Enduring Relationships in an Economy with Capital

Aubhik Khan
Federal Reserve Bank of Philadelphia

B. Ravikumar*
Pennsylvania State University

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Abstract

We introduce capital accumulation into an economy where individuals have private information with respect to productivity shocks and risk-sharing is implemented through long term contracts. In contrast to the risky unobservable endowment model, individual's welfare under the contract always exceeds the value of autarchy in our model. Furthermore, the contract exhibits two-sided voluntary long-term participation: neither the agent nor the intermediary providing insurance has an incentive to terminate the contract at the onset of any period. These properties follow from the result that an individual's expected lifetime utility under the contract is increasing in his stock of capital.

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*Corresponding author: B. Ravikumar, Department of Economics, Pennsylvania State University, 616 KERN Graduate Bldg., State College, PA 16802; email: ravikumar@psu.edu. We would like to thank Fernando Alvarez, Julia Thomas and the participants of the Theory workshop at the University of Virginia. All remaining errors are our sole responsibility.

1 Introduction

Many economic environments are characterized by repeated trade between a principal and an agent, where the latter has private information with respect to the value of trade. As noted by Townsend (1982), such dynamic incentive problems naturally lead to long-term contractual relationships which allow the principal to condition the agent's current and future payo®s on his reports of privately observed variables. By exploiting intertemporal trade-o®s in the agent's preferences, the contract allows for incentive-compatible trade. Partial equilibrium analyses of such enduring relationships, which abstract from factor accumulation, include the unobserved income shock models of Green (1987), Thomas and Worrall (1990) and Taub (1990), the unobservable e®ort models of Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) and Phelan and Townsend (1991), and the unobservable investment model of Atkeson (1991). Recently, Atkeson and Lucas (1992, 1995) have extended the study of such contracting environments to general equilibrium. The primary purpose of this paper is to extend the equilibrium study of economies with long term contracts to allow for capital accumulation.[1]

In our economy, agents produce using linear technologies that are subject to idiosyncratic productivity shocks which are private information. Individuals share risk through long-term contractual arrangements with competitive intermediaries. While investment is e®ectively determined by the intermediary, the returns to investment are observed only by the agent. This creates an incentive for the agent to under-report his income which eliminates the possibility of complete risk-sharing. However, as discussed above, the contract allows for incentive-compatible allocations which provide partial insurance through transfers from the intermediary; in our context, the transfers a®ect both consumption and investment. At any time, an intermediary may hold a positive or negative surplus with respect to the set of contracts he operates. We assume that intermediaries may borrow or lend amongst themselves and determine the equilibrium interest rate for such transactions through a market-clearing condition.

There are several reasons for extending the endowment model of long-term contracts with private information to allow for capital accumulation. First, such a framework endogenizes the extent of market incompleteness and allows an exploration of

[1] Our work is also a natural extension of stochastic one-sector capital accumulation models to the case of private information.

phenomena involving savings and investment or economic growth that appear to be at odds with the assumption of complete markets. For instance, Feldstein and Horioka (1980), Backus, Kehoe, and Kydland (1992) and Baxter and Crucini (1993), among others, have presented evidence that the savings to investment correlation is positive within a large number of economies. In related work, Khan and Ravikumar (2000) have examined the implications of this model for the relation between risk-sharing and economic growth.

A second motivation lies in the implications of the endowment model itself. As discussed above, in the presence of private information, insurance is incomplete and negative income shocks imply a reduction in lifetime consumption. Given convexity of individual preferences, e±ciency implies that the contract distributes the e®ect of such shocks over the lifetime of the agent. This, in turn, leads to an increasing dispersion of lifetime utility or wealth across agents. As a result, for almost all individuals, welfare under the contract eventually falls below the autarchy value of the risky endowment process. Strong assumptions concerning the enforceability of the contract are then required to rule out individuals' reversion to autarchy. In contrast, the long-term contract in our economy exhibits a one-to-one mapping from an individual's stock of capital to his lifetime utility: whenever there is a decrease in the agent's lifetime utility, the contract implements this change through an equivalent decrease in capital. These changes in capital a®ect the value of autarchy as well. In particular, reductions in capital within the contract reduce the value of the autarchy option for the individual. At the onset of any period in our economy, agents always prefer to remain in the contract since it o®ers insurance i.e., no individual wants to revert to autarchy.

Another feature of our model is that, in every period, intermediaries' expected discounted expenditure with respect to each contract satis¯es a zero-pro¯t condition. This implies that intermediaries never have an incentive to reject the continued provision of insurance. Hence, the contract exhibits two-sided voluntary participation: at the onset of any period, prior to the revelation of current productivity, neither the intermediary nor the agent prefers to end their relationship. Thus, long-term commitment is a property of our contract.[2] If one-period contractual agreements are enforceable, then there is no additional enforcement required to ensure that the long-term contract is honoured over time. In other words, if, after productivity is observed,

[2] We use term `commitment' in the sense of Phelan (1995).

neither the agent nor the intermediary can renege during the period on the previously agreed terms of trade, then no additional assumptions concerning enforceability are required to ensure continuation of trade.

Finally, while our production economy retains the characteristic of increasing dispersion of wealth or utility entitlements, this is also characteristic of the autarchic production economy. In numerical examples, we see that the private information economy reduces the rate at which inequality increases in the absence of trade. The key to this result is that the contract dampens consumption and investment °uctuations relative to autarchy.

In allowing for capital accumulation under private information, our work is related to Marcet and Marimon (1993) who examine the implications of unobservable investment shocks in the context of a two-agent environment where a risk-neutral investor lends to a risk-averse producer operating neoclassical production technology. Marcet and Marimon focus on transitional dynamics and abstract from the equilibrium determination of the interest rate at which the investor may borrow or lend. Capital accumulation also exists in the equilibrium model of Aiyagari and Williamson (1997) who study the allocation of credit in a random matching environment where agents have private information about risky endowments. Focusing on the nature of e±cient credit arrangements, they abstract from capital accumulation within the contract and instead assume that a social planner directly accumulates capital.

In our model, investment is observable, so we are able to retain the analytical approach used to examine long-term contracts in endowment economies. In particular, adapting the technique of Green (1987), we ensure that the contract is incentive compatible by imposing his variational equivalent, temporary incentive compatibility, on feasible allocations. Next, we solve the contract using a recursive dual approach which minimizes an intermediary's present-discounted expenditure of providing an agent with a pre-determined level of expected lifetime utility. The zero pro¯t condition for intermediaries, in conjunction with a market-clearing condition which determines their equilibrium discount factor, allows us to infer the corresponding solution to the original problem of maximizing the agents' lifetime utility.

Similar to the endowment model of contracts, expected lifetime utility is a state variable in the recursive dual problem; however, our framework expands the state vector with capital. For CRRA preferences we are able to reduce the dimension of the state vector. A by-product of this method is that it establishes the one-to-one mapping between capital and expected lifetime utility discussed above. Thus, capital is sufficient to determine an individual's welfare. In the endowment model, an individual's lifetime wealth is determined by his utility entitlement, but the wealth is distinct from the value of his endowment stream. In our production and capital accumulation economy, the one-to-one mapping allows us to `physically' link the individual's wealth with his expected utility.

In the next section we describe the environment and formally derive the incentive compatible contract as a solution to recursive expenditure minimization problem; we relegate all proofs to Appendix A. In Section 3, we characterize several properties of the contract and contrast these with the long-term contract in endowment economies. A series of numerical examples is presented in section 4.2 which serve to further illustrate the properties of the long term contract. Appendix B contains a brief review of certain properties of autarchy; these results are used in sections 3 and 4.2.

2 Preliminaries

In this section, we first describe the environment and the typical contract between an agent and an intermediary. We then develop two standard results on incentive compatibility and duality which will allow us to solve the contract using a recursive dual approach.

