their portfolios necessary to reach optimality. In the presence of noise, the uninformed update their beliefs differently than the informed, essentially because they are updating their beliefs on different information. Trade obtains because agents are essentially "betting" on who has the correct interpretation.

In this paper, we argue that by casting this model in an ongoing securities market setting makes clear the role of information in generating trade and trade volume when markets are complete or incomplete. It also sheds light on the issue of the value of information for a particular trader. If the arrival of signals does not alter security payoffs or endowments so that it's role is purely informational, then we show that agents will alter their portfolio choice after information arrives essentially if only if markets are incomplete. We show that we can do away with the assumption of noise traders in Grossman and Stiglitz and still obtain that information is valuable to some traders (in a sense suitably defined later), even though prices are fully revealing. So, by weakening the assumption of completeness of markets, we recover the intuitive result that information should be valuable even if it is revealed to all and even if all agents attach the same meaning to that information. We also discuss the issue of the volume of trade generated by the arrival of information and show that it is increasing in the "degree" of market incompleteness.

In section 2, we define market completeness, market incompleteness and state the first welfare theorem in the exponential-normal setting. In section 3, we describe the 3−period economy and give necessary and sufficient conditions for information-based trade. In section 4, we define and give necessary and sufficient conditions for information to be valuable to a given agent. In section 5, we define and give an example of volume of trade and show in this example that it increases with the degree of market incompleteness. In section 6, we conclude.

2 Market Completeness

We're interested in showing the importance of market incompleteness in generating trade in the Grossman-Stiglitz framework. Market incompleteness in finite economies has been extensively studied (see [Magill and Quinzii, 1998] for an overview). However, the economy we're interested in analyzing has uncertainty modelled as a continuum of states. While the issue of characterizing market completeness in these economies is difficult in general, the Grossman-Stiglitz framework offers a tractable definition of market completeness. Consider a two period economy where agents get utility out of consuming the unique consumption good and their utility function is a negative exponential with constant absolute risk aversion (CARA) factor A^h.

$$\mathbb{E}u^h(x^h) = -\mathbb{E}\exp\left[-A^h x^h\right] \tag{1}$$

Suppose that agents can trade J securities in the first period that pay off normally distributed dividends V_j in the second period. They can also trade a riskless security V_0. Suppose that agents' second period endowment e^h and security payoffs are jointly normal. If q is a vector of security prices, z^h a vector of security trades for agent h, and $\mathbb{V} = [1, V_1, .., V_J]$ the vector of payoffs then the agent's optimization problem is to maximize expected utility (1) subject to $qz^h = 0$. In equilibrium, $\sum_h z^h = 0$. Since all variables are jointly normal, it follows that agent h's consumption $c^h = e^h + \mathbb{V}z^h$ must also be normal. We write the following

Definition 1 *An 2-period equilibrium is a collection $\{x, z, q\}$ so that*

(i) Agents maximize $\mathbb{E}(-e^{-A^h x^h})$ subject to

$$\begin{aligned} qz^h &= 0 \\ x^h_{t+2} &= e^h + \mathbb{V}z^h \end{aligned} \tag{2}$$

(ii) $\sum_h z^h = 0$

We adopt a tractable definition of optimality where transfers are restricted to be jointly normal with other random variables.

Definition 2 *An 2-period equilibrium allocation $\{x, z, q\}$ is Pareto Optimal if there doesn't exist normal random variables a^h so that:*

$$\sum_h a^h = 0$$

And such that for all agents:

$$\mathbb{E}(-e^{-A^h x^h + a^h}) \geq \mathbb{E}(-e^{-A^h x^h})$$

with a strict inequality for at least one agent.

Proposition 3 *Given the security structure* $\mathbb{V}$*, a 2-period equilibrium is Pareto optimal if for all agents:*

$$e^h = a^h \mathbb{V} \text{ for some } a^h \in \mathbb{R}^{J+1} \tag{3}$$

Proof. *In the Appendix.* ■

This proposition states that if all endowments lie in the space spanned by security payoffs, then the first welfare theorem must hold. When endowments satisfy the condition stated in the above proposition, we say that markets are complete. If they don't, then markets are incomplete. The latter term suggests that in general, if the condition is not satisfied, the securities present are not enough to allow all agents to share risk optimally.

Example 4 *Given* $\mathbb{V} = [1, \theta + \varepsilon]$*, suppose that for all* h*,* e^h *has the following form:*

$$e^h = a^h + b^h (\theta + \varepsilon) \tag{4}$$

Then markets are complete. So adding more non-redundant securities than are already present will never enhance any traders' welfare.

Example 5 *In general, suppose the security and endowments are jointly normal. Project* e^h *onto the space spanned by the risky security. Then:*

$$e^h = \mathbb{E}e^h + Cov\left(e^h, \frac{\theta + \varepsilon - \mu_\theta - \mu_\varepsilon}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\right)\left(\frac{\theta + \varepsilon - \mu_\theta - \mu_\varepsilon}{\sqrt{\sigma_\theta^2 + \sigma_\varepsilon^2}}\right) + \gamma^h \tag{5}$$

where γ^h *is orthogonal to* $(\theta + \varepsilon)$*, has zero mean and arbitrary variance. If* $Var\left(\gamma^h\right) > 0$ *for some* γ^h *then markets are incomplete. Suppose that securities with payoffs perfectly correlated to* γ^h *for each* h *were added to the securities market. Then the resulting competitive equilibrium allocation would be a Pareto improvement over the competitive equilibrium with the old security structure.*

Adding only some of these securities is not necessarily Pareto improving. See [Hart, 1975] for counterexamples.

Example 6 *As we shall see in the next section, we're not interested in all forms of market incompleteness. In particular, we're not interested in risk components of endowments that are orthogonal to all the individual risk factors of dividends. Project* e^h *onto the space spanned by* θ *and* ε*. Then:*

$$\begin{aligned} e^h &= \mathbb{E}e^h + Cov(e^h, \frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon})\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right) + Cov(e^h, \frac{\theta-\mu_\theta}{\sigma_\theta})\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right) \quad (6) \\ &\quad +\delta^h \quad (7) \\ &= \mathbb{E}e^h + Cov(e^h,\varepsilon)\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon^2}\right) + Cov(e^h,\theta)\left(\frac{\theta-\mu_\theta}{\sigma_\theta^2}\right) + \delta^h \end{aligned}$$

Suppose that $Var(\delta^h)=0$*, then markets are incomplete if for some agent h,*

$$\frac{Cov(e^h,\varepsilon)}{\sigma_\varepsilon^2} \neq \frac{Cov(e^h,\theta)}{\sigma_\theta^2} \quad (8)$$

This is, in fact, the kind of market incompleteness we're interested in. Indeed if $\frac{Cov(e^h,\varepsilon)}{\sigma_\varepsilon^2} = \frac{Cov(e^h,\theta)}{\sigma_\theta^2}$ *for all agents but* $Var(\delta^h) > 0$*, then portfolio choices will not depend on the magnitude of this variance. However, the resulting equilibrium will likely not be Pareto optimal.*

