# Banking and Markets in a Monetary Model[*]

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October 17, 2004

## Abstract

We model a dynamic economy in which two types of aggregate shocks are present. A liquidity shock affecting the demand for real balances, and a real shock that affecting the outcome of investment. Banks arise to insure depositors against liquidity risk, which is by nature intertemporal. Markets provide insurance against the risk induced by real shocks, which, unlike liquidity shocks, determine the total amount of resources available in the economy at a given time. We study how the demand for liquididy of banks interacts with asset prices and interest rates in different institutional settings. We show that the economy in which banks, markets, and a central bank are present reaches the same equilibrium of an economy with complete asset markets and a safe asset, and that this is the only case leading to a Pareto optimal equilibrium.

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[*] For helpful conversations, we would like to thank, without implicating for any shortcoming, Costas Azariadis, Abhijit Banerjee, Martín González Eiras, Paula Hernandez, Jim Peck, Rob Reed, Paolo Siconolfi, Steve Spear, Anne Villamil, Ping Wang, as well as participants at several universities and conferences where this paper was presented. We gratefully acknowledge financial support from the Weidenbaum Center on the Economy, Government, and Public Policy at Washington University.

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# 1 Introduction

Financial systems differ substantially across countries. The difference is often summarized by the relative weight of banks and markets in overall financial activity. For example, the U.S. financial system is considered a market oriented one, while the German financial system is characterized by a strong relevance of (universal) banking. Interestingly, banks and markets are present, albeit to a different extent, in all industrialized economies, and both contribute to economic activity. Moreover, the different mix of markets and other intermediaries has not prevented countries from reaching high levels of industrial development: The United Sates, France, and Germany, for example, are all very rich countries. It is reasonable to argue that banks and markets play different yet in part complementary roles in the economy, even though both share the common general goal of transferring resources from lenders to borrowers. In this paper, we build a model to provide a characterization of these roles.

We postulate that banks specialize in insuring the economy against liquidity risk. When lenders face risk over the relative distribution of income and consumption over time, they use intermediation for two reasons: smooth consumption given its intertemporal price, and pool risk. Part of aggregate savings constitute a reserve of liquid assets, real balances, available to lenders on demand, but lenders as a group can economize over the total amount of investment in (non-productive) real balances. Banks perform these roles for depositors. Markets, on the other hand, allow investors to share risk over the risk generated by investment projects. In general, investment projects bear risk and their realization, at a certain future date, can be more or less successful. In as much as projects' realizations exhibit some negative correlation, there is a social benefit in trading risk with securities. Note that, even though banks and asset markets provide specialized services by allowing trades of different classes of risks, their roles can interact to jointly determine interest rates and asset prices in the economy.

We study an overlapping generations model in which at each date the population is partitioned into two groups of two-period-lived agents: lenders and borrowers (or entrepreneurs). Agents are born at either of two identical locations (islands). At the end of each period, a fraction of lenders born in one island is relocated to the other island. Spatial separation and limited communication prevent trade across islands, and relocated agents must carry (fiat) currency[1]. This friction generates a stochastic demand for real balances. Banks in this setting arise to insure depositors against liquidity shocks, and face stochastic withdrawals at the end of each period. For this reason, they hold precautionary reserves of real balances, which can be dominated in rate of return. Banks also make loans to borrowers. Borrowers are of two types in each island[2]. Each type is endowed with a stochastic investment project, and a random fraction of borrowers in each type results in an unsuccessful project in each period. We assume that the investment projects are negatively correlated across types. In addition, we assume that a bank cannot lend simultaneously to both types of borrowers, and therefore are not allowed to perfectly diversify. Following Allen and Gale (2004) we assume the existence of a (complete) set of Arrow securities on the set of states of nature generated by real shocks, in which banks are allowed to trade[3]. Banks face real risk because they cannot perfectly diversify across borrowers. Hence, they have an incentive to trade risk with other banks in the Arrow security market.

We study this environment in three different settings. Initially, we do not allow the existence of the Arrow securities market, and show that real shocks and liquidity shocks interact by affecting banks' precautionary demand for real balances. We then introduce Arrow securities and let banks trade assets to share risk. We study how real and liquidity shocks interact in this setting. We show how banks' demand for precautionary reserves as well as interest rates are affected, and how in turn the presence of liquidity shocks affects asset prices. We study specifically how asset prices are affected in equilibrium. Finally, we introduce a central bank that makes one-period liquidity loans to banks and show that in this case the equilibrium outcome is the same that obtains in an economy with complete markets and a safe asset. Only in this case the equilibrium is Pareto optimal. Therefore, in our setting banks, markets, and the central bank, all contribute distinctively to the determination of equilibrium.

In the section following the literature review we introduce the general framework of our model. In the following three sections we study the model under the three specifications of financial structure discussed above. The progressive inclusion of different financial institution is motivated in part by

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[1] This part of the model is as in Townsend (1987).

[2] Another interpretations is to think of these regions as two sectors of economic activity.

[3] We also follow Allen and Gale (2004) in restricting market participation: individual depositors are not allowed to trade in contingent claims.

pedagogical considerations, as it allows to clearly present the various parts of the model. In the conclusions, we discuss some limitations of our setting and indicate directions of future research and potential applications.