In our economy, individuals operate linear technologies subject to privately observed stochastic productivity. Capital is the sole input in production and is accumulable. Each agent enters into a permanent contract with a risk-neutral competitive intermediary.

Let z_t describe an individual's productivity at time $t = 0, 1, \ldots$. Productivity is independently and identically distributed over time and across individuals. Specifically, $z_t \in Z = \{z_1, z_2\}$ where $0 < z_1 < z_2$. Let $\mathcal{Z} \equiv 2^Z$ be the complete σ-algebra and define $\mu : \mathcal{Z} \to [0, 1]$ as a probability measure on Z, summarized by $\mu(z_i) = \mu_i > 0$. The contract requires that the agent report his productivity in each period. A reporting strategy is $\sigma = (\sigma_0, \sigma_1, \ldots)$ such that $\sigma_t : Z^{t+1} \to Z$ is the time t report of productivity by the agent.[3] Let Σ represent the set of all measurable reporting

[3] Throughout, we use the following notation: for any element $x \in X$, where X is a vector space, we define the time t value of x as x_t, the partial history of values through period t as $x^t = (x_0, \ldots, x_t) \in X^{t+1}$ and a sequence of values as $x = (x_0, x_1, \ldots) \in X^\infty$. It is understood

strategies.

In each period, the intermediary will assign the agent both a transfer for current consumption and investment for future production. Let $B_t(\sigma^t) : Z^{t+1} \to \mathbb{R}$ be the net transfer to the agent as a function of his reported history of productivity shocks and $K_{t+1}(\sigma^t) : Z^{t+1} \to \mathbb{R}_+$ be investment in the agent's technology for the next period. Output at time t is $z_t K_t(\sigma^{t-1})$ and consumption at time t is $C_t(z_t, \sigma^t) = z_t K_t(\sigma^{t-1}) + B_t(\sigma^t)$. We assume that $K_0(\emptyset) = K_0 \in \mathbb{R}_+$ is given. A plan is a vector-valued sequence of functions (K, B), where $K = \{K_t\}_{t=0}^{\infty}$ and $B = \{B_t\}_{t=0}^{\infty}$, which allows for a feasible level of consumption at every point in time, in every state of nature i.e.,

$$z_t K_t(z^{t-1}) + B_t(z^t) \geq 0, \; t = 0, 1, \ldots.$$

Let $\Gamma(K_0)$ be the set of all plans and let $\Sigma(K, B) \subseteq \Sigma$ be the set of all measurable reporting strategies which allow for feasible consumption given a plan $(K, B) \in \Gamma(K_0)$.

The agent has a strictly increasing and continuous von Neumann - Morgenstern period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ and subjective discount factor $\beta \in (0, 1)$. The expected lifetime utility from a plan $(K, B) \in \Gamma(K_0)$ is a function of the reporting strategy, $\sigma \in \Sigma(K, B)$, adopted by the agent and denoted $V(K, B, \sigma)$.[4]

$$V(K, B, \sigma) = \sum_{s=0}^{\infty} \beta^s \int_{Z^{s+1}} u\left(z_s K_s\left[\sigma^{s-1}(z^{s-1})\right] + B_s\left[\sigma^s(z^s)\right]\right) \mu^{s+1}(dz^s).$$

A plan $(K, B) \in \Gamma(K_0)$ is incentive compatible if it offers the agent a level of expected lifetime utility, when he reports each period's productivity accurately, that is no less than the level of utility from any other reporting strategy:

$$V(K, B, z) \geq V(K, B, \sigma), \; \forall \sigma \in \Sigma(K, B). \tag{1}$$

The optimal contract discounts future expenditure by the rate $q^{-1} - 1$.[5] Let $E_0(K, B)$ be the discounted cost of the lifetime consumption implied by the plan

that $x^{-1} = \{\emptyset\}$. Finally, it is convenient to define the history from t through s, where $t < s$ as ${}_t x^s = (x_t, \ldots, x_s) \in X^{s-t+1}$.

[4]The probability space $(Z, \mathcal{Z}, \mu)$ generates the finite dimensional product probability spaces $(Z^t, \mathcal{Z}^t, \mu^t)$, $t = 0, 1, \ldots$, as well as the infinite dimensional probability space of all measurable sequences of technology shocks $(Z^\infty, \mathcal{Z}^\infty, \mu^\infty)$ in the standard manner.

[5]We will establish later that the discount factor for future expenditures is constant in an equilibrium.

$(K,B) \in \Gamma(K_0)$.

$$E_0(K,B) \equiv \sum_{s=0}^{\infty} q^s \int_{Z^{s+1}} [K_{s+1}(z^s) + B_s(z^s)]\, \mu^{s+1}(dz^s). \tag{2}$$

Definition 1 The contract between an intermediary and agent is the solution to the following problem: given K_0, maximize $V(K,B,z)$ over the set of all plans $(K,B) \in \Gamma(K_0)$ satisfying the incentive compatibility constraint (1) and the financial constraint $E_0(K,B) \leq 0$.

Let the value of the optimum, be denoted $V^*(K_0)$.

2.1 Temporary incentive compatibility

Having defined incentive compatibility, we now develop the well known equivalent, temporary incentive compatibility. This will allow us to characterize a solution to the contract using a recursive approach. Below, we define temporary incentive compatibility constraints and prove their equivalence to incentive compatibility for the above contract. Our proof draws heavily on the work of Green (1987); we adapt his techniques to an environment with investment.

Given a plan $(K,B) \in \Gamma(K_0)$, and truthful reporting of a partial history $z^{t-1} \in Z^t$, we define the expected value of continued truthful reporting as

$$V(K,B,z,z^{t-1}) = \sum_{s=t}^{\infty} \beta^{s-t} \int_{Z^{s+1-t}} u\left(z_s K_s\left[z^{t-1}, {}_t z^s\right] + B_s\left[z^{t-1}, {}_t z^s\right]\right) \mu^{s+1-t}(d_t z^s) \tag{3}$$

where note that $\mu^k = \mu^{k-1} \times \mu$. Given (3), it follows that preferences will satisfy

$$V(K,B,z,z^{t-1}) = \int_Z \left[u\left(z_t K_t\left[z^{t-1}\right] + B_t\left[z^{t-1}, z_t\right]\right) + \beta V\left(K,B,z,\left[z^{t-1}, z_t\right]\right)\right] \mu(dz_t). \tag{4}$$

A plan is $(z,t)-$incentive compatible (i.c.) if having accurately reported, z^{t-1} through date $t-1$ the agent has no incentive to misrepresent future productivity. Formally, a plan (K,B) is $(z,t)-$i.c. if

$$V(K,B,z,z^{t-1}) = \max_{\sigma \in \{\sigma \in \Sigma(K,B) | \sigma^{t-1}(z^{t-1}) = z^{t-1}\}} V(K,B,\sigma,z^{t-1}). \tag{5}$$

Note that incentive compatibility is equivalent to $(z, 0)$-incentive compatibility. If the agent has truthfully reported his productivity through date $t-1$, and is constrained to do so from date $t+1$, we define a plan to be $(z, t)-$temporarily incentive compatible (t.i.c.) if it provides no incentives for misrepresentation of z_t. A plan (K, B) is $(z, t)-$ t.i.c. if given $z_i, z_j \in Z$ with $i \neq j$,

$$\begin{aligned} & u\left[z_i K_t(z^{t-1}) + B_t(z^{t-1}, z_i)\right] + \beta V\left(K, B, z, \left[z^{t-1}, z_i\right]\right) \\ \geq\ & u\left[z_i K_t(z^{t-1}) + B_t(z^{t-1}, z_j)\right] + \beta V\left(K, B, z, \left[z^{t-1}, z_j\right]\right). \end{aligned} \tag{6}$$

There are two basic results. First, a plan (K, B) is $(z, t)-$ i.c. if and only if it is $(z, t)-$ t.i.c. and $(z, t+1)-$ i.c. Second, given that (K, B) satisfies the boundary condition

$$\lim_{t \to \infty} \beta^t V(K, B, z, z^t) = 0, \ \forall z \in Z^\infty, \tag{7}$$

(K, B) is incentive compatible if and only if it is t.i.c. at every node.