3 The Economy and the Main Result

In the standard Grossman-Stiglitz model, there are H agents trading a riskless and a risky asset in a securities market. For the sake of clarity, we also assume the existence of only two securities, although it should be clear from the derivation of the propositions below that none of the results are affected if we assumed an arbitrary number of securities. In this model, there are three periods denoted $s = t, t+1, t+2$. Endowments are stochastic and materialize in period $t+2$. Asset payoffs also materialize in period $t+2$. The riskless asset pays off 1 unit of the consumption good. The risky asset pays off $V = \theta + \varepsilon$ units of the consumption good with the usual assumptions that $\varepsilon \text{\textasciitilde} N(\mu_\varepsilon, \sigma_\varepsilon^2)$ and $\theta \text{\textasciitilde} N(\mu_\theta, \sigma_\theta^2)$ and where $Cov_t(\varepsilon, \theta) = 0$. Traders have identical beliefs about the payoffs of the assets and trade away endowment risk in period t subject to the net value of their trade being zero and a zero net trade condition in equilibrium. Agents correctly forecast period $t+1$ prices and take these prices as given in their period t decisions. In period $t+1$, the signal θ is revealed publicly to all agents, and securities markets open again. In period $t+2$ consumption takes place.

For tractability, we assume that endowments and dividends are jointly normal. We also assume that all agents act as price takers and have rational expectations. By price taking behavior, we mean that agents do not act strategically to influence prices. By rational expectations, we mean that agents extract all possible information from equilibrium prices. As we shall see later, the rational expectations and identical beliefs assumptions rule out period $t+1$ retrade based on disagreements about the meaning or interpretation of the signal θ. We formally define our concept of an equilibrium, akin to a Walrasian equilibrium wherein the price formation process is not an issue. For this purpose, let $\mathbb{V} = [1; V]$ denote the vector of security payoffs, $q_t = (q_t(1), q_t(2))$ denote security prices, $z_t^h = (z_t^h(1), z_t^h(2))$ denote agent h's period t portfolio of securities and e_{t+2}^h denote his endowment.

Definition 7 *An (3-period) equilibrium is a collection* $\{x, z, q\}$ *so that* *(i) Agents maximize* $\mathbb{E}_t(-e^{-A^h x^h_{t+2}})$ *subject to*

$$\begin{aligned} q_t z^h_t &= 0 \qquad (9) \\ q_{t+1} z^h_{t+1} &= q_{t+1} z^h_t \\ x^h_{t+2} &= e^h_{t+2} + \mathbb{V} z^h_{t+1} \end{aligned}$$

(ii) $\sum_h z^h_t = 0$ *for* $t = 0, 1$

(iii) z^h_t *is measurable with respect to period* t *information.*

The first equation in the budget constraint states that the net value of all trades must be zero. This means that the value of securities short-sold finances the value of the securities bought. The second equation states that the value of the new portfolio z^h_{t+1} must equal the value at the new prices q_{t+1} of the old portfolio. In this model, we say there is no retrade if for all agents and all realizations of the random variable θ, $z^h_t = z^h_{t+1}$. This model assumes that all agents receive the signal θ publicly at time $t+1$. An alternative interpretation is that some agents receive the signal and it is revealed through prices to the other agents. Indeed the relative price of the risky security is (see equation 56 in the Appendix):

$$q_{t+1} = \theta + \mu_\varepsilon - \frac{\sum_h \frac{Cov_t(e^h_{t+2}, \varepsilon)}{\sigma_\varepsilon}}{\sum_h \frac{1}{A^h}} \qquad (10)$$

If some agents are not directly informed of θ in period $t+1$, q_{t+1} will reveal it in equilibrium.

We're interested in seeing how market risk *vis-à-vis* endowment risk influences retrade in period $t+1$. For this purpose, it is useful to project endowments on the span of dividend risks θ and ε. We obtain (just as in example 6):

$$e^h_{t+2} = \mathbb{E}_t e^h_{t+2} + Cov_t(e^h_{t+2}, \varepsilon)\left(\frac{\varepsilon - \mu_\varepsilon}{\sigma^2_\varepsilon}\right) + Cov_t(e^h_{t+2}, \theta)\left(\frac{\theta - \mu_\theta}{\sigma^2_\theta}\right) + \gamma^h_{t+2} \qquad (11)$$

Where γ^h_{t+2} has the following properties:

$$\begin{aligned} \mathbb{E}_t \gamma^h_{t+2} &= 0 \\ Cov_t(\gamma^h_{t+2}, \theta) &= 0 \\ Cov_t(\gamma^h_{t+2}, \varepsilon) &= 0 \end{aligned} \tag{12}$$

If θ is revealed to all agents in the economy, then the following proposition fully describes retrade behavior in period $t+1$. Henceforth, we use the terms information-based trade and retrade in period $t+1$ interchangeably.

Proposition 8 *There is no retrade in period* $t+1$ *if and only if for all* $h' \in H$:

$$\frac{1}{A^{h'}} \sum_h \frac{Cov_t(e^h_{t+2}, \theta)}{\sigma^2_\theta} - \left(\sum_h \frac{1}{A^h} \right) \frac{Cov_t(e^{h'}_{t+2}, \theta)}{\sigma^2_\theta} = \tag{13}$$

$$\frac{1}{A^{h'}} \sum_h \frac{Cov_t(e^h_{t+2}, \varepsilon)}{\sigma^2_\varepsilon} - \left(\sum_h \frac{1}{A^h} \right) \frac{Cov_t(e^{h'}_{t+2}, \varepsilon)}{\sigma^2_\varepsilon}$$

Proof. In the Appendix. ■

This expression has a clear meaning in light of market completeness. Indeed, one easily obtains the following

Corollary 9 *If markets are complete, then there is no retrade in period* $t+1$.

Proof. Market completeness means that for all h:

$$e^h_{t+2} = \alpha^h + \beta^h(\theta + \varepsilon) \text{ for some } \left(\alpha^h, \beta^h\right) \in \mathbb{R}^2$$

But then:

$$\begin{aligned} Cov_t\left(e^h_{t+2}, \varepsilon\right) &= \beta^h Var_t(\varepsilon) \\ Cov_t\left(e^h_{t+2}, \theta\right) &= \beta^h Var_t(\theta) \end{aligned}$$

So:

$$\beta^h = \frac{Cov_t\left(e^h_{t+2}, \varepsilon\right)}{\sigma^2_\varepsilon} = \frac{Cov_t\left(e^h_{t+2}, \theta\right)}{\sigma^2_\theta}$$

So expression 13 is satisfied. ■

The intuition for this result is that market completeness implies that after one round of trade, agents have already reached a Pareto optimal outcome so the need for period $t+1$ retrade is obviated, as long as all agents observe θ perfectly (if that's not the case, some agents have an informational advantage over others and the trade will not be individually rational from some trader's perspective). We formalize this intuition in the following

Proposition 10 *If markets are complete and if (x, z, q) is a 3-period equilibrium then $(\hat{x}_t, z_t, q_t)$ is a Pareto optimal 2-period equilibrium where*

$$\hat{x}_t = \left(\hat{x}^h_t\right)_h = \left(e^h_{t+2} + \mathbb{V} z^h_t\right)_h \tag{14}$$