## 1.1 A brief discussion of the literature

The literature on banks and markets, and in general financial systems, is very extensive. Here we only discuss some contributions that are particularly relevant to our work. Several empirical studies have analyzed the presence in the economy of banks and market-based intermediation. Levine (1997) surveys a large sample of the literature on finance and growth: the evidence that emerges from various studies is that financial systems participate in the growth process, and that both banks and market-based intermediation individually contribute to the growth of output. La Porta et al. (1998) take an alternative approach and investigate how the protection that different legal systems, common-law and French-civil-law systems, shapes the evolution of the financial system. Through empirical analysis they show that the legal protection of small investors, marked in countries with a common-law based legal infrastructure, is an important determinant of the development of strong asset markets. On the other hand, civil-law based systems show a strong domination of banks in their financial sector. La Porta et al. (1998) do not fail to note, however, that some countries where asset markets did not develop nonetheless experienced robust and persistent economic development, and therefore their analysis should not be intended to determine whether one system is superior to the other.

Rajan and Zingales (2002) expand and deepen the scope of the former analysis from a theoretical point of view, and introduce other elements besides the evolution of the legal system. They study what characteristics of the economic, legal, and technological environment explain the relative advantages of banks and markets. Weak legal protection of minority shareholders, small markets and firms, incremental technological progress, little transparency, tend to favor banks, that is relationship-based financial intermediation. Vice versa, strong protection of minority shareholders, large markets and firms, wave-like technological progress, and transparency, tend to favor market-based financial systems. Note that Rajan and Zingales, like La Porta et al. (1998), do not intend to pick a winner, but describe the characteristics of the environment that make one type of financial system better fit to intermediate funds from savers to investors.

Other authors concentrate on the properties of the institutions themselves rather than the environment in which they operate. For example, Allen and Gale (2000) study, among other issues, how markets and banks collect information in different ways and hence provide different services. Note that also in the approach that we take in our analysis banks and markets provide different services: banks specialize in insuring the economy against liquidity shocks, and markets in insuring the economy against cross-sectional risk. Hence, banks and markets perform different, albeit complementary, roles in the transfer of resources from savers to investors.

A common theme in the study of financial systems is the emphasis on liquidity provision. Allen and Gale (1997), for instance, have in common with our analysis the separation between liquidity shocks and "real shocks", and the overlapping generation structure of the economy. They study the different reaction of the economy in response to real shocks, and show that the volatility of consumption is different is bank-based and market-based financial systems. In their model banks produce a Pareto-optimal allocation while markets do not. A similar conclusion, albeit for different reasons, is reached by Holmstrom and Tirole (1997) who study the optimal provision of liquidity in the economy. They study a moral hazard model in which banks provide liquidity, and are superior to markets in this role. In their framework, unlike the others mentioned here, firms suffer liquidity shocks, and in the presence of aggregate risk a commitment problem needs the coordinating role of banks for the efficient allocation of liquidity (which markets are not able to provide). In one case, when there is only aggregate risk, Holmstrom and Tirole (1997) show that government debt is needed as a tool for optimal liquidity provision. The reader will recognize the similarity with the role that the central bank plays in our model.

Allen and Gale (2004) present a framework conceptually closely related to ours. They study a model with liquidity and real shocks, and relate the existence of different intermediary contracts to their ability to complete markets. This idea is close to the one in the present paper. One way to reformulate our results is to emphasize that we start from an economy in which markets are incomplete, both relative to liquidity risk and real shocks. In this formulation banks and the central bank complete the market with respect to liquidity shocks and asset markets complete markets with respect to "real" risk.

Finally, Diamond and Rajan (2001) and Diamond (2004) make a distinction about the different roles of markets and banks which is different from the one present in our analysis. They point to different type of commitments that are involved in market contracts and bank based contracts (deposits), and the resulting incentives for banks, firms, and depositors.

Unlike the other papers cited above, we study a monetary model where liquidity shocks are denominated in fiat currency. This means that the preference shock that hits certain consumers does not take the form of an urgency to consume, but the urgency to hold a liquid asset, fiat money. We want to study the role of banks and markets at the aggregate level, and this formulation seems more appropriate for this purpose. From a historical perspective, the evolution of banks has led to financial tensions arising from the different structures of the asset side and the liability side of banks balance sheets. Central banks have been introduced (in part) to bear the task of moderating the risk for banks originating from the liability side of their balance sheets. Markets, on the other hand, help banks moderating the risks originating from the asset side of their balance sheet. These are the features of the financial structure that we want to capture.

## 2 The model

We study an economy populated by a sequence of two-period lived, overlapping generations, and an initial old generation. There is a unique consumption good in the economy. There are two separate but identical islands, $A$ and $B$, and two regions in each island, called region 1 and region 2.[4] At every period $t = 0, 1, 2, ...$, a new generation in each island and region is born. Each generation consists of two groups, each of a continuum of agents of unit mass. One group consists of risk neutral *entrepreneurs*, who invest when young and value consumption only when old. The second group consists of a continuum of risk averse lenders who also value consumption only when old. At time $t = 0$ there is an initial old generation of lenders each endowed with $M$ units of fiat money. We assume that goods cannot be transported between islands, and limited communication also prevents the transfer of assets. Only money, which is universally recognizable, can be transferred between islands[5].

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[4] An alternative interpretation is to think about these as sectors.

[5] For a more extended description of this environment see Antinolfi, Huybens and Keister (2001), and Champ, Smith, and Williamson (1996).