Lemma 1 The plan $(K, B) \in \Gamma(K_0)$ is $(z, t)-$ i.c. if and only if it is $(z, t)-$ t.i.c. and $(z, t+1)-$ i.c.

Towards proving an equivalence of t.i.c. and i.c., it is useful to rewrite (3) as

$$\begin{aligned} V(K, B, z, z^{t-1}) =\ & \sum_{s=t}^{n-1} \beta^{s-t} \int_{Z^{s+1-t}} u\left(z_s K_s\left[z^{t-1}, {}_t z^s\right] + B_s\left[z^{t-1}, {}_t z^s\right]\right) \mu^{s+1-t}(d_t z^s) \\ & + \int_{Z^{n+1-t}} \beta^{n-t} V\left(K, B, z, \left[z^{t-1}, {}_t z^n\right]\right) \mu^{n+1-t}(d_t z^n). \end{aligned} \tag{8}$$

Lemma 2 Given (7) holds, the plan (K, B) is $(z, 0)-$ i.c. if and only if it is $(z, t)-$ t.i.c. for all $t = 0, 1, \ldots$.

2.2 Duality

We now develop a duality result similar to that in Green (1987) and Oh and Green (1992). The dual expenditure minimization problem examined below will be shown to be equivalent, given the appropriate initial condition, to the utility maximization problem solved in the contract. Given this result, we will exploit the simplicity of the dual approach to help us characterize the contract. In the dual problem, allocations are constrained so that the agent receives some pre-determined level of expected lifetime utility entitlement, which we denote V^*.

$$V(K;B;z) \geq V^*. \tag{9}$$

Given this level of utility, the dual involves the minimization of expenditure by the intermediary.

Definition 2 The dual contract minimizes $E_0(K;B)$ subject to $(K;B) \in \Gamma(K_0)$ satisfying (1) and (9).

Define the value of the optimum as $E^*(V^*;K_0)$. Our duality result, which is standard, states that if $(K^*;B^*)$ solves the contract and the value of the optimum is $V^*(K_0)$, then given $V^*(K_0)$, $(K^*;B^*)$ solves the dual problem. Furthermore, if $(\hat{K};\hat{B}) \in \Gamma(K_0)$ solves the dual problem given $V^*(K_0)$, then $V(\hat{K};\hat{B};z) = V^*(K_0)$ and $(\hat{K};\hat{B})$ solves the contract.

Lemma 3 Let $(K^*;B^*) \in \Gamma(K_0)$ solve the contract and define $V^* = V(K^*;B^*;z)$, then $(K^*;B^*)$ solves the dual contract. Furthermore, if there exists $(\hat{K};\hat{B}) \in \Gamma(K_0)$ that also solves the dual contract, then $(\hat{K};\hat{B})$ solves the contract.

3 The equilibrium recursive contract

We are now in a position to solve the contract. At this point we restrict our analysis to the case of CRRA period utility. For any $C > 0$, let

$$u(C) = \begin{cases} (1-\beta)\frac{C^\alpha}{\alpha}, & \text{for } \alpha \neq 0; \alpha < 1 \\ (1-\beta)\log C & \text{for } \alpha = 0. \end{cases}$$

As discussed above, we follow Green (1987) and characterize the contract using a recursive approach to the dual and ensure that the resulting allocation is incentive compatible by constraining it to be temporarily incentive compatible in every state of nature. In this section, we first describe the recursive dual and then solve the resulting contract for the case of logarithmic and iso-elastic preferences. Our analysis will highlight an important property of our economy: the contract is characterized by a monotonically increasing relationship between an individual's expected lifetime utility and his stock of capital.

Hereafter, we suppress time subscripts and replace them with subscripts which identify the agent's current productivity. When the current reported level of productivity is z_i, $i = 1, 2$, let $C_i = z_i K + B_i$ represent current consumption where B_i is the productivity contingent net transfer. Furthermore, define U_i to satisfy $(1 - \beta) U_i = u(C_i)$ and let V_i represent expected lifetime utility entitlement from the onset of the next period. We define the period expenditure function associated with a level of current utility $(1 - \beta) U_i$ as $C(U_i) = u^{-1}((1 - \beta) U_i)$. When $\alpha = 0$, logarithmic period utility implies $C(U_i) = \exp(U_i)$ while for $\alpha \neq 0$, iso-elastic period utility yields an expenditure function $C(U_i) = (\alpha U_i)^{\frac{1}{\alpha}}$. Note that $B_i = C(U_i) - z_i K$.

The temporary incentive compatibility constraints may be written as follows: For $i, j = 1, 2$ with $i \neq j$, if $[z_i - z_j] K + C(U_j) > 0$ then

$$(1 - \beta) U_i + \beta V_i \geq u([z_i - z_j] K + C(U_j)) + \beta V_j. \tag{10}$$

Given an initial lifetime expected utility entitlement of V, an agent's utility entitlement is subject to the following promise-keeping constraint.

$$V \leq \sum_{i=1}^{2} \pi_i ((1 - \beta) U_i + \beta V_i) \tag{11}$$

The lifetime expenditure function for the intermediary then satisfies the following functional equation:

$$E(V; K) = \min \sum_{i=1}^{2} \pi_i \left(C(U_i) - z_i K + K_i' + qE(V_i; K_i') \right) \tag{12}$$
$$\text{subject to (10) and (11).}$$

An equilibrium for our economy is a set of contracts such that total consumption plus investment is equal to total production. We will show below this market clearing condition endogenizes intermediaries' discount factor q.

3.1 The logarithmic intensive form

In this subsection, we examine the contract under the assumption that agents have logarithmic period utility. It is useful to define some notation which allows the introduction of an intensive form for the contract. This intensive form reduces the state vector and makes explicit the monotonicity property. Let γ_i be the rate of growth of the capital stock when productivity is z_i, $i = 1, 2$, while v is the initial lifetime utility entitlement net of the logarithm of the initial capital stock and u_i is the current period's utility similarly deflated. Proceeding in a consistent manner, let v_i be the future utility entitlement when current productivity is z_i, net of the logarithm of the capital determined for the next period. Finally, it is convenient to define a term $x_i = v_i + \log \gamma_i$, which may be interpreted as expected lifetime utility from the onset of the next period, given productivity z_i, deflated by the logarithm of the current stock of capital.

Definition 3 When $\alpha = 0$, let $\gamma_i = \frac{K_i'}{K}$, $v = V - \log K$, $u_i = U_i - \log K$, $v_i = V_i - \log K_i'$, $x_i = v_i + \log \gamma_i$.

For the logarithmic case, the one-to-one mapping discussed above implies that lifetime expected utility, in the contract, is a linear function of the logarithm of the capital stock. This property, which is shared by the contract with the value functions that arise under either complete risk-sharing or autarchy, allow us to simplify the problem as follows. Dividing both sides of (12) by K we conjecture that, for some unknown function $W : R \to R$,

$$\frac{E(V; K)}{K} = W(v). \tag{13}$$

Equation (13) essentially states that the objective function depends only on the difference between lifetime utility entitlement and the logarithm of the capital stock. This conjecture is motivated by the observation that constraint correspondence is parameterized by $V - \log K$. Thus, the composite state variable, v, is sufficient to determine the feasible set. As a result, the intensive form expenditure function, W, depends only upon this composite variable.