Proof. The proof uses proposition 3. Refer to the Appendix for the proof. ■

When markets are incomplete, endowments take the general form (11) where it may be that $Var_t(\gamma^h_{t+2}) > 0$ and that

$$\frac{Cov_t\left(e^h_{t+2}, \varepsilon\right)}{\sigma^2_\varepsilon} \neq \frac{Cov_t\left(e^h_{t+2}, \theta\right)}{\sigma^2_\theta} \tag{15}$$

As mentioned in example (6), not all forms of market incompleteness are interesting. When $Var(\gamma^h_{t+2}) > 0$ and the above expression holds with equality then the endowment risk factor orthogonal to the space spanned by securities cannot be hedged against. Therefore, no risk-sharing of the γ^h_{t+2} risk factor will ever take place. When expression (15) holds for at least some agent, then typically expression (13) will not hold and information-based trade obtains. This is formalized in the following proposition that looks at a particular form of market incompleteness (the orthogonal term γ_{t+2}^h is omitted for clarity and without loss of generality).

Proposition 11 *Suppose that for all agents, endowments have the form:*

$$e_{t+2}^h = \alpha^h + \beta^h \theta + \eta^h \varepsilon$$

Where $(\alpha^h, \beta^h, \eta^h,)$ *are scalars. Then there is an open set of full measure* $\Omega \subset \mathbb{R}^{3H}$ *such that if* $(\alpha, \beta, \eta) \in \Omega$*, then retrade in period* $t+1$ *will occur.*

Proof. Expression (13) becomes:

$$\frac{1}{A^{h'}} \sum_h \beta^h - \left(\sum_h \frac{1}{A^h} \right) \beta^{h'} = \frac{1}{A^{h'}} \sum_h \eta^h - \left(\sum_h \frac{1}{A^h} \right) \eta^{h'} \tag{16}$$

We call this expression $P(h')$.Construct the set

$$\tilde{\Omega} = \{(\alpha, \beta, \eta) \in \mathbb{R}^{3H} | P(h') \text{ holds for all } h' \in H\}$$

This set has zero Lebesgue measure and is closed. Set $\Omega = \mathbb{R}^{3H} - \tilde{\Omega}$. ■

4 Value of Information

Since there is no retrade in period $t+1$ when markets are complete, we focus on the case of market incompleteness. In general, we may write an agent's endowment as follows:

$$e_{t+2}^h = \alpha^h + \beta^h \theta + \eta^h \varepsilon + \gamma_{t+2}^h$$

$\left(\alpha^h, \beta^h, \eta^h\right)$ are scalars while γ_{t+2}^h is a random variable jointly normal and orthogonal with θ and ε with mean 0 and arbitrary variance. Markets are incomplete if there is at least one agent h so that $\beta^h \neq \eta^h$.

Given that all other agents have rational expectations, information has value for agent h if at time t he is willing to pay a strictly positive cost for the acquisition of signal θ in period $t+1$. Since there is no noise in the economy, all other agents will deduce the signal from the relative price of the risky security in period $t+1$ (or, equivalently from the change in relative prices). So, it suffices to compare agent h's ex-ante utility in the $2-$period equilibrium to that in the $3-$period equilibrium. If the latter is strictly greater than the former, then information is valuable to this agent.

It is useful to introduce the following notation:

$$\frac{1}{A} \equiv \sum_h \frac{1}{A^h};\ \beta \equiv \sum_h \beta^h;\ \eta \equiv \sum_h \eta^h$$

In a $2-$period economy, using the budget constraint yields the following expression for consumption.

$$x^h_{t+2} = e^h_{t+2} + (\theta + \varepsilon - q_t) z^h_t(2)$$

One finds that:

$$\begin{aligned} q_t &= \mu_\theta - A\beta\sigma_\theta - A\eta\sigma_\varepsilon \\ z^h_t(2) &= \frac{\sigma_\theta \left(\frac{A}{A^h}\beta - \beta^h\right) + \sigma_\varepsilon \left(\frac{A}{A^h}\eta - \eta^h\right)}{\sigma_\theta + \sigma_\varepsilon} \end{aligned}$$

In a $3-$period economy, the following level of consumption obtains, using the budget constraints.

$$X^h_{t+2} = e^h_{t+2} + \left(Q_{t+1} - Q_t\right) Z^h_t(2) + (\theta + \varepsilon - Q_{t+1}) Z^h_{t+1}(2)$$

Where:

$$\begin{aligned} Q_t &= \mu_\theta - A\beta\sigma_\theta - A\eta\sigma_\varepsilon \\ Q_{t+1} &= \theta - A\eta\sigma_\varepsilon \\ Z_t^h(2) &= \frac{A}{A^h}\beta - \beta^h \\ Z_{t+1}^h(2) &= \frac{A}{A^h}\eta - \eta^h \end{aligned}$$

These equations may be understood in the context of mean-variance pricing. We easily obtain:

$$\begin{aligned} Z_t^h(2) &= \frac{1}{A^h}\left[\frac{\mathbb{E}_t Q_{t+1} - Q_t}{\sigma_\theta}\right] - \beta^h \qquad (17) \\ Z_{t+1}^h(2) &= \frac{1}{A^h}\left[\frac{\theta - Q_{t+1}}{\sigma_\varepsilon}\right] - \eta^h \end{aligned}$$

In this model, trading away $(\theta + \varepsilon)$ risk in period t and then trading away ε risk in period $t+1$ given θ is like trading sequentially two securities: In period t, agents trade a security whose expected pay off in period $t+1$ is $\mathbb{E}_t Q_{t+1}$, it's price is Q_t and the variance of it's payoff is $\sigma_\theta^2 = Var_t(Q_{t+1})$. They each own β^h of it. In period $t+1$, given θ agents trade a security that pays off ε in period $t+2$. It's expected payoff is $\mu_\varepsilon = 0$, it's price is $(Q_{t+1} - \theta)$ and the variance of it's payoff is σ_ε^2. They each own η^h of it. Because period $t+1$ prices reflect changes in θ one to one, net trade in the risky security is not a function of θ. However, period $t+1$ trade in the riskless security does in fact depend on θ because of the budget constraint.

Note that:

$$\begin{aligned} q_t &= Q_t \\ z_t^h(2) &= \frac{\sigma_\theta Z_t^h(2) + \sigma_\varepsilon Z_{t+1}^h(2)}{\sigma_\theta + \sigma_\varepsilon} \end{aligned}$$

So that:

$$X_{t+2}^h = x_{t+2}^h + a_{t+2}^h$$

Where:

$$a_{t+2}^h = \left(Z_t^h(2) - Z_{t+1}^h(2)\right)\left(Q_{t+1} - \frac{Q_t\sigma_\varepsilon + (\theta+\varepsilon)\,\sigma_\theta}{\sigma_\theta + \sigma_\varepsilon}\right)$$

One can show that:

$$-\mathbb{E}_t e^{-A^h X_{t+2}^h} \geq -\mathbb{E}_t e^{-A^h x_{t+2}^h}$$

If and only if:

$$\mathbb{E}_t a_{t+2}^h \geq \frac{A^h}{2}\left(Var_t(x_{t+2}^h + a_{t+2}^h) - Var_t(x_{t+2}^h)\right)$$

This expression states that information is valuable to agent h if the objective unconditional expected value of the gain from information-based trade is sufficiently large *vis-à-vis* his tolerance to the added risk imposed by another round of trade. When markets are complete, it's easy to check that the expression holds with equality so information is not valuable in the case of optimal risk sharing and rational expectations.