## 2.1 Entrepreneurs

Entrepreneurs are risk neutral and value consumption only when old. Each has access to a (risky) technology, and has no endowment. An investment of $k$ units of the consumption good at time $t$ yields $g(k)$ units at $t+1$ with positive probability, or else 0. We assume that $g(k)$ is strictly increasing, strictly concave, $C^2$ and satisfies the standard Inada conditions. In addition, we assume that the real shocks that affects investment projects are perfectly inversely correlated between the two regions: either entrepreneurs in region 1 get $g(k)$ and entrepreneurs in region 2 get 0, or the converse is true. We indicate with $s_1$ and $s_2$ the states of nature in which the real shock is favorable to entrepreneurs in region 1 and 2 respectively. We let $\eta(s_j)$ be the probability that the investment is successful only in region $j = 1, 2$. Let $S \equiv \{s_1, s_2\}$.

Because entrepreneurs have no endowment, they need to borrow to invest. In case of successful projects, entrepreneurs in region $j$ pay back the amount they borrowed one period earlier with interest. We denote $R_t^j$ the gross interest rate that the entrepreneur in region $j = 1, 2$ pays in the favorable state.

An entrepreneur in region $j$ solves the following expected income maximization problem:

$$\max_{k_t^j} \quad \eta(s_j)\left[g\left(k_t^j\right) - R_t^j k_t^j\right].$$

The first-order condition for this problem is

$$g'\left(k_t^j\right) = R_t^j.$$

We can express the demand for funds by an entrepreneur of region $j$ as

$$k_t^{j*} = \psi\left(R_t^j\right) \equiv (g')^{-1}\left(R_t^j\right).$$

## 2.2 Lenders

All lenders receive an endowment vector $(\omega_1, \omega_2) = (x, 0)$, with $x > 0$. At the end of each period a fraction $\pi_t$ of young agents in each island is relocated to a different island. The fraction $\pi_t$ represents the size of the aggregate liquidity shock in each island, and determines the existence of banks in a similar fashion as in Diamond and Dybvig (1983). In addition, because money is the only asset that can be transported between islands, the random liquidity shock determines a transaction role for money as in Townsend (1987). We assume that $\pi_t$ is drawn from the distribution function $F(\pi_t)$, which is twice continuously differentiable and has density $f(\pi_t)$.

Lenders have preferences given by $U(c_1, c_2) = \ln c_2$, where $c_2$ represents consumption of an agent when old. Because lenders face the possibility of a liquidity shock, they deposit their endowment in a local bank. Banks promise a rate of return to depositors contingent on three factors: the state of nature prevailing in the region where the bank is located; the depositor's relocation status; and finally the fraction of total population relocated.

### 2.3 Banks

Banks take deposits, decide their portfolio of loans and reserves, announce rates of return on deposits, and trade in asset markets. We assume perfect competition in the banking sector: banks act as Nash competitors and maximize the expected utility of depositors. Both liquidity and real shocks are realized at the same instant.

We assume that banks in each region can lend only to entrepreneurs of the *same* region. This assumption does not allow banks to perfectly diversify credit risk. We maintain the assumption for simplicity, but it is possible to make this feature of the model endogenous, for example by introducing a cost function for the intermediation process that would limit the number of entrepreneurs whom a bank finds profitable to lend to[6].

We study the problem of a bank in three economies. In the first scenario, the bank faces only the problem of determining its demand for monetary reserves. In the second scenario we open a market for contingent claims where banks in different regions (but not different islands) are allowed to trade in the contingent claim market. Finally, we also add a central bank that provides an elastic currency to the economy through discount window loans.

## 3 The case of no asset market or central bank

In this section, banks take deposits, choose their portfolio of loans and reserves, and announce rates of return on deposits. Their problem is to choose

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[6] For analysis in which the bank size is determined endogenously see Krasa and Villamil (1992, 1994).

the fraction of deposits to invest in real balances to maximize the expected utility of depositors. Let $\gamma_t^j$ be the fraction of deposits invested in real balances, and let $\alpha_t^j(s,\pi)$ be the fraction of cash balances that a bank of region $j$ uses to pay relocated depositors. We denote $r_t^m(s,\pi)$ to be the real return on deposits to a mover when the aggregate state is $(s,\pi)$. Likewise, we indicate with $r_t(s,\pi)$ the real rate of return on deposits promised to a lender who does not leave the island. The first constraint that the bank faces is that it can only use real balances to satisfy the demand for withdrawals of the $\pi$ relocated depositors. Formally,

$$\pi r_t^{mj}(s,\pi) \leqq \gamma_t^j \alpha_t^j(s,\pi)\frac{p_t}{p_{t+1}}. \tag{1}$$

Banks use the remaining resources, possible remaining real balances and return on loans, to repay deposits and provide the promised return to the $1-\pi$ depositors who are not relocated to the other island. The constraint is given by

$$(1-\pi)\, r_t^j(s,\pi) \leqq \gamma_t^j\left(1-\alpha_t^j(s,\pi)\right)\frac{p_t}{p_{t+1}} + (1-\gamma_t)\, R_t^j(s), \tag{2}$$

where $R_t^j(s) = R_t^j$ if $s = s_j$, and 0 otherwise, with $j = 1, 2$. The problem of the bank is to choose $r_t^{mj}(s,\pi)$ and $r_t^j(s,\pi)$ to maximize the utility of depositors, taking the amount deposited, $x$, as given. The problem is:

$$\max_{\gamma^j,\alpha^j,r^{m,j},r^j} \sum_{s\in S}\eta(s)\int_0^1 \left[\pi \ln r_t^{mj}(s,\pi)\, x + (1-\pi)\ln r_t^j(s,\pi)\, x\right] f(\pi)\, d\pi$$