The intensive form representation of the t.i.c. constraints, for i versus j, with $z_i - z_j + \exp(u_j) > 0$; is shown in (14).

$$(1 - \beta)\, u_i + \beta x_i \geq (1 - \beta) \log (z_i - z_j + \exp(u_j)) + \beta x_j \tag{14}$$

The promise keeping constraint on utility entitlements, in intensive form, may be written as

$$v \leq \sum_{i=1}^{2} \pi_i \left((1-\beta) u_i + \beta x_i\right). \qquad (15)$$

The contract, in intensive form, involves the solution to the following functional equation.

$$W(v) = \min \sum_{i=1}^{2} \pi_i \left(\exp(u_i) - z_i + \exp(x_i - v_i)\left[1 + qW(v_i)\right]\right) \qquad (16)$$

subject to (14) - (15).

The proposition below follows immediately.

Proposition 4 When preferences are characterized by logarithmic utility ($\alpha = 0$), the dynamic program described by (16) is equivalent to that described in (12).

In order to solve the economy, we suppress (14) for $i = 1$, $j = 2$ (the upward t:i:c:); we will show later that this constraint does not bind. The reduced problem, which solves (16) subject to (14) for $i = 2$ and (15) given v and q, involves a closed, convex-valued constraint correspondence since $z_2 - z_1 > 0$. Note that this reduced problem implies an unconstrained choice of v_i which solves the following problem.

$$\min_{v_i} \exp(-v_i)\left[1 + qW(v_i)\right].$$

Let v^* be the optimal value of v_i, $i = 1, 2$. Since v^* is independent of v, standard methods may be used to prove that $W(v)$ is well-defined, strictly increasing, convex and differentiable. This in turn implies that v^* solves

$$-1 - qW(v^*) + qW'(v^*) = 0 \qquad (17)$$

Let λ be the multiplier for equation (14), where $i = 2$, and θ be the multiplier for (15). The first-order conditions with respect to $(u_i, x_i)_{i=1}^{2}$ imply the following restrictions.

$$\exp(u_1) = (1-\beta)\left(\theta - \lambda\frac{1}{\pi_1}\frac{\exp(u_1)}{z_2 - z_1 + \exp(u_1)}\right) \tag{18}$$

$$\exp(u_2) = (1-\beta)\left(\theta + \lambda\frac{1}{\pi_2}\right) \tag{19}$$

$$\exp(x_1) = \beta\frac{\exp(v^*)}{[1 + qW(v^*)]}\left(\theta - \lambda\frac{1}{\pi_1}\right) \tag{20}$$

$$\exp(x_2) = \beta\frac{\exp(v^*)}{[1 + qW(v^*)]}\left(\theta + \lambda\frac{1}{\pi_2}\right) \tag{21}$$

The Benveniste-Scheinkman condition implies that $W'(v) = \theta$. It is straightforward to show that $\lambda > 0$; if not, (14), for $i = 2$, is violated. Thus, the downward t.i.c. constraint binds. Using the necessary conditions and the downward t.i.c. we can establish the following lemma which proves that insurance is incomplete within the contract.

Lemma 5 The contract allocation is characterized by $\lambda > 0$, $u_1 < u_2$, $x_1 < x_2$ and $v_i = v^*$, $i = 1, 2$.

Since equation (14) binds when $i = 2$, an implication of lemma 5 is that $\exp(u_1) - z_1 > \exp(u_2) - z_2$. This is sufficient to ensure that (14) for $i = 1$ does not bind, as we had conjectured earlier.

3.1.1 Equilibrium

We now explore equilibrium for the economy with logarithmic preferences. Given any individual identified by an initial level of capital K, expected lifetime utility is determined by the intermediary's zero-profit condition: $E(V; K) = 0$. Since $E(V; K) = KW(V - \log K)$, this condition implies, given strict monotonicity of W, that $v = V - \log K$ is constant across different levels of K and solves $W(v) = 0$. Now let v denote this level of the composite state variable which is common across agents and let $\exp(u_i)$ and γ_i describe consumption per unit capital and growth in capital, given i, associated with this value of v. It follows that consumption and investment, given productivity z_i, for any individual identified by K, will be $\exp(u_i)K$ while investment will be $\gamma_i K$.

To examine market-clearing, let $\psi(K)$ represent the distribution of capital across agents over the space of current capital holdings, K. For the group of agents identified with some level of capital $K \in \mathcal{K}$, average output will be ξK and average current expenditure will be $\sum_{i=1}^{2} \mu_i (\exp(u_i) + \gamma_i) K$. Economy-wide market clearing requires that aggregate output equal the sum of aggregate consumption and investment. That is,

$$\int_{\mathcal{K}} \xi K \psi(dK) = \int_{\mathcal{K}} \left(\sum_{i=1}^{2} \mu_i (\exp(u_i) + \gamma_i) \right) K \psi(dK).$$

It is easy to see that the equilibrium restriction on aggregate allocations implies an equivalent restriction on the expected or average current expenditure within the contract i.e., $\sum_{i=1}^{2} \mu_i (\exp(u_i) + \gamma_i) = \xi$ or $\sum_{i=1}^{2} \mu_i (\exp(u_i) - z_i + \gamma_i) = 0$. Note that this implies that on average, savings is equal to investment.

Next, recalling $\gamma_i = \exp(x_i - v_i)$ and $W(v) = 0$, the dynamic program (16) implies that $q \sum_{i=1}^{2} \mu_i \gamma_i W(v_i) = 0$. Since $v_i = v^*$, where v^* is the solution to (17), we must have $W(v_i) = 0$, for $i = 1, 2$. This determines q from (17) such that q satisfies $qW'(v^*) = 1$. Note that v is then constant over time and across states of nature. The recursive equilibrium is stationary in the sense that $(u_i, \gamma_i)_{i=1}^{2}$ and q are also time-invariant. This verifies our earlier conjecture that q is constant over time.

3.2 The iso-elastic case

When preferences are characterized by iso-elastic period utility functions we may apply an intensive form to reduce the dimension of the state vector, similar to the approach we have taken for the logarithmic case. In particular, for a different composite state variable, v, defined below, the conjecture in equation (13) continues to hold. The following definitions are useful for the iso-elastic case.

Definition 4 When $\alpha \neq 0$, let $\gamma_i = \frac{K_i'}{K}$, $v = \frac{V}{K^\alpha}$, $u_i = \frac{U_i}{K^\alpha}$, $v_i = \frac{V_i}{(K_i')^\alpha}$, $x_i = v_i (\gamma_i)^\alpha$.

The intensive form representation of the t.i.c. constraints, for i versus j, with $z_i - z_j + (\alpha u_j)^{\frac{1}{\alpha}} > 0$, is shown in (22).

$$(1 - \beta) u_i + \beta x_i \geq (1 - \beta) \frac{\left(z_i - z_j + (\alpha u_j)^{\frac{1}{\alpha}} \right)^\alpha}{\alpha} + \beta x_j \tag{22}$$

The intensive form contract for the case of iso-elastic preferences is given by the solution to

$$W(v) = \min \sum_{i=1}^{2} \mu_i \left((\alpha u_i)^{\frac{1}{\alpha}} - z_i + \left(\frac{x_i}{v_i}\right)^{\frac{1}{\alpha}} [1 + qW(v_i)] \right) \tag{23}$$

subject to (22) and (15).

Proposition 6 When preferences are characterized by iso-elastic utility ($\alpha \neq 0$), the dynamic program described by (23) is equivalent to that described in (12).