Proposition 12 *If markets are incomplete and given that the parameters of the* *economy are such that for some agent h:*

$$A\left(\eta-\beta\right)>A^h\left(\eta^h-\beta^h\right)$$

Then information is valuable to agent h if and only if

$$A\left[\left(\frac{1}{\sigma_\theta}+\frac{1}{\sigma_\varepsilon}\right)(\eta-\beta)-\left(\frac{\sigma_\varepsilon}{\sigma_\theta}\eta-\frac{\sigma_\theta}{\sigma_\varepsilon}\beta\right)\right]>A^h\left(\eta^h-\beta^h\right) \tag{18}$$

An identical result holds by reversing both inequalities.

Proof. *In the Appendix.* ■

Notice that this expression doesn't involve the terms γ_{t+2}^h. This is because this risk component is completely undiversifiable since it is orthogonal to the space spanned by security payoffs.

Example 13 *To make things transparent, suppose that all agents have the same CARA coeffecient, normalized to* 1*. Also, suppose that* $\sigma_\theta=\sigma_\varepsilon$ *also normalized to*1*. Finally, suppose that for some agent h:*

$$\frac{1}{H}\left(\beta-\eta\right)>\beta^h-\eta^h$$

Then inequality (18) becomes precisely the expression above, which is interpreted to mean that the market carries more θ *than* ε *risk than agent h does. So, agent h has an incentive to disaggregate his risk between* θ *and* ε *risk since the market values* θ *risk relatively more than he does. The right hand side can be thought of as a measure of aggregate market incompleteness, while the left hand side is an individual measure of market incompleteness.*

In [Grossman and Stiglitz, 1980], information is valuable only if some traders have an informational advantage over others. The noise introduced in the aggregate supply of the risky security makes the signal extraction problem nontrivial.

This model shows that information may be valuable to some agent even if it is fully revealed through prices to all others. This is because risk sharing in this market is imperfect. After one round of trade, agents find it useful to engage in additional information-based risk-sharing since the market does not offer optimal risk-sharing. For some agents, this additional risk-sharing may make them worse off and so their actions are not individually rational. However, the above proposition and example show that in fact some agents will be strictly better off from an ex-ante point of view if they decided to purchase the signal at a positive cost, despite the fact that they will not have monopoly over that information.

Note two features of this model: a) the Grossman-Stiglitz result is recovered if markets are complete. b) Although initial endowment levels are arbitrary, the overall wealth that traders can use to engage in information-based trade is partly determined by market forces and therefore reflects aggregate endowment risk and preferences toward risk. This feature idealizes the notion that agents are trading away "endogenized" risk in an ongoing securities market and that the result does not rely on the arbitrary nature of endowments but rather on limited risk-sharing opportunities offered to agents.

5 Volume of Trade

We define volume of trade as the sum of all shares changing hands over time.

Definition 14 *Per Capita Volume of Trade for security* $j = 1, 2$ *is*

$$V_j = \frac{1}{2H}\mathbb{E}_t\left[\left|\sum_h \left|Z_{t+1}^h(j)\right| - \sum_h \left|Z_t^h(j)\right|\right|\right]$$

We wish to show that trade volume increases with the degree of market incompleteness. Suppose there are two agents, one of whom has endowments (β^1, η^1) of θ and ε risk while the other has endowments (β^2, η^2) of θ and ε risk.

Then, we easily obtain:

$$V_2 = \frac{1}{4} \left| \, \left|\beta^1 - \beta^2\right| - \left|\eta^1 - \eta^2\right| \, \right|$$

In general, if markets are complete, then the volume of trade is zero for any security. Here, we see that if markets are incomplete and endowments are in general position (meaning that, in general, one would expect $\beta^1 \neq \beta^2$ and/or $\eta^1 \neq \eta^2$), then volume of trade will be positive[1]. It is increasing in the degree of market incompleteness and in differences in θ or ε risk across agents, given that markets are incomplete.

6 Concluding Comments

In this paper, we have demonstrated that by embedding the Grossman-Stiglitz framework in a three period ongoing securities market setting, we can do away with the noise trader assumption and still get information-based trade even if prices are fully revealing. We show that if agents have correct expectations then market completeness implies no retrade and market incompleteness implies that retrade obtains in period $t+1$. We also give necessary and sufficient conditions for information to be valuable to agents in the context of fully revealing prices. Our example shows that the value of information can be made arbitrarily large if markets are allowed to be sufficiently incomplete. We also discuss the increasing relationship between trade volume and market incompleteness. This is in contrast to the idea that trade volume is generated through information or interpretation differentials across agents.

These results are similar to those found in a companion paper [Blume et al., 2002]

---

[1] It is also easy to prove that, generically in the space of endowments, market incompleteness implies a non-zero trade volume. While the proof is trivial, the statement of the proposition is tedious.

which discusses the No Trade theorem in [Milgrom and Stokey, 1982]. That result is often used to argue that information-based trade is not possible in an ongoing securities market when agents begin at a Pareto optimal allocation. In [Blume et al., 2002], we recast the Milgrom and Stokey setup in a three period finite market economy and drop common knowledge of rationality assumptions since they are not needed in an anonymous price-taking environment. We get that information-based trade is possible essentially if and only if markets are incomplete and/or agents disagree on the meaning of signals. Note that in these models, the assumption of shared beliefs or at least shared likelihood functions is essential. If it is not satisfied, then trade will typically take place even if markets are complete and there is common knowledge of rationality.

This paper also shows that the assumption of noise trading in the finance literature is unnecessary insofar as it is a device used to generate trade. Since it is generally agreed that imperfect risk sharing is a more realistic assumption than perfect risk sharing, the assumption of market incompleteness is a weakening of the usual assumptions in the right direction. In addition, trade volume in this model is directly related to the degree of market incompleteness and so information revelation through prices is an ancillary issue, the central issue being continued information-based risk sharing in ongoing markets. Finally, as a modelling issue, market incompleteness is a deviation from the assumption of perfect risk sharing and does not affect assumptions about agents' rationality and therefore this assumption does not deviate from the axiomatically founded subjective expected utility theory, unlike the assumption of noise trading.