subject to (1) and (2) in addition to the non-negativity constraints $0 \leq \alpha \leq 1$ and $0 \leq \gamma \leq 1$. Substituting the constraints (1) and (2), which will hold with equality in equilibrium, and deleting irrelevant constants, the problem can be equivalently written as

$$\begin{aligned}\max_{\gamma^j,\alpha^j(s,\pi)} \sum_{s\in S}\eta(s)\int_0^1 &\left[\pi \ln \gamma_t^j\alpha_t^j(s,\pi) + \right.\\ &\left.(1-\pi)\ln\left(\gamma_t^j\left(1-\alpha_t^j(s,\pi)\right)\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s)\right)\right] f(\pi)\, d\pi\end{aligned}$$

subject to $0 \leq \alpha^j \leq 1$ and $0 \leq \gamma^j \leq 1$, and where $\eta(s_1) = \eta$ and $\eta(s_2) = 1-\eta$.

Note that $\alpha$, the fraction of real balances used to repay relocated depositors, is chosen after the shocks are observed. Therefore, the optimal value of $\alpha$ is contingent of the choice of $\gamma$, the total amount of real balances available. On the other hand, the optimal amount of real balances is chosen before the observation of the realization of the shocks, and cannot be contingent on their value. The solution to the problem of the bank is given by

$$\alpha_t^j(s,\pi) = \begin{cases} \frac{\pi\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]}{\gamma_t^j \frac{p_t}{p_{t+1}}}; & \pi < \pi_j^*\left(\gamma_t^j, s\right) \\ 1; \quad \pi_j^*\left(\gamma_t^j, s\right) \leq \pi < 1 \end{cases}$$

where

$$\pi_j^*\left(\gamma_t^j, s\right) \equiv \frac{\gamma_t^j \frac{p_t}{p_{t+1}}}{\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)}$$

In practice, $\pi_j^*$ indicates the critical value of the liquidity shock such that a bank in region $j$ exhausts the whole amount of real balances held as reserves. For shocks higher than $\pi_j^*$ banks in region $j$ face a liquidity shortage, and movers and non-movers receive different returns on their deposits. Notice that when $R_t^j(s) = 0$, then $\pi_j^*\left(\gamma_t^j, s\right) = 1$. The reason for this property of the solution is that real balances are the only source of funds for banks when the real shock is unfavorable to borrowers in that region, and therefore the bank will never give all its currency reserves to movers. The optimal choice of real balances is given by

$$\gamma_t^j = 1 - \eta(s_j) \int_{\pi^*\left(\gamma_t^j, s_j\right)}^{1} F(\pi)\, d\pi.$$

Because of the aggregate nature of the liquidity shock, the bank provides partial liquidity insurance to its depositors. The rate of return on money is lower than the expected return on loans, and at the margin banks balance the insurance benefit of holding currency reserves and their opportunity cost due to the higher expected returns on loans to entrepreneurs. Notice that cash reserves are used by the bank to provide insurance to *both* movers and non-movers, who do not suffer the liquidity preference shock. In other words, the presence of credit risk increases the demand for real balances. The presence of credit risk makes money an attractive asset because money has value in every future state of the world. Lack of credit risk would mean that $\eta(s_j) = 1$, and

$\gamma_t^j = 1 - \int_{\pi^*\left(\gamma_t^j, s_j\right)}^1 F(\pi)\, d\pi$, which is the same result obtained in Antinolfi, Huybens and Keister (2001) and Champ, Smith and Williamson (1996) when the interest rate on bank loans is deterministic.

## 3.1 Equilibrium

In equilibrium, the money and credit markets have to clear in each island. The real money supply in each period is $\frac{M}{p_t}$, therefore the market clearing condition on the money market is given by

$$\frac{M}{p_t} = \left(\gamma_t^1 + \gamma_t^2\right) x, \tag{3}$$

where superscripts indicate the demand for real balances by banks in region 1 and 2 respectively. Equation (3) implies that

$$\frac{p_t}{p_{t+1}} = \frac{\left(\gamma_{t+1}^1 + \gamma_{t+1}^2\right)}{\left(\gamma_t^1 + \gamma_t^2\right)}.$$

Credit markets in region 1 and 2 also must clear, hence demand and supply of credit must be equal, that is in region $j = 1, 2$ we must have

$$\psi\left(R_t^j\right) = \left(1 - \gamma_t^j\right) x \tag{4}$$

Under the assumption that $f(k) = k^\alpha$, $0 < \alpha < 1$ for all $j$, we have $\psi\left(R_t^j\right) = \left(\frac{\alpha}{R_t^j}\right)^{\frac{1}{1-\alpha}}$. Hence in equilibrium:

$$\left(\frac{\alpha}{R_t^j}\right)^{\frac{1}{1-\alpha}} = \left(1 - \gamma_t^j\right) x.$$

Letting $\phi \equiv \frac{\alpha}{x^{1-\alpha}}$, it follows that $\left(1 - \gamma_t^j\right) R_t = \phi\left(1 - \gamma_t^j\right)^\alpha$. Therefore in equilibrium

$$\gamma_t^j = 1 - \eta(s_j) \int_{\frac{\frac{\gamma_t^j}{\gamma_t^1+\gamma_t^2}\left(\gamma_{t+1}^1+\gamma_{t+1}^2\right)}{\frac{\gamma_t^j}{\gamma_t^1+\gamma_t^2}\left(\gamma_{t+1}^1+\gamma_{t+1}^2\right)+\phi\left(1-\gamma_t^j\right)^\alpha}}^1 F(\pi)\, d\pi. \tag{5}$$

When both regions are considered, the resulting two-dimensional, first-order system of difference equations defines the equilibrium dynamics for the economy.