We proceed using the same approach implemented in section 3.1. Equation (22) is suppressed for $i = 1$, $j = 2$ (the upward t.i.c.); as in the log case, this will not bind. The reduced problem solves (23) subject to (22) for $i = 2$ and (15) given v and q, involves a closed, convex-valued constraint correspondence since $z_2 - z_1 > 0$. As before, this reduced problem implies an unconstrained choice of v_i. Let v^* be the optimal value of v_i, $i = 1, 2$. The properties of W discussed in the log case continue; the intensive form expenditure function is well-defined, strictly increasing, convex and differentiable. This in turn implies that the optimal choice v^* solves

$$-1 - qW(v^*) + \alpha q v^* W'(v^*) = 0. \tag{24}$$

As before, let λ be the multiplier for (22), where $i = 2$, and θ be the multiplier for (15). The first-order conditions with respect to $(u_i, x_i)_{i=1}^2$ yield the following.

$$(\alpha u_1)^{\frac{1-\alpha}{\alpha}} = (1-\beta)\left(\theta - \lambda \frac{1}{\mu_1} \left[\frac{(\alpha u_1)^{\frac{1-\alpha}{\alpha}}}{z_2 - z_1 + (\alpha u_1)^{\frac{1}{\alpha}}} \right]_{1-\alpha} \right) \tag{25}$$

$$(\alpha u_2)^{\frac{1-\alpha}{\alpha}} = (1-\beta)\left(\theta + \lambda \frac{1}{\mu_2}\right) \tag{26}$$

$$\left(\frac{x_1}{v^*}\right)^{\frac{1-\alpha}{\alpha}} = \beta \frac{\alpha v^*}{[1 + qW(v^*)]}\left(\theta - \lambda \frac{1}{\mu_1}\right) \tag{27}$$

$$\left(\frac{x_2}{v^*}\right)^{\frac{1-\alpha}{\alpha}} = \beta \frac{\alpha v^*}{[1 + qW(v^*)]}\left(\theta + \lambda \frac{1}{\mu_2}\right) \tag{28}$$

Again, the Benveniste-Scheinkman condition implies that $W'(v) = \theta$. Equation (22), when $i = 2$, then implies that $(\alpha u_1)^{\frac{1}{\alpha}} - z_1 > (\alpha u_2)^{\frac{1}{\alpha}} - z_2$. This is sufficient to ensure that (22) for $i = 1$ does not bind, as we had conjectured. This in turn ensures that lemma 5 continues to apply for the case of iso-elastic utility.

It is straightforward to adapt the discussion in section 3.1.1 to examine equilibrium for the economy characterized by iso-elastic period utility. In particular the intermediary's initial zero profit condition implies, as before, that $W(v) = 0$ for each contract. Since $v = \frac{V}{K^{\alpha}}$, this leads to a proportionality between V and K^{α} establishing the monotonicity property for the iso-elastic case. The market-clearing condition then requires that v be constant over time and across states of nature. This value of v is given by v^* where the latter is the solution to $\alpha q v^* W'(v^*) = 1$.

4 Implications

In this section, we begin by discussing the implications of the contract for the response of consumption and savings to productivity shocks. We follow this discussion by examining the effect of the monotonicity property for the value of autarchy.

Recall that $v^* = v$ in equilibrium, so the Benveniste and Scheinkman condition yields $q\theta = 1$ in the logarithmic case and $\alpha v^* q\theta = 1$ in the iso-elastic case. This implies that $\gamma_1 < \gamma_2$.[6] Consumption for an agent identified with a current level of capital K and productivity z_i is $C_i \equiv C(U_i)$, which equals $\exp(u_i)K$ for the log case and $(\alpha u_i)^{\frac{1}{\alpha}}K$ for the iso-elastic case while investment is $K'_i = \gamma_i K$ for both cases. From lemma 5, we see that both consumption and investment fall during times of lower than expected income. Private information allows only limited risk sharing, and reports of low productivity result in a reduction in lifetime consumption. Given diminishing marginal utility, the cost minimizing intermediary will spread this fall in lifetime consumption over time. Consequently, low productivity results in both lower current consumption and reduced investment.

The net transfer from the intermediary to the agent, defined as consumption minus income, is B_i; thus, $-B_i$ may be interpreted as the agent's savings or payment to the intermediary. In the log case, $B_i = (\exp(u_i) - z_i)K$ while in the iso-elastic case, $B_i = \left[(\alpha u_i)^{\frac{1}{\alpha}} - z_i\right]K$. As discussed in sections 3.1 and 3.2, an implication of the contract is that $-B_1 < -B_2$. Since $C_1 < C_2$, we immediately see that the contract exhibits co-insurance in the sense that the payment to the intermediary and

[6] For the log case, equations (20) and (21) imply that $\gamma_1 = \beta\left(q^{-1} - \lambda\frac{1}{\pi_1}\right)$ while $\gamma_2 = \beta\left(q^{-1} + \lambda\frac{1}{\pi_2}\right)$. For the isoelastic case, equations (27) and (28) yield $\gamma_1 = \left[\beta\alpha v\left(q^{-1} - \lambda\frac{1}{\pi_1}\right)\right]^{\frac{1}{1-\alpha}}$ and $\gamma_2 = \left[\beta\alpha v\left(q^{-1} + \lambda\frac{1}{\pi_2}\right)\right]^{\frac{1}{1-\alpha}}$.

the individual's consumption move together.

Finally, interpreting $-B_i$ as the agent's level of savings, we note that savings and investment are positively correlated. Periods of above average productivity lead to an increase in both the level of savings and the quantity of investment. In a two-country model with complete markets, productivity spillovers and capital adjustment costs, Baxter and Crucini (1993) have reproduced the positive savings-investment correlation. Their quantitative result emphasizes country size. We are able to generate this qualitative result even though locations are small and productivity is independently distributed.

4.1 A comparison with autarchy

The above analysis isolates lifetime utility per unit of capital, v, that is constant across states and over time. Expected lifetime utility for an individual with K units of capital is $V^*(K) = v + \log K$ for the logarithmic case and $V^*(K) = vK^\alpha$ for the iso-elastic case. An implication of the monotonicity property, in equilibrium, is that the net expected expenditure by the intermediary with respect to any contract at the onset of any period is zero since $E(V;K) = KW(v) = 0$. This in turn implies that, on average, savings equals investment, $\sum_{i=1}^{2} \pi_i \left(B_i + K'_i \right) = 0$.

To see whether an individual may eventually be better off without the contract, we briefly consider the autarchic allocation. In autarchy, savings is equal to investment at every point in time and in every state of nature i.e., $-B_t(z^t) = K_{t+1}(z^t)$ for all $z^t \in Z^{t+1}$, $t = 0, 1, \ldots$. Let $V^A(z_i;K)$ denote the expected lifetime utility of an individual with K units of capital given z_i as the current level of productivity. The functional equation, in the absence of trade, is given by

$$V^A(z_i;K) = \max_{K' \in [0, z_i K]} \left(u(z_i K - K') + \beta \sum_{j=1}^{2} \pi_j V^A(z_j;K') \right). \tag{29}$$

It is well known that the optimal policy is $K' = \beta z_i K$ for logarithmic utility and $K' = \left[\beta \sum_{j=1}^{2} \pi_j z_j^\alpha \right]^{\frac{1}{1-\alpha}} z_i K$ for iso-elastic utility.

Next we take the expectation of $V^A(z_i;K)$ over z_i to define $V^A(K) = \sum_{i=1}^{2} \pi_i V^A(z_i;K)$ which yields expected lifetime utility. Note that $V^A(z_i;K)$ represents the value of K units of capital conditional on the current productivity shock, while $V^A(K)$ values the current capital stock prior to the realization of the productivity shock. The latter function ensures that the value function for the autarchic problem is consistent with the timing of events used to solve the contract. Once $V^A(K)$ is determined, the autarchic analogue of v is given by $v^A = V^A(K) - \log K$ for the logarithmic period utility and $v^A = V^A(K)/K^{\alpha}$ for the iso-elastic utility.

We show in Appendix B that the autarchy allocation is distinct from the contract allocation and that the autarchy allocation is feasible for the contract. It then follows that the contract allocation cannot yield lower expected lifetime utility at any value of K. Since the monotonicity property implies that individuals are completely identified by their stock of capital, the contract is always at least as good as autarchy at the onset of any period.