7 Appendix

Proof. (Proposition 3)

To simplify notation, we assume without loss of generality that $\{V_j\}_{j\in J}$ form an orthonormal basis for the subspace spanned by the securities. Then (by part (b) of theorem 4.14 in [Rudin, 1987]):

$$e^h - \mathbb{E}e^h = \sum_j Cov(e^h, V_j)V_j + \gamma^h \tag{19}$$

where γ^h has zero mean and is orthogonal to all V_j. So, by assumption, $Var\left(\gamma^h\right) = 0$. Let x^h be a Pareto optimal allocation. Then

$$x^h = e^h + \alpha^h \tag{20}$$

where α^h are random variables and

$$\sum_h \left(x^h - e^h\right) = 0 \tag{21}$$

We have:

$$\alpha^h - \mathbb{E}\alpha^h = \sum_j Cov(\alpha^h, V_j)V_j + \delta^h \tag{22}$$

where δ^h is of zero mean and is orthogonal to all V_j. Note that (by computing the m.g.f of a multinormal distribution. See pp 39 of [Johnson and Kotz, 1972]):

$$\begin{aligned} -\mathbb{E}\left\{\exp[-A^h x^h]\right\} &= -\exp\left[\mathbb{E}\left(-A^h x^h\right) + \frac{1}{2}Var\left(-A^h x^h\right)\right] \qquad (23) \\ &= -\exp\left[-A^h\mathbb{E}x^h + \frac{\left(A^h\right)^2}{2}Var\left(x^h\right)\right] \\ &= -\exp\left[-A^h\mathbb{E}x^h\right]\exp\left[\frac{\left(A^h\right)^2}{2}Var\left(x^h\right)\right] \end{aligned}$$

Where:

$$\mathbb{E}x^h = \mathbb{E}e^h + \mathbb{E}\alpha^h \tag{24}$$

$$\begin{aligned} Var\left(x^h\right) &= Var\begin{pmatrix} \mathbb{E}e^h + \sum_j Cov(e^h, V_j)V_j + \mathbb{E}\alpha^h + \sum_j Cov(\alpha^h, V_j)V_j \\ +\delta^h + \gamma^h \end{pmatrix} \qquad (25) \\ &= Var\begin{pmatrix} \mathbb{E}e^h + \sum_j Cov(e^h, V_j)V_j + \mathbb{E}\alpha^h + \sum_j Cov(\alpha^h, V_j)V_j \\ +\gamma^h \end{pmatrix} \\ &\quad +Var(\delta^h) + 2Cov(\gamma^h, \delta^h) \end{aligned}$$

Note that if $Var(\gamma^h) = 0$ (as we assume), then the optimal value of δ^h is $Var(\delta^h) = 0$. So the optimal choice of α^h lies in the subspace spanned by the securities, and not outside.

However, when $Var(\gamma^h) > 0$, the $2Cov(\gamma^h, \delta^h)$ term adds ambiguity to the sign of

$$Var(\delta^h) + 2Cov(\gamma^h, \delta^h)$$

Given our assumption, and given Pareto weights λ^h, the Pareto optimal problem is to choose a vector $\beta^h \in \mathbb{R}^{J+1}$ for each agent so as to

$$\max -\sum_h \lambda^h \mathbb{E}\exp[-A^h x^h]$$

subject to

$$\begin{aligned} x^h &= e^h + \beta^h \mathbb{V} \qquad (26) \\ \sum_h \beta_j^h &= 0 \text{ for all } j \end{aligned}$$

Define the Lagrangian for this problem:

$$\mathcal{L} = -\sum_h \lambda^h \mathbb{E}\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)] + \sum_j \left\{\xi_j\left(\sum_h \beta_j^h\right)\right\} \tag{27}$$

The First Order Conditions for this problem are:

$$\frac{\partial \mathcal{L}}{\partial \beta_j^h} = 0 \Leftrightarrow \lambda^h \mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\frac{\partial\left(-A^h\left(e^h+\beta^h\mathbb{V}\right)\right)}{\partial \beta_j^h}\right\} = \xi_j \tag{28}$$

So:

$$\begin{aligned} -A^h\lambda^h\mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\right\} &= \xi_j \text{ if } j=0 \\ -A^h\lambda^h\mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]V_j\right\} &= \xi_j \text{ if not} \end{aligned} \tag{29}$$

Where, using the first lemma found in [Stein, 1981]:

$$\begin{aligned} \mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]V_j\right\} &= \mathbb{E}\left\{\frac{d\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]}{dV_j}\right\} \\ &= \mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\frac{\partial[-A^h\left(e^h+\beta^h\mathbb{V}\right)]}{\partial V_j}\right\} \\ &= -A^h\left(Cov(e^h,V_j)+\beta_j^h\right)\mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\right\} \end{aligned} \tag{30}$$

So that:

$$\frac{\left(A^h\right)^2\lambda^h\left(Cov(e^h,V_j)+\beta_j^h\right)\mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\right\}}{-A^h\lambda^h\mathbb{E}\left\{\exp[-A^h\left(e^h+\beta^h\mathbb{V}\right)]\right\}} = \frac{\xi_j}{\xi_0} \tag{31}$$

So that:

$$A^h\lambda^h\left(Cov(e^h,V_j)+\beta_j^h\right) = -\frac{\xi_j}{\xi_0} \tag{32}$$

The competitive equilibrium problem is to maximize expected utilities subject to budget constraints. Write the Lagrangian for each agent:

$$\mathcal{L}^h = \mathbb{E}\exp[-A^h\left(e^h + z^h\mathbb{V}\right)] + \eta^h q z^h \tag{33}$$

The FOCs are:

$$\frac{\partial \mathcal{L}^h}{\partial z^h_j} = 0 \Leftrightarrow \mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\frac{\partial\left(-A^h\left(e^h + z^h\mathbb{V}\right)\right)}{\partial z^h_j}\right\} = \eta^h q_j \tag{34}$$

So, by the same argument as in the Pareto optimal case:

$$A^h\left(Cov(e^h, V_j) + z^h_j\right) = -\frac{q_j}{q_0} \tag{35}$$

Suppose that (x, z, q) is a competitive equilibrium, then the above equation must hold. Let

$$\begin{aligned} \beta^h &= z^h \qquad (36) \\ \xi_j &= q_j \\ \lambda^h &= 1 \end{aligned}$$

Then the FOCs of the Pareto optimal problem are satisfied. The equilibrium condition $\sum_h \beta^h_j = 0$ is also satisfied as a consequence of the securities market clearing condition. So the first welfare theorem holds as long as endowments lie in the space spanned by the securities. ■

Proof. (Proposition 8)

There are two agents h and H. They have negative exponential utilities of period $t+2$ consumption:

$$-e^{-A^h x^h_{t+2}}$$

All random variables are jointly normally distributed. Set:

$$\hat{\theta} = \frac{\theta - \mu_\theta}{\sigma_\theta}; \hat{\varepsilon} = \frac{\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon}$$

Then we can write endowments as (by part (b) of theorem 4.14 in [Rudin, 1987]):

$$e^h_{t+2} = \mathbb{E}_t e^h_{t+2} + Cov_t(e^h_{t+2}, \hat{\varepsilon}) \left(\frac{\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon} \right) + Cov_t(e^h_{t+2}, \hat{\theta}) \left(\frac{\theta - \mu_\theta}{\sigma_\theta} \right) + \gamma^h_{t+2} \quad (37)$$

γ^h_{t+2} is a normally distributed random variable that is orthogonal to both θ and ε and has zero mean. The optimization problem can be solved by backward induction. In period $t+1$, agents maximize expected utilities conditional on the publicly observed signal θ.