### 3.1.1 Steady State analysis

We focus our analysis on steady-state equilibria. In steady state equation (5) becomes

$$\gamma^j = 1 - \eta(s_j) \int_{\frac{\gamma^j}{\gamma^j + \Phi(1-\gamma^j)^\alpha}}^{1} F(\pi)\, d\pi$$

for $j = 1, 2$. Notice that in the steady state each equation becomes independent from the other. We state the following

**Proposition 1** *There exists a unique* $(\gamma^{1*}, \gamma^{2*}) \in (0,1)^2$ *which satisfies both equilibrium equations, hence the steady-state equilibrium is unique.*

**Proof.** See appendix. ■

Furthermore it is not difficult to see that this equilibrium is never Pareto optimal[7]. Intuitively, optimal risk-sharing dictates that a bank equalize the rate of return for both movers and non-movers. In fact, the economy does not face a random amount of resources relative to the liquidity shock. However, banks cannot adjust the amount of currency holdings after observing the liquidity shock. A bank must choose monetary reserves before observing the liquidity shock, even though ex-post, in the state of nature in which borrowers repay their loans with interest, the bank would be able to borrow money, for example, from a central bank. This is the role that the central bank will play. Before introducing the central bank, however, we allow banks to trade contingent claims to trade credit risk on a given island.

# 4 The model with asset markets but no central bank

In this section we consider the problem of the bank and the equilibrium of the economy when banks can trade credit risk on asset markets. We model asset markets by opening a market for Arrow securities in each island in which banks can trade immediately after young agents make their deposits. Arrow securities of course are not a perfect representation of asset markets. For example, Arrow securities markets are self-financing. However, we believe

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[7] A classical reference is Balasko and Shell (1980); see Champ, Smith and Williamson (1996) for discussion in a similar setting.

that they are good representation in that they allow the trade of goods across states of nature. Allen and Gale (2004) follow the same approach in modeling asset markets.

Let $\theta^j(s)$ denote the Arrow security traded by a bank in region $j$ at the beginning of time $t$, which pay at time $t+1$ one unit of the consumption good when the state is $s$ (with $s \in S$). Arrow securities are traded before the observation of the shocks, and the determination of $\gamma's$ and $\theta's$ is simultaneous. As in the previous sections, the fraction of real balance reserves used to repay movers is determined after the observation of the shocks. The constraints that the bank faces in this case are:

$$\pi r^m(s,\pi) = \gamma_t^j \alpha_t^j(s,\pi) \frac{p_t}{p_{t+1}} \tag{6}$$

and

$$(1-\pi) r(s,\pi) = \gamma_t^j (1-\alpha_t(s,\pi)) \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s) + \theta^j(s) \tag{7}$$

The difference from the previous section is the presence of the term representing the Arrow security obligation of the bank in state $s$. Intuitively, the bank now has an additional tool that can be used to transfer consumption across states of nature for depositors who are *not* relocated. In this sense the value of monetary reserves is affected because the bank can cover some of the credit risk it is facing through asset markets. The problem of the bank in this case is

$$\max_{\gamma^j, \alpha^j, r^{m,j}, r^j} \ln \sum_{s \in S} \eta(s) \int_0^1 \left[\pi \ln r_t^{mj}(s,\pi) d + (1-\pi) \ln r_t^j(s,\pi) d\right] f(\pi) d\pi$$

subject to (6) and (7), the usual non-negativity constraints, and

$$q_1 \theta^j(s_1) + q_2 \theta^j(s_2) = 0$$

where $q_1$ and $q_2$ are the prices of the Arrow securities that pay in state $s_1$ and $s_2$ respectively. Normalizing,

$$\theta^j(s_1) + q\theta^j(s_2) = 0, \tag{8}$$

where $q \equiv \frac{q_1}{q_2}$ is the relative price of Arrow securities. Equation (8) is the self-financing constraint typical of Arrow securities trading. Again, we solve the problem of the bank by first determining the optimal liquidation of real balance reserves. The solution to this problem sets

$$\alpha^j(s,\pi) = \begin{cases} \dfrac{\pi\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s) + \theta^j(s)\right]}{\gamma_t^j \frac{p_t}{p_{t+1}}}; & \pi < \pi_j^*\left(\gamma_t^j, s\right) \\ \\ 1; \quad \pi_j^*\left(\gamma_t^j, s\right) \leq \pi < 1 \end{cases} \tag{9}$$

where the critical values of the liquidity shocks, depending on the state $s$, are given by

$$\pi_j^*\left(\gamma_t, s\right) = \frac{\gamma_t^j \frac{p_t}{p_{t+1}}}{\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R(s) + \theta^j(s)\right]}. \tag{10}$$

Having determined the liquidation policy of the bank once the state of the economy is realized, we need to determine the (ex-ante) choices of $\gamma_t^j$, $\theta^j(s_1)$, and $\theta^j(s_2)$.