4.2 A numerical example

The introduction of production and investment into the long-term contract, plus the endogeneity of the discount factor used by intermediaries, makes further analytical characterization difficult. However, computation of the contract economy is relatively easy. We do not have to solve for the value function or track the distribution of wealth in order to obtain the equilibrium contract allocation. The monotonicity property, necessarily absent in the endowment model, allows us to solve a small number of non-linear equations which describe the typical contract. This yields $(u_i, x_i)_{i=1}^2$ and v, which in turn allows us to calculate $(\gamma_i)_{i=1}^2$. Through numerical examples below we highlight several additional features of the production economy which distinguish it from the endowment model.

In order to numerically simulate our economy, we set the parameters as follows. We allow productivity to vary around its mean value ξ, symmetrically. In other words, let $\xi - z_1 = z_2 - \xi = \chi$ and $\mu_1 = 0.5$. After choosing a plausible value for $\xi = 1.05$, we set χ to as to imply a reasonably high coefficient of variation, 0.15, for the productivity process. Finally, let $\beta = 0.97$; this implies that the economy with logarithmic preferences, under complete risk-sharing, would have a real interest rate of 5 per cent and a growth rate of roughly 2 per cent per annum. We examine the contract for three different values of α: 0.5, 0, and -1. It is important to emphasize that while we illustrate these features of our private information economy using specific parameters, these findings hold across all parameterizations that we have examined.

As noted in the introduction, an individual's lifetime wealth in the endowment model is determined by his utility entitlement. Since our purpose here is to contrast the long-term contract under production and capital accumulation with the endowment model, we adopt individual utility entitlements as our metric even though our model has a distinct, and perhaps preferable, measure of wealth, capital. To begin with, consider the evolution of lifetime utility under the enduring relationship. Our strategy is to examine an economy with a unit measure of agents who are ex-ante identical, each with a unit of capital, in the initial period. A fraction μ_1 of these agents will experience a capital growth γ_1 and a fraction μ_2 will experience a growth rate γ_2, so the population is no longer homogeneous in period 1. In period 2, each of these two groups are again split into proportions μ_1 and μ_2. Each period, we calculate the population mean and standard deviation of expected lifetime utility under the assumption of long-term contracts. We then contrast the results with those under autarchy.

Figure 1 displays the economy-wide mean level of utility, for both the contract and autarchic environments, for the first 50 periods. In contrast to the endowment economy, we see that expected lifetime utility in the enduring relationship actually rises above that in autarchy over time. This is despite the fact that in some cases the growth rate of wealth is higher under autarchy.[7] For example, when $\alpha = -1$, which corresponds to a coefficient of relative risk aversion $(1 - \alpha)$ of 2, the strong self-insurance motive present under autarchy leads to a doubling of the mean rate of growth of capital, relative to the contract economy. However, the insurance offered through long-term contracts implies reduced variability in investment rates, relative to autarchy. This is sufficient to ensure that the mean level of expected lifetime utility grows in the contract economy while it falls in autarchy.

In Figure 2, we examine the standard deviation of lifetime utility. Dispersion in lifetime utility grows both settings. However, the long-term contract dampens the inequality in lifetime utility relative to autarchy, and that the dampening is dramatic when α is sufficiently low ($\alpha = -1$). The contract provides insurance by dampening variability in both consumption and investment. As a result the inequality rises more slowly than in autarchy.

[7] The rates of growth for the contract economies, when $\alpha = -1$, 0 and 0.5, are 0.98, 1.95 and 3.95 per cent, respectively. The corresponding growth rates under autarchy are 2.15, 2.0 and 3.45 percent.

5 Concluding remarks

We have examined the qualitative behaviour of an economy with long-term contracts with production and capital accumulation. Insurance is implemented in our environment by exploiting di®erences in agents' willingness to intertemporally substitute consumption: individuals with low productivity receive a net transfer of resources at the expense of reduced future expected utility. Under the assumption of linear technology and CRRA preferences, we establish a one-to-one mapping between an individual's stock of capital and his expected lifetime utility. In particular, we show that adjustments to lifetime utility are implemented through changes in an individual's capital. This feature provides an interesting contrast between our model and the long-term contracts model with risky endowments.

Both the endowment environment and our environment exhibit increasing dispersion in lifetime utility (or wealth) across agents. For the endowment economy, however, wealth in the typical contract eventually falls below the autarchy value of the endowment process. In the production economy with capital accumulation, the adjustment in individuals' stocks of capital a®ect the value of autarchy. We ¯nd that the contract is able to maintain welfare above autarchy levels for any agent. Furthermore, the one-to-one mapping, in equilibrium, implies that the expected pro¯t from providing insurance is zero for each intermediary. Hence, unlike the endowment model our contract exhibits two-sided voluntary participation: provided any agreement is enforceable during the period, it is never rejected by either party at the onset of any period.[8]

Recently, Atkeson and Lucas (1995) and Phelan (1995) have shown that the increasing dispersion of wealth property in the endowment model can be eliminated if the set of possible lifetime utility entitlements is bounded below. Atkeson and Lucas motivate their bound as re°ecting the fact that, in a dynasty model, the current generation cannot promise away the utility entitlement of future generations. In our model, lifetime utility is proportional to capital and future utility is determined by current investment. Bounding future utility below is equivalent to forcing a lower bound on the agent's current savings. This is far stronger than the assumption that agents cannot sell the utility of future generations. Phelan assumes that at the onset

[8] Thus, our contract may be interpreted as allowing for a two-sided lack of commitment, thereby extending Phelan's (1995) one-sided lack of commitment.

of any period, the agent is free to recontract with another intermediary. Again, the one-to-one mapping in our model implies that if agents were free to do so, they would never want to exercise this option. Hence, the methods in Atkeson and Lucas and Phelan do not generate an invariant limiting distribution of wealth in our environment. In related work, Aiyagari and Alvarez (1995) have shown that the presence of an invariant distribution in the endowment model relies on individuals' consumption possibilities sets being compact. In our model with capital accumulation, the possibility of long run growth makes the assumption of upper bounds on consumption unattractive.

A limitation of our approach is the assumption of observable capital accumulation. Cole and Kocherlakota (1997) have emphasized the e®ects of introducing hidden storage in an environment where individuals have risky, privately observed, endowment shocks and may operate a risk-free, unobservable storage technology. Assuming that the intermediary has the ability to independently invest in a separate, publicly observable storage technology, they show that this leads to an elimination of state-contingent payo®s or insurance. Our framework, which does not provide intermediaries with the ability to accumulate wealth independently of the agent, assumes that capital accumulation by the agent is both essential and risky. Thus, the Cole and Kocherlakota results do not easily extend to our environment. While we view the assumption of observable investment as restrictive in certain contexts, we also view the lack of insurance arrangements which arises under the Cole and Kocherlakota model as equally problematic. In future research, we intend to examine the implications of risky, unobservable investment by the agent in a setting where there is no publicly observable investment by the intermediary.

Appendix

A Proofs

Lemma 1: ( )) Let (K; B) be a plan that is (z; t)-i:c: and assume that it is not (z; t + 1)-i:c: Then there exists ¾$_a$ 2 § (K; B) such that ¾$_a^t$(z^t) = z^t, 8z^t 2 Z^{t+1} and V (K; B; ¾$_a$; b^t) > V (K;B; z; b^t) for some b^t 2 Z^{t+1}. Consider the plan ¾$_b$ where ¾$_b^s$(z^s) = z^s for s = 0; 1; : : : ; t and ¾$_b^s$(z^s) = z^s 8s ¸ t+1 unless z^s = [(b$_0$; : : : ; b$_t$);$_{t+1}$ z^s] for some $_{t+1}$z^s 2 Z$^{s\,i\,t}$. In this case, let ¾$_b^s$(z^s) = ¾$_a^s$(z^s). Equation (4) implies that V (K; B; ¾$_b$; z$^{t\,i\,1}$) > V (K;B; z; z$^{t\,i\,1}$) contradicting (z; t)-i:c: of (K; B). Thus given (K; B) is (z; t)-i:c: it is also (z; t + 1)-i:c: Next assume that (K; B) is not (z; t)-t:i:c:, then for some z$_i$ 2 Z, (6) fails. Given (4), this again contradicts (5).