$$\max -\mathbb{E}_t e^{-A^h x^h_{t+2}}$$

subject to:

$$\begin{aligned} w^h_{t+1} &= q_{t+1}(1) z^h_{t+1}(1) + q_{t+1}(2) z^h_{t+1}(2) \quad (38) \\ x^h_{t+2} &= e^h_{t+2} + z^h_{t+1}(1) + V z^h_{t+1}(2) \end{aligned}$$

Recall that:

$$
\begin{aligned}
V &= \theta + \varepsilon \qquad (39)\\
x^h_{t+2} &= \left(\mathbb{E}_t e^h_{t+2} + Cov_t(e^h_{t+2}, \hat{\varepsilon})\left(\frac{\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon}\right) + Cov_t(e^h_{t+2}, \hat{\theta})\left(\frac{\theta - \mu_\theta}{\sigma_\theta}\right) + \gamma^h_{t+2}\right)\\
&\quad + z^h_{t+1}(1) + (\theta + \varepsilon)\, z^h_{t+1}(2)
\end{aligned}
$$

This is a standard optimization problem where the Kuhn Tucker conditions are necessary if an equilibrium exists. We write the Lagrangian:

$$
\mathcal{L}^h_{t+1} = -\mathbb{E}_{t+1} e^{-A^h x^h_{t+2}} + \lambda^h_{t+1}\left[w^h_{t+1} - \left(q_{t+1}(1) z^h_{t+1}(1) + q_{t+1}(2) z^h_{t+1}(2)\right)\right] \qquad (40)
$$

The First Order Conditions are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}^h_{t+1}}{\partial z^h_{t+1}(1)} &= 0 \Leftrightarrow \lambda^h_{t+1} q_{t+1}(1) = \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} \frac{\partial\left(-A^h x^h_{t+2}\right)}{\partial z^h_{t+1}(1)}\right] \qquad (41)\\
\frac{\partial \mathcal{L}^h_{t+1}}{\partial z^h_{t+1}(2)} &= 0 \Leftrightarrow \lambda^h_{t+1} q_{t+1}(2) = \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} \frac{\partial\left(-A^h x^h_{t+2}\right)}{\partial z^h_{t+1}(2)}\right]
\end{aligned}
$$

Or:

$$
q_{t+1} \equiv \frac{q_{t+1}(2)}{q_{t+1}(1)} = \frac{\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} V\right]}{\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]} \qquad (42)
$$

But:

$$
\begin{aligned}
\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} V\right] &= \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} (\theta + \varepsilon)\right] \qquad (43)\\
&= \theta \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right] + \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}} \varepsilon\right]
\end{aligned}
$$

The second part can be computed using equation (2.3) in [Stein, 1981] or by integration by parts.

$$\begin{aligned}
\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\varepsilon\right] &= \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right]\sigma_\varepsilon+\mu_\varepsilon\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right] \qquad (44)\\
&= \mathbb{E}_{t+1}\left[\frac{d\left(e^{-A^h x^h_{t+2}}\right)}{d\varepsilon}\right]\sigma_\varepsilon+\mu_\varepsilon\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]\\
&= \mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\frac{\partial\left(-A^h x^h_{t+2}\right)}{\partial\varepsilon}\right]\sigma_\varepsilon+\mu_\varepsilon\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]\\
&= -A^h\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\left(\frac{Cov_t(e^h_{t+2},\hat{\varepsilon})}{\sigma_\varepsilon}+z^h_{t+1}(2)\right)\right]\sigma_\varepsilon+\mu_\varepsilon\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]\\
&= -A^h\sigma_\varepsilon\left(\frac{Cov_t(e^h_{t+2},\hat{\varepsilon})}{\sigma_\varepsilon}+z^h_{t+1}(2)\right)\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]+\mu_\varepsilon\mathbb{E}_{t+1}\left[e^{-A^h x^h_{t+2}}\right]
\end{aligned}$$

So that:

$$q_{t+1}=\theta+\mu_\varepsilon-A^h\sigma_\varepsilon\left(\frac{Cov_t(e^h_{t+2},\hat{\varepsilon})}{\sigma_\varepsilon}+z^h_{t+1}(2)\right) \qquad (45)$$

This equation and the budget constraint 38 define z^h_{t+1} as a function of wealth w^h_{t+1} and the relative price q_{t+1} given a realization of θ. In period t, agents maximize the unconditional expected value of their indirect utility function:

$$v^h_{t+1}(s^h_{t+2})=-e^{-A^h x^h_{t+2}(s^h_{t+2})} \qquad (46)$$

Where:

$$s^h_{t+2}=(e^h_{t+2},z^h_{t+2},w^h_{t+1},q_{t+1}) \qquad (47)$$

The optimization problem is then:

$$\max\mathbb{E}_t v^h_{t+1}(s^h_{t+2})$$

subject to:

$$
\begin{aligned}
q_t(1)z_t^h(1) + q_t(2)z_t^h(2) &= 0 \qquad (48)\\
z_t^h(1) + q_{t+1}z_t^h(2) &= w_{t+1}^h
\end{aligned}
$$

This problem's FOCs are necessary if an equilibrium exists. We write the time t Lagrangian:

$$
\mathcal{L}_t^h = \mathbb{E}_t v_{t+1}^h(s_{t+2}^h) - \lambda_t^h \left[q_t(1)z_t^h(1) + q_t(2)z_t^h(2) \right] \qquad (49)
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}_t^h}{\partial z_t^h(1)} &= 0 \Leftrightarrow \lambda_t^h q_t(1) = \mathbb{E}_t \left[\frac{\partial v_{t+1}^h(s_{t+2}^h)}{\partial w_{t+1}^h} \bigg|_{w_{t+1}^h = z_t^h(1) + q_{t+1}z_t^h(2)} \frac{\partial w_{t+1}^h}{\partial z_t^h(1)} \right] \qquad (50)\\
\frac{\partial \mathcal{L}_t^h}{\partial z_t^h(2)} &= 0 \Leftrightarrow \lambda_t^h q_t(2) = \mathbb{E}_t \left[\frac{\partial v_{t+1}^h(s_{t+2}^h)}{\partial w_{t+1}^h} \bigg|_{w_{t+1}^h = z_t^h(1) + q_{t+1}z_t^h(2)} \frac{\partial w_{t+1}^h}{\partial z_t^h(2)} \right]
\end{aligned}
$$

Or:

$$
q_t \equiv \frac{q_t(2)}{q_t(1)} = \frac{\mathbb{E}_t \left[\frac{\partial v_{t+1}^h(s_{t+2}^h)}{\partial w_{t+1}^h} q_{t+1} \right]}{\mathbb{E}_t \left[\frac{\partial v_{t+1}^h(s_{t+2}^h)}{\partial w_{t+1}^h} \right]} \qquad (51)
$$