Let us solve the case of bank $j = 1$ (The case of bank 2 is symmetric). Recall that $\eta(s_1) = \eta$ and $\eta(s_2) = 1 - \eta$. Using the optimal values for $\alpha's$ we can formulate this problem as

$$\max_{\gamma,\theta_1,\theta_2} \eta \int_0^{\pi^*(s_1)} \log\left[\gamma_t^1 \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^1\right) R(s_1) + \theta^1(s_1)\right] f(\pi)\, d\pi +$$

$$+\eta \int_{\pi^*(s_1)}^{1} \left\{\pi \log \gamma_t^1 + (1-\pi) \log\left[\left(1-\gamma_t^1\right) R(s_1) + \theta^1(s_1)\right]\right\} f(\pi)\, d\pi +$$

$$(1-\eta) \int_0^{\pi^*(s_2)} \log\left[\gamma_t^1 \frac{p_t}{p_{t+1}} + \theta^1(s_2)\right] f(\pi)\, d\pi$$

$$+(1-\eta) \int_{\pi^*(s_2)}^{1} \left[\pi \log \gamma_t^1 + (1-\pi) \log \theta^1(s_2)\right] f(\pi)\, d\pi$$

subject to:

$$q_1\theta_1^1 + q_2\theta_2^1 = 0.$$

where $\theta_i^1 \equiv \theta^1(s_i)$, $i = 1,\ 2$. The solution to the problem of the bank in region 1 gives

$$\gamma_t^1 = 1 - \eta \int_{\pi_1^*(s_1)}^{1} F(\pi)\, d\pi - (1-\eta) \int_{\pi_1^*(s_2)}^{1} F(\pi)\, d\pi,$$

and

$$\theta_t^1(s_2) = \frac{R_t^1(s_1)}{q_t} (1-\eta) \int_{\pi_1^*(s_2)}^{1} F(\pi)\, d\pi.$$

The symmetric solution to the problem of the bank in region 2 gives

$$\gamma_t^2 = 1 - \eta \int_{\pi_2^*(s_1)}^{1} F(\pi)\, d\pi - (1-\eta) \int_{\pi_2^*(s_2)}^{1} F(\pi)\, d\pi$$

and

$$\theta_t^2(s_1) = q_t R^2(s_2) \left( \eta \int_{\pi^*_{(s_1)}}^{1} F(\pi)\, d\pi \right)$$

Two observations are important about the solution to the bank's problem. First, the presence of asset markets affects the demand for real balance reserves of the bank. Asset markets give the bank a new tool for transferring real risk (that is, the credit risk generated by real shocks on investment). The bank still insures relocated depositors against liquidity risk, for which it needs currency, but now it has another tool that provides resources to repay non-relocated depositors when borrowers do not pay back their loans. In general, asset markets will in part substitute for real balances. However, the presence of the liquidity shock will affect the demands (and prices) for Arrow securities. Specifically, the amount of Arrow security that pays off in state $s_2$ for bank in region 1 depends on the ratio of the interest rate on loans and the price of the Arrow security, which is an indication of the relative cost of obtaining consumption in state of nature $s_2$, multiplied by the likelihood of the event that $s_2$ will occur *and* the bank will suffer a shortage of liquidity. The form that the dependence of the demand for assets on interest rates and prices takes is intuitively clear: interest rates represent the opportunity cost of holding real balances, but real balances play a dual role as they are the only alternative asset available to insure depositors who are *not* relocated. The second observation concerns the efficiency of the equilibrium. As it will be shown below, the equilibrium is not Pareto optimal for the case analyzed in this section.

## 4.1 Steady state equilibrium

In equilibrium, the money market, loans markets, and Arrow securities market must clear. The market clearing conditions are the same as in the previous section, with the addition of the Arrow securities market. In order, money markets clearing requires

$$\left(\gamma_t^1+\gamma_t^2\right)x=\frac{M}{p_t}$$

which implies that

$$\frac{\left(\gamma_{t+1}^1+\gamma_{t+1}^2\right)}{\left(\gamma_t^1+\gamma_t^2\right)}=\frac{p_t}{p_{t+1}}.$$

Loan markets clearing conditions imply that

$$\left(1-\gamma_t^1\right)x=\left(\frac{\alpha}{R_1}\right)^{\frac{1}{1-\alpha}}$$

$$\left(1-\gamma_t^2\right)x=\left(\frac{\alpha}{R_2}\right)^{\frac{1}{1-\alpha}}$$

Arrow securities markets clear when

$$q_1\theta_1^1+q_2\theta_2^1=0$$

$$q_1\theta_1^2+q_2\theta_2^2=0$$

and

$$\theta_1^1=-\theta_1^2$$
$$\theta_2^1=-\theta_2^2$$

It is easy to show that equilibria always exist. In particular, there is always an equilibrium where $\pi^*_{1,1} = \pi^*_{2,2} = 1$ and $\pi^*_{ij} \in (0,1)$ , $i \neq j$[8]. This means that a bank in region $j$ never exhausts the amount of real balances held as reserves when the real shock in region $j$ is favorable. Therefore, the bank optimally sets the consumption of movers and non-movers to be the same as long as the realization of the real shock is favorable. Hence, in this case there is complete risk risk sharing with respect to the liquidity shock region-wise. This was never possible when there were no Arrow securities. Note, however, that in the region where the real shock is not favorable depositors do not get full insurance. Hence risk sharing is not complete island-wise.

One remaining question is whether there are other equilibria in the economy analyzed in this section. We do not have a proof of global uniqueness of the steady state equilibrium, even though in all the examples we produced only one steady-state equilibrium exists. It is important to notice, however, that equilibria are never Pareto optimal in this case as well. Depositors who are relocated and depositors who are not still face risk about their consumption when old, even though ex-ante a bank no longer faces no risk about the availability of resources in the economy in different states of nature. When there is an excess demand for liquidity movers pay a cost in terms of lower return on their deposits.