(( ) If (K; B) is a plan that is both (z; t)-t:i:c: and (z; t + 1)-i:c:, then (4) immediately establishes (5). ■

Lemma 2: ( )) Repeated applications of lemma 1 proves that if (K; B) is (z; t)-i:c: then it is (z;t)-t:i:c:, for t = 0; 1; : : :.

(( ) Let (K; B) be (z; t)-t:i:c: for all t = 0; 1; : : :, but assume that it is not (z; 0)-i:c: Then there exists b¾ 2 § (K; B) such that " = V (K; B; b¾; ;) ¡ V (K; B; z; ;) > 0. Given (7) and (8), 9n 2 N such that V (K; B; e¾; ;) > V (K; B; z; ;) where e¾^t(z^t) = b¾^t(z^t) for t < n and e¾$_t$(z^t) = z$_t$ for t ¸ n. This strategy involves misrepresentation at only a ¯nite number of nodes, a maximum of n^2. Let z^l 2 Z^{l+1} describe the last node at which there is misrepresentation. As (K; B) is t:i:c: at z^l, truth-telling dominates any other strategy, as it does at all other z̄^{l} 2 Z^{l+1}. We see that e¾ allows for misrepresentation at a maximum of (n ¡ 1)^2 nodes. Backwards induction proves that e¾ is (z; 0)-i:c: This contradicts the existence of b¾. ■

Lemma 3: ( )) Since (K$^¤$; B$^¤$) solves the contract, it satis¯es (1), (9) and E$_0$(K$^¤$; B$^¤$) · 0 by de¯nition. Assume there exists (K̂; B̂) 2 ¦ (K$_0$) which also satis¯es (1) and (9) where " ´ E$_0$(K$^¤$; B$^¤$) ¡ E$_0$(K̂; B̂) > 0. As (K̂; B̂) satis¯es (1), it is (z; 0)-i:c: which by lemma 1 implies that it is (z; 0) ¡ t:i:c: and (z; 1) ¡ i:c:. For (6) to hold for both i = 1; 2, given concavity of v, it must be true that B$_0$(z$_1$) ¸ B$_0$ (z$_2$). Next, temporary incentive compatibility for i = 2, requires V (K; B; z; z$_1$) · V (K; B; z; z$_2$). There are three possibilities: (i) equation (6) does not bind for either i = 1 or 2, (ii) equation (6) binds only for i = 1 or

2, and (iii) equation (6) binds for both $i = 1$ and $i = 2$. We design an alternative plan $(\hat{K}, \bar{B})$ as follows. Let $\bar{B}_t(z^t) = \hat{B}_t(z^t)$ for all $t > 0$ and $\bar{K}_t(z^t) = \hat{K}_t(z^t)$ for all t. If temporary incentive compatibility for $(\hat{K}, \hat{B})$ is characterized by case (i), then let $\bar{B}_0(z_i) > \hat{B}_0(z_i)$ for either $i = 1$ or 2 until (6) binds for either $i = 1$ or 2 or $\bar{B}_0(z_i) = \hat{B}_0(z_i) + \varepsilon$. Alternatively if case (ii) applies, let $\bar{B}_0(z_i) > \hat{B}_0(z_i)$ until the t.i.c. for $j \neq i$ binds or $\bar{B}_0(z_i) = \hat{B}_0(z_i) + \varepsilon$. Finally, if case (iii) applies, then it must be true that $\hat{B}_0(z_1) = \hat{B}_0(z_2)$; let $\bar{B}_0(z_i) = \hat{B}_0(z_i) + \varepsilon/2$. The alternate plan $(\hat{K}, \bar{B})$ is temporary incentive compatible at $t = 0$ and, by construction, $(z, 1) -$ i.c., thus it satisfies (1). Furthermore, $E_0(\hat{K}, \bar{B}) \le 0$ and $V(\hat{K}, \bar{B}; z) > V^*(K_0)$. This is a contradiction.

($\Leftarrow$) Next assume that $(\hat{K}, \hat{B})$ solves the dual problem, but not the primal. Since $(\hat{K}, \hat{B}) \in \Gamma(K_0)$ by definition, and satisfies both (1) and (9), then it must be that $E_0(\hat{K}, \hat{B}) > 0 \ge E_0(K^*, B^*)$. This contradicts $(\hat{K}, \hat{B})$ solving the dual. ■

Proposition 4: When $\alpha = 0$, recall that $C(U) = \exp(U)$. The result then follows from substituting $v + \log K$ for V, $u_i + \log K$ for U_i, $v_i + \log K_i'$ for V_i', γ_i for $\frac{K_i'}{K}$ and x_i for $v_i + \log \gamma_i$, in (13) - (15) and simplifying. ■

Lemma 5: If $\lambda = 0$ then (18) - (21) jointly imply that $u_1 = u_2$ and $x_1 = x_2$ which violates (14), hence $\lambda > 0$ and (14) binds. Equations (20) and (21) then yield $x_1 < x_2$ while (18) and (19) yield $u_1 < u_2$. ■

Proposition 6: When $\alpha \neq 0$, note that $C(U) = (\alpha U)^{\frac{1}{\alpha}}$. Using the definitions of v, u_i, v_i, x_i and γ_i for the iso-elastic case, the proof is identical in method to that used to prove lemma 4. ■

B Autarchy

In this appendix, we show that the autarchic allocation, characterized by $-B_t(z^t) = K_{t+1}(z^t)$ for all $z^t \in Z^{t+1}$, is feasible for the contract. As shown below, the solution to (29) implies that $z_i K - K' > 0$ when utility is either logarithmic or iso-elastic. Denoting the autarchy allocation as $\left(K^A, B^A\right)$, it then follows that $\left(K^A, B^A\right) \in \Gamma(K_0)$. Equation (2), given the definition of autarchy, implies that $E\left(K^A, B^A\right) = 0$, for any q. It remains to be shown that $\left(K^A, B^A\right)$ is incentive compatible. We establish by proving that $\left(K^A, B^A\right)$ is temporarily incentive compatible at every (z, t) for all

$z \in Z^\infty$ and $t = 0, 1, \ldots$, and applying lemma 2. This is shown for the logarithmic case in the next section, and, in the following section, for the iso-elastic case.

B.1 Logarithmic case

In any period t, let current productivity be z_i, $i = 1, 2$, and the capital stock be K. As is well known, when $u(C) = (1-\beta)\log C$ this yields the optimal policy $K' = s_0 z_i K$ where $s_0 = \beta$. The autarchic analogue of v is given by $v^A = V^A(K) - \log K$ or

$$v^A = \log\left((1-\beta)\left[z_1^{\mu_1} z_2^{\mu_2} \beta^{\beta} \right]^{\frac{1}{1-\beta}} \right).$$

Provided $(z_i - z_j) + (1 - s_0) z_j > 0$, temporary incentive compatibility in autarchy requires

$$\begin{aligned}(1-\beta)\log[(1-s_0) z_i K] + \beta v^A + \beta \log s_0 z_i K \geq \\ (1-\beta)\log[(z_i - z_j)K + (1-s_0) z_j K] + \beta v^A + \beta \log s_0 z_j K\end{aligned}$$

where $z_i \neq z_j$ and $i, j = 1, 2$. Fix i and define δ so that $z_j = \delta z_i$. It is then feasible to misrepresent the productivity level z_i if $0 < \delta < \beta^{-1}$. The temporary incentive compatibility constraint is satisfied if $f(\delta) \geq 0$ where

$$f(\delta) \equiv (1-\beta)\log(1-\beta) - (1-\beta)\log(1-\delta\beta) - \beta\log\delta.$$

Clearly $\lim_{\delta \to 0} f(\delta) = \infty$, $f(1) = 0$ and $\lim_{\delta \to \beta^{-1}} f(\delta) = \infty$. Further

$$f'(\delta) = \frac{\beta[\delta - 1]}{\delta(1-\delta\beta)}.$$

Note that there is a unique value, $\delta = 1$, such that $f'(\delta) = 0$. Furthermore,

$$f''(\delta) = \frac{\beta^2\delta(\delta-2) + \beta}{[\delta(1-\beta\delta)]^2}$$

so that

$$f''(\delta)\,|_{\delta=1} = \frac{\beta(1-\beta)}{(1-\beta)^2} > 0.$$

As a result, $f(\delta) \geq 0$ for all δ.