Where:

$$
\begin{aligned}
v_{t+1}^h(s_{t+2}^h) &= -\exp\left[-A^h \left(e_{t+2}^h + z_{t+1}^h(1) + V z_{t+1}^h(2) \right) \right] \qquad (52)\\
&= -\exp\left[-A^h \left(\begin{array}{c} \mathbb{E}_t e_{t+2}^h + Cov_t(e_{t+2}^h, \hat{\varepsilon}) \frac{\varepsilon - \mu_\varepsilon}{\sigma_\varepsilon} + Cov_t(e_{t+2}^h, \hat{\theta}) \left(\frac{\theta - \mu_\theta}{\sigma_\theta} \right) + \gamma_{t+2}^h \\ +(w_{t+1}^h - q_{t+1}z_{t+1}^h(2)) + (\theta + \varepsilon) z_{t+1}^h(2) \end{array} \right) \right]
\end{aligned}
$$

So that:

$$
\frac{\partial v_{t+1}^h(s_{t+2}^h)}{\partial w_{t+1}^h} = -A^h v_{t+1}^h(w_{t+1}^h, q_{t+1}) \qquad (53)
$$

So that:

$$\mathbb{E}_{t+1}\left[\frac{\partial v^h_{t+1}(s^h_{t+2})}{\partial w^h_{t+1}} q_{t+1}\right] = -A^h \mathbb{E}_{t+1}\left[v^h_{t+1}(s^h_{t+2}) q_{t+1}\right] \tag{54}$$

To make this expression tractable, we compute the equilibrium price that must hold if any θ is realized. Dividing expression 45 by A^h and summing over $h \in H$, we obtain:

$$q_{t+1}\sum_h \frac{1}{A^h} = (\theta + \mu_\varepsilon)\sum_h \frac{1}{A^h} - \sum_h Cov_t(e^h_{t+2}, \hat{\varepsilon}) - \sum_h z^h_{t+1}(2) \tag{55}$$

In equilibrium, $\sum_h z^h_{t+1} = 0$. So:

$$q_{t+1} = \theta + \mu_\varepsilon - \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}} \tag{56}$$

Using expression 45 gives us the trade in the risky asset in period $t+1$:

$$z^h_{t+1}(2) = \frac{1}{A^h \sigma_\varepsilon}\frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}} - \frac{Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sigma_\varepsilon} \tag{57}$$

Therefore:

$$\begin{aligned} \mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2}) q_{t+1}\right] &= \mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2})\theta\right] \\ &+ \left(\mu_\varepsilon - \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}}\right)\mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2})\right] \\ &= \mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2})\frac{\theta - \mu_\theta}{\sigma_\theta}\right]\sigma_\theta + \mu_\theta \mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2})\right] \\ &\quad + \left(\mu_\varepsilon - \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}}\right)\mathbb{E}_t\left[v^h_{t+1}(s^h_{t+2})\right] \end{aligned} \tag{58}$$

Where:

$$\begin{aligned}
\mathbb{E}_t\left[v_{t+1}^h(s_{t+2}^h)\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)\right]\sigma_\theta &= \sigma_\theta\mathbb{E}_t\left[\frac{dv_{t+1}^h(s_{t+2}^h)}{d\theta}\right] \qquad (59)\\
&= -A^h\sigma_\theta\mathbb{E}_t\left[v_{t+1}^h(s_{t+2}^h)\left(\begin{array}{c}\frac{Cov_t(e_{t+2}^h,\hat{\theta})}{\sigma_\theta}+z_t^h(2)\\ -z_{t+1}^h(2)+z_{t+1}^h(2)\end{array}\right)\right]\\
&= -A^h\sigma_\theta\mathbb{E}_t\left[v_{t+1}^h(s_{t+2}^h)\left(\frac{Cov_t(e_{t+2}^h,\hat{\theta})}{\sigma_\theta}+z_t^h(2)\right)\right]\\
&= -A^h\sigma_\theta\left(\frac{Cov_t(e_{t+2}^h,\hat{\theta})}{\sigma_\theta}+z_t^h(2)\right)\mathbb{E}_t\left[v_{t+1}^h(s_{t+2}^h)\right]
\end{aligned}$$

So:

$$q_t=\mu_\theta+\mu_\varepsilon-A^h\sigma_\theta\left(\frac{Cov_t(e_{t+2}^h,\hat{\theta})}{\sigma_\theta}+z_t^h(2)\right)-\frac{\sum\limits_h Cov_t(e_{t+2}^h,\hat{\varepsilon})}{\sum\limits_h\frac{1}{A^h}} \qquad (60)$$

Dividing the above expression by $A^h\sigma_\theta$, summing over all h and using the equilibrium condition $\sum\limits_h z_t^h=0$ yields:

$$q_t=\mu_\theta+\mu_\varepsilon-\frac{\sum\limits_h Cov_t(e_{t+2}^h,\hat{\theta})}{\sum\limits_h\frac{1}{A^h}}-\frac{\sum\limits_h Cov_t(e_{t+2}^h,\hat{\varepsilon})}{\sum\limits_h\frac{1}{A^h}} \qquad (61)$$

Using 60, we can deduce the amount of the risky security agents purchase:

$$z_t^h(2)=\frac{1}{A^h\sigma_\theta}\frac{\sum\limits_h Cov_t(e_{t+2}^h,\hat{\theta})}{\sum\limits_h\frac{1}{A^h}}-\frac{Cov_t(e_{t+2}^h,\hat{\theta})}{\sigma_\theta} \qquad (62)$$

So:

$$z_t^h(2)\neq z_{t+1}^h(2)$$

if and only if

$$\frac{1}{A^h\sigma_\theta}\frac{\sum\limits_h Cov_t(e^h_{t+2},\hat{\theta})}{\sum\limits_h \frac{1}{A^h}} - \frac{Cov_t(e^h_{t+2},\hat{\theta})}{\sigma_\theta} \neq \frac{1}{A^h\sigma_\varepsilon}\frac{\sum\limits_h Cov_t(e^h_{t+2},\hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}} - \frac{Cov_t(e^h_{t+2},\hat{\varepsilon})}{\sigma_\varepsilon} \tag{63}$$

■

Proof. (Proposition 10)

Suppose that markets are complete and that (x, z, q) is a 3-period equilibrium.We will show that $(e^h_{t+2} + \mathbb{V}z^h_t, z^h_t, q_t)$ is a 2-period equilibrium. Proposition 3 does the rest. Since markets are complete, we write:

$$e^h = a^h + b^h V$$

In the 2-period economy, agents' Lagrangian is:

$$\mathcal{L}^h = \mathbb{E}\exp[-A^h\left(e^h + z^h\mathbb{V}\right)] + \eta^h q z^h \tag{64}$$