### 4.1.1 Cash-position of banks with Arrow securities.

An important question raised by the analysis in this section is whether the steady state equilibrium demands for real balances in an economy with Arrow securities are greater or smaller than their corresponding quantities in the stationary equilibrium of the economy without Arrow securities.

The particular equilibrium obtained in the case with Arrow securities, in which $\pi^*_{j,j} = 1$ for $j = 1, 2$, can be characterized by the following system of

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[8] There is a technical issue that must be clarified. The objective function for bank 1 includes an expression such that, after replacing by the equilibrium expressions, and when evaluated at $\pi^*_{1,1} = 1$, it would implies an indeterminancy of the form $0 \cdot \infty$. However it can be shown that this indeterminacy can be solved by using L'Hopital rule (see appendix for details). Hence this equilibrium exists as long as we define the equilibrium value of the objective function equal to its limit.

equations:

$$\gamma^1 = 1 - (1-\eta) \int_{\frac{\gamma^1}{\gamma^1+\Phi(1-\gamma^2)^\alpha}}^{1} F(\pi)\, d\pi$$

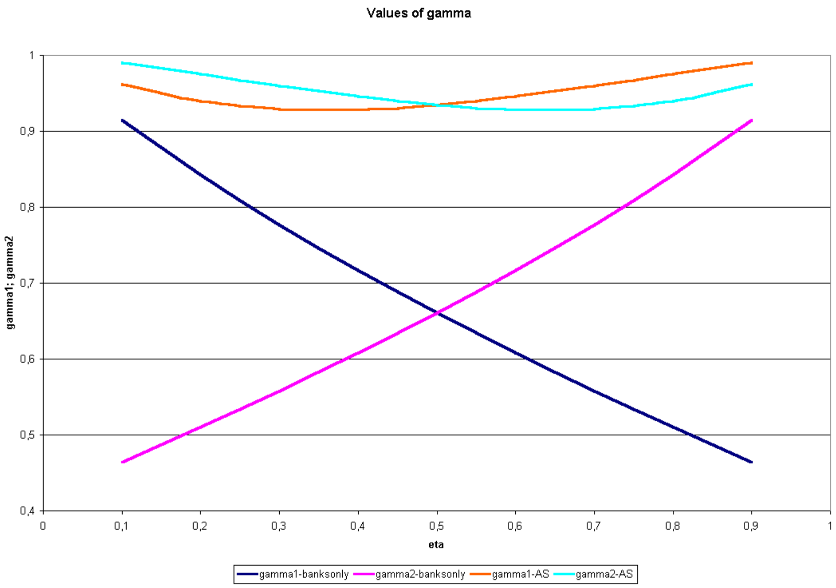
$$\gamma^2 = 1 - \eta \int_{\frac{\gamma^2}{\gamma^2+\Phi(1-\gamma^1)^\alpha}}^{1} F(\pi)\, d\pi$$

Denote $(\gamma_A^1, \gamma_A^2)$ the solution to this system. An important question raised by the analysis in this section is whether the values of $(\gamma_A^1, \gamma_A^2)$ in this stationary equilibrium (in an economy with Arrow securities) are greater or less than their corresponding values in the stationary equilibrium in the economy without Arrow securities. Recall that this equilibrium is characterized by

$$\begin{aligned} \gamma^1 &= 1 - \eta \int_{\frac{\gamma^1}{\gamma^1+\Phi(1-\gamma^1)^\alpha}}^{1} F(\pi)\, d\pi \\ \gamma^2 &= 1 - (1-\eta) \int_{\frac{\gamma^2}{\gamma^2+\Phi(1-\gamma^2)^\alpha}}^{1} F(\pi)\, d\pi \end{aligned}$$

Denote $(\gamma_B^1, \gamma_B^2)$ the solution to this system. The question is whether we can say that $\gamma_A^j$ is greater, equal or less than $\gamma_B^j$. A general characterization of this kind is somewhat cumbersome. However some results are not difficult to obtain. First, it is straightforward to show that $\frac{\partial \gamma_B^1}{\partial \eta} < 0$ and $\frac{\partial \gamma_B^2}{\partial \eta} > 0$. (This is obtained by applying the Implicit Function Theorem to the latter system). It is also not difficult to prove that $\lim_{\eta \to 0} \gamma_A^j = \lim_{\eta \to 0} \gamma_A^j = 1$ for $j = 1, 2$. Then, for $\eta$ close to 1 it is obvious that $\gamma_A^1 > \gamma_B^1$. Similarly, for $\eta$ close to 0 then $\gamma_A^2 > \gamma_B^2$. However, we were not able (so far) to show these two inequalities for more general values of $\eta$.

When we specialize the distribution of $\pi$ to the uniform case, we were able to draw a chart with the values of $(\gamma_A^1, \gamma_A^2)$ and $(\gamma_B^1, \gamma_B^2)$ as a function of $\eta$. We assumed that $\alpha = 0.65$ and that the value of $x$ is such that $\Phi = 0.9$.



This chart reveals that, at least for the uniform distribution case, and for the numerical values given above, it is true that $\gamma_A^j > \gamma_B^j$ for every $\eta \in [0,1]$. This is so even though in the first case banks have access to two more security markets (the Arrow securities) than in the second case. A suggested intuition is that, given that the return on deposits when Arrow securities are available are greater than the return with no Arrow securities, and given that each bank faces a solvency risk, they are forced to keep more cash than when those securities are not present in order to satisfy such deposit returns.