B.2 Iso-elastic case

As above, let current productivity be z_i, $i = 1, 2$, the capital stock equal K and assume that $u(C) = (1 - \beta)\frac{C^\alpha}{\alpha}$, where $\alpha < 1$ and $\alpha \neq 0$. For this case, the optimal policy is to set $K' = s_\alpha z_i K$ where $s_\alpha = \left[\beta \sum_{j=1}^{2} \pi_j z_j^\alpha\right]^{\frac{1}{1-\alpha}}$. The corresponding intensive form term is given by $v^A = V^A(K)/K^\alpha$ or

$$v^A = \frac{(1-\beta) s_\alpha^{1-\alpha}}{\alpha\beta(1-s_\alpha)^{1-\alpha}}. \tag{30}$$

Provided $(z_i - z_j) + (1 - s_\alpha) z_j > 0$, temporary incentive compatibility for the autarchy allocation requires

$$(1-\beta)\frac{[(1-s_\alpha)z_i K]^\alpha}{\alpha} + \beta v^A [s_\alpha z_i K]^\alpha \geq$$
$$(1-\beta)\frac{[(z_i - z_j)K + (1-s_\alpha)z_j K]^\alpha}{\alpha} + \beta v^A [s_\alpha z_j K]^\alpha$$

where $z_i \neq z_j$ and $i, j = 1, 2$. As above, fix $i = 1, 2$ and define δ so that $z_j = \delta z_i$. An equivalent statement of the t.i.c. constraint is to require, whenever $0 < \delta < s_\alpha^{-1}$, that $f(\delta) \geq 0$ where

$$f(\delta) \equiv (1-\beta)\frac{(1-s_\alpha)^\alpha}{\alpha} + \beta v^A s_\alpha^\alpha - (1-\beta)\frac{(1-s_\alpha\delta)^\alpha}{\alpha} + \beta v^A (s_\alpha\delta)^\alpha.$$

As in the log case, $f(1) = 0$. Further,

$$\begin{aligned} f'(\delta) &= (1-\beta)(1-s_\alpha\delta)^{\alpha-1}s_\alpha - \alpha v^A\beta(s_\alpha\delta)^{\alpha-1}s_\alpha \\ &= (1-\beta)s_\alpha\left[\frac{1}{(1-s_\alpha\delta)^{1-\alpha}} - \frac{1}{(\delta - s_\alpha\delta)^{1-\alpha}}\right]. \end{aligned}$$

where we have used (30) to eliminate v^A in the second line. It is apparent that $f'(1) = 0$. Since

$$f''(\delta) = (1-\alpha)s_\alpha^2\left[(1-\beta)(1-s_\alpha\delta)^{\alpha-2} + \alpha\beta v^A(s_\alpha\delta)^{\alpha-2}\right] > 0,$$

we have $f(\delta) \geq 0$ for all δ.

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Figure 1: Mean Expected Lifetime Utility

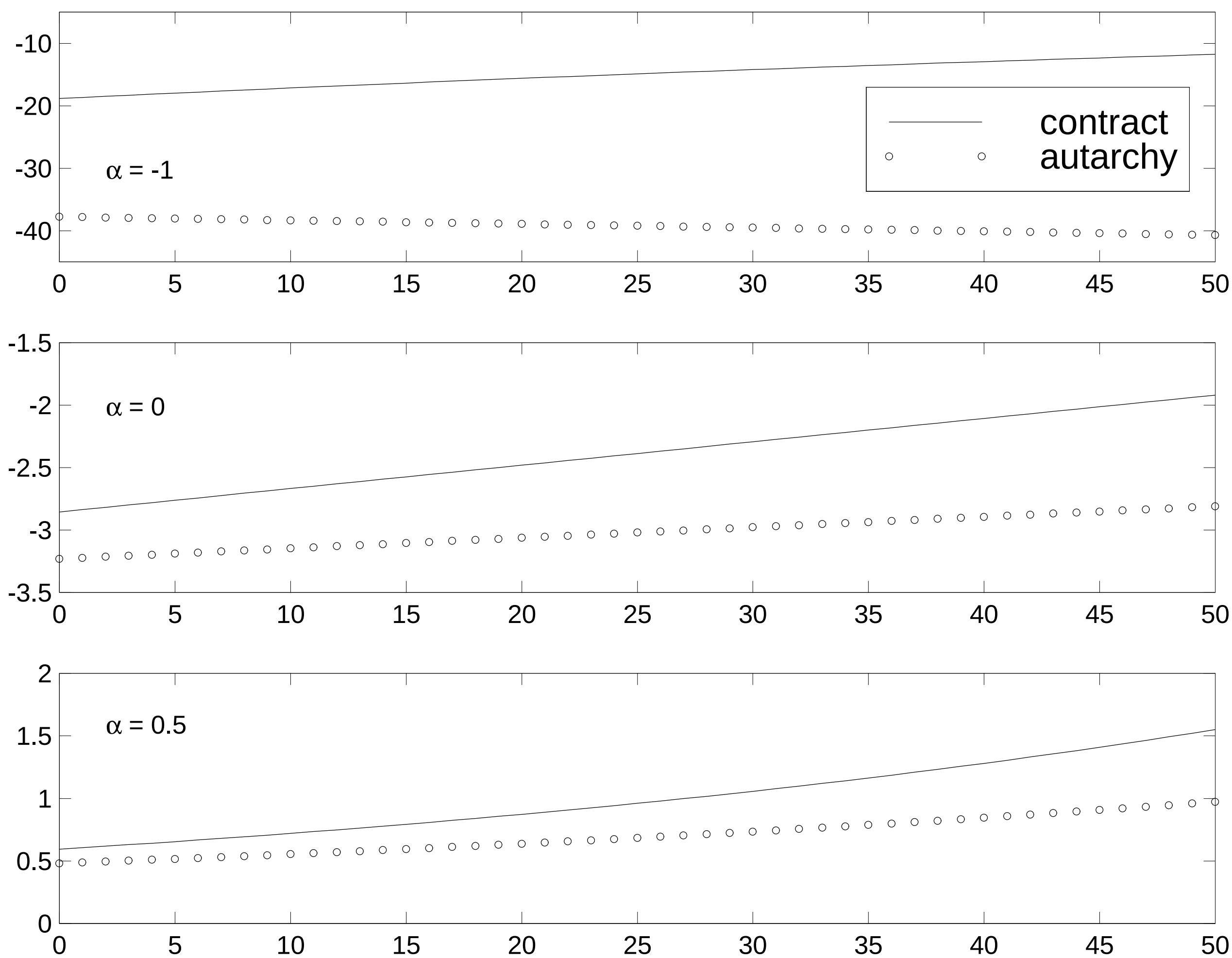

Figure 2: Standard Deviation of Expected Lifetime Utility