The FOCs are:

$$\begin{aligned} \frac{\partial \mathcal{L}^h}{\partial z^h(1)} &= 0 \Leftrightarrow -A^h\mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\right\} = \eta^h q(1) \\ \frac{\partial \mathcal{L}^h}{\partial z^h(2)} &= 0 \Leftrightarrow -A^h\mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)](\theta+\varepsilon)\right\} = \eta^h q(2) \end{aligned} \tag{65}$$

$$\begin{aligned} \mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)](\theta+\varepsilon)\right\} &= \mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\left(\frac{\theta-\mu_\theta}{\sigma_\theta}\right)\right\}\sigma_\theta \\ &\quad + \mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)\right\}\sigma_\varepsilon \\ &\quad + (\mu_\theta + \mu_\varepsilon)\,\mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\right\} \\ &= -A^h\mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)](b^h + z^h(2))\right\}(\sigma_\theta + \sigma_\varepsilon) \\ &\quad + (\mu_\theta + \mu_\varepsilon)\,\mathbb{E}\left\{\exp[-A^h\left(e^h + z^h\mathbb{V}\right)]\right\} \end{aligned} \tag{66}$$

So:

$$q = \mu_\theta + \mu_\varepsilon - A^h(b^h + z^h(2))\,(\sigma_\theta + \sigma_\varepsilon) \tag{67}$$

So, in equilibrium:

$$q = \mu_\theta + \mu_\varepsilon - \frac{(\sigma_\theta + \sigma_\varepsilon) \sum\limits_h b^h}{\sum\limits_h \frac{1}{A^h}} \tag{68}$$

So:

$$z^h(2) = \frac{1}{A^h} \frac{\sum\limits_h b^h}{\sum\limits_h \frac{1}{A^h}} - b^h \tag{69}$$

Compare this to the three period economy (see proof of proposition (8)):

$$q_t = \mu_\theta + \mu_\varepsilon - \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\theta})}{\sum\limits_h \frac{1}{A^h}} - \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\varepsilon})}{\sum\limits_h \frac{1}{A^h}} \tag{70}$$

$$z^h_t(2) = \frac{1}{A^h \sigma_\theta} \frac{\sum\limits_h Cov_t(e^h_{t+2}, \hat{\theta})}{\sum\limits_h \frac{1}{A^h}} - \frac{Cov_t(e^h_{t+2}, \hat{\theta})}{\sigma_\theta} \tag{71}$$

But:

$$b^h = \frac{Cov_t\left(e^h_{t+2}, \varepsilon\right)}{\sigma^2_\varepsilon} = \frac{Cov_t\left(e^h_{t+2}, \theta\right)}{\sigma^2_\theta} \tag{72}$$

So:

$$\begin{aligned} q_t &= \mu_\theta + \mu_\varepsilon - \frac{\sum\limits_h \frac{Cov_t(e^h_{t+2}, \theta)}{\sigma_\theta} + \sum\limits_h \frac{Cov_t(e^h_{t+2}, \varepsilon)}{\sigma_\varepsilon}}{\sum\limits_h \frac{1}{A^h}} \\ &= \mu_\theta + \mu_\varepsilon - \frac{(\sigma_\theta + \sigma_\varepsilon) \sum\limits_h b^h}{\sum\limits_h \frac{1}{A^h}} = q \end{aligned} \tag{73}$$

And:

$$\begin{aligned} z_t^h(2) &= \frac{1}{A^h}\frac{\sum\limits_h \frac{Cov_t(e_{t+2}^h,\theta)}{\sigma_\theta^2}}{\sum\limits_h \frac{1}{A^h}} - \frac{Cov_t(e_{t+2}^h,\theta)}{\sigma_\theta^2} \\ &= z^h(2) = \frac{1}{A^h}\frac{\sum\limits_h b^h}{\sum\limits_h \frac{1}{A^h}} - b^h = z^h(2) \end{aligned} \tag{74}$$

$z_t^h(1) = z^h(1)$ by virtue of the budget constraints. ■

Proof. (Proposition 12)

The following inequality

$$\mathbb{E}_t a^h \geq \frac{A^h}{2}\left(Var_t(x_{t+2}^h + a^h) - Var_t(x_{t+2}^h)\right)$$

is equivalent to:

$$\mathbb{E}_t a^h \geq \frac{A^h}{2}\left(Var_t(a^h) + 2Cov_t(x_{t+2}^h, a^h)\right) \tag{75}$$

Where:

$$\mathbb{E}_t a^h = A\left(\frac{A}{A^h}(\eta-\beta) - \left(\eta^h - \beta^h\right)\right)\frac{\sigma_\theta\sigma_\varepsilon(\eta-\beta)}{\sigma_\theta+\sigma_\varepsilon}$$

$$Var_t(a^h) = \left(\frac{\frac{A}{A^h}(\eta-\beta) - \left(\eta^h-\beta^h\right)}{\sigma_\theta+\sigma_\varepsilon}\right)^2 2\sigma_\theta^2\sigma_\varepsilon^2$$

$$\begin{aligned} & Cov_t(x_{t+2}^h, a^h) \\ = & \left(\frac{\frac{A}{A^h}(\eta-\beta) - \left(\eta^h-\beta^h\right)}{\sigma_\theta+\sigma_\varepsilon}\right)\left[\sigma_\theta Cov_t\left[x_{t+2}^h, \varepsilon\right] - \sigma_\varepsilon Cov_t\left[x_{t+2}^h, \theta\right]\right] \end{aligned}$$

Where:

$$Cov_t\left(x_{t+2}^h, \varepsilon\right) = \left(\eta^h + \frac{A}{A^h}\frac{\sigma_\theta\beta + \sigma_\varepsilon\eta}{\sigma_\theta+\sigma_\varepsilon} - \frac{\sigma_\theta\beta^h + \sigma_\varepsilon\eta^h}{\sigma_\theta+\sigma_\varepsilon}\right)\sigma_\varepsilon^2$$

$$Cov_t\left(x_{t+2}^h, \theta\right) = \left(\beta^h + \frac{A}{A^h}\frac{\sigma_\theta\beta + \sigma_\varepsilon\eta}{\sigma_\theta+\sigma_\varepsilon} - \frac{\sigma_\theta\beta^h + \sigma_\varepsilon\eta^h}{\sigma_\theta+\sigma_\varepsilon}\right)\sigma_\theta^2$$

The inequality becomes:

$$
\begin{aligned}
& A\left(\eta-\beta\right)\xi\frac{\sigma_\theta\sigma_\varepsilon}{\sigma_\theta+\sigma_\varepsilon} \\
\geq \quad & \frac{A^h}{2}\xi\left(\frac{2\sigma_\theta^2\sigma_\varepsilon^2\xi}{\left(\sigma_\theta+\sigma_\varepsilon\right)^2}+2\frac{\sigma_\theta\sigma_\varepsilon}{\sigma_\theta+\sigma_\varepsilon}\left[\begin{array}{c}\frac{-\sigma_\theta\sigma_\varepsilon\xi+\sigma_\theta^2\left(\beta^h-\frac{A}{A^h}\beta\right)-\sigma_\varepsilon^2\left(\eta^h-\frac{A}{A^h}\eta\right)}{\sigma_\theta+\sigma_\varepsilon} \\ +\sigma_\varepsilon\eta^h-\sigma_\theta\beta^h\end{array}\right]\right)
\end{aligned}
$$

Where:

$$\xi=\left(\frac{A}{A^h}\left(\eta-\beta\right)-\left(\eta^h-\beta^h\right)\right)$$

Assuming that markets are incomplete and $\xi \neq 0$, the inequality in the proposition obtains after simplification. ■

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