# 5 The economy with asset markets and a Central Bank

In this section we complete the financial structure of our simple economy and add a central bank. The central bank operates a discount window to provide short-term (i.e. one-period) loans of currency to banks facing an excessively high amount of withdrawals at the end of period $t$. Note that these loans are made after shocks are realized. Therefore they constitute pure-liquidity loans, the bank will be solvent in period $t+1$, when borrowers repay their loans and Arrow securities trades clear. The difference with the previous case is that a bank knows ex-ante that it will be able to take contingent loans from the central bank.

We assume that the central bank charges a zero net nominal (and real in steady state) interest rate on discount window loans. We let $\delta_t^j(s,\pi)$ denote the amount of real balances that a bank in region $j$ borrows from the Central Bank at date $t$. This amount of currency is used to pay movers in period $t$ and will be repaid in period $t+1$ to the central bank. The constraints of the bank are given by the following equations:

$$\begin{aligned} \pi r_t^{mj}(s,\pi) &= \gamma_t^j \alpha_t^j(s,\pi)\frac{p_t}{p_{t+1}} + \delta_t^j(s,\pi)\frac{p_t}{p_{t+1}} \\ (1-\pi) r_t^j(s,\pi) &= \gamma_t^j\left[1-\alpha_t^j(s,\pi)\right]\frac{p_t}{p_{t+1}} - \delta_t^j(s,\pi)\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s) + \theta_t^j(s) \end{aligned}$$

In this case the problem of the bank is to choose optimally a liquidation policy $\alpha_t^j(s,\pi)$, a borrowing policy $\delta_t^j(s,\pi)$, the amounts of Arrow securities $\theta_t^j(s)$ to trade, and the fraction of deposits $\gamma_t$ to hold as reserves. As in the previous sections the bank chooses $\alpha_t^j(s,\pi)$ and $\delta_t^j(s,\pi)$ after observing the shocks:

$$\begin{aligned} &\max_{0\leqq\alpha_t^j(s,\pi)\leqq 1,\, \delta_t^j(s,\pi)} \pi \ln\left(\gamma_t^j\alpha_t^j(s,\pi)\frac{p_t}{p_{t+1}} + \delta_t^j(s,\pi)\frac{p_t}{p_{t+1}}\right) + \\ &(1-\pi)\ln\left(\gamma_t^j\left[1-\alpha_t^j(s,\pi)\right]\frac{p_t}{p_{t+1}} - \delta_t^j(s,\pi)\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s) + \theta_t^j(s)\right) \end{aligned}$$

In the appendix we show that one solution to this problem sets

$$\alpha_t^j(s,\pi) = \begin{cases} \dfrac{\pi\left[\gamma_t^j\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s) + \theta_t^j(s)\right]}{\gamma_t^j\frac{p_t}{p_{t+1}}}; & \pi \leqq \pi_j^*\left(\gamma_t^j, s\right) \\ 1; \quad \pi > \pi_j^*\left(\gamma_t^j, s\right) \end{cases}$$

and

$$\delta_t^j(s,\pi) = \begin{cases} 0; \quad \pi \leqq \pi_j^*\left(\gamma_t^j, s\right) \\ \frac{p_{t+1}}{p_t}\left[\pi\left[\left(1-\gamma_t^j\right)R_t^j(s) + \theta_t^j(s)\right] - \gamma_t^j\frac{p_t}{p_{t+1}}(1-\pi)\right]; & \pi > \pi_j^*\left(\gamma_t^j, s\right) \end{cases}$$

where $\pi_j^*$ is given by (10). The bank holds a certain amount of reserves, and uses them to pay movers as long as $\pi \leqq \pi_j^*\left(\gamma_t^j, s\right)$. For larger values of the relocation shock, the bank borrows currency for one period from the discount window. This is not the only solution to the bank's problem. In fact, the liquidation and borrowing policies depend on the total amount of currency reserves the bank decided to acquire before observing the liquidity and productivity shocks, and this amount is indeterminate with zero-nominal-rate discount window lending. If it were not, the bank would hold only currency when currency's rate of return dominated other rates of return. Vice versa, the demand for currency reserves would be zero if money were dominated in rate of return by other portfolios of assets.

We can make these statements because in this section, with a central bank operating a discount window, we have a complete set of markets. The easiest way to note this fact is by rewriting the maximization problem of the bank subject to only one budget constraint. Solving for $\delta_t^j(s,\pi)$ and substituting we obtain:

$$\pi r_t^{mj}(s,\pi) + (1-\pi) r_t^j(s,\pi) = \gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s) + \theta_t^j(s),$$

which implies

$$\theta_t^j(s) = \pi r_t^{mj}(s,\pi) + (1-\pi) r_t^j(s,\pi) - \left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]. \quad (11)$$

Recall that the self-financing condition at the beginning of date $t$ is given by:

$$\theta_t^j(s_1) + q\theta_t^j(s_2) = 0.$$

Replacing in this equation the expression for $\theta_t^j(s)$ gives the sole budget constraint for maximization problem of the bank:

$$\sum_{s\in S} q_s \left[\pi r_t^{mj}(s,\pi) + (1-\pi) r_t^j(s,\pi)\right] = \sum_{s\in S} q_s \left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right] \quad (12)$$

which holds for every $\pi$. The problem of the bank can then be written as

$$\max \sum_{s\in S} \eta(s) \left[\int_0^1 \left(\pi \ln\left(r_t^{mj}(s,\pi)\right) + (1-\pi)\ln\left(r_t^j(s,\pi)\right)\right) f(\pi)\, d\pi\right]$$

subject to (12). In the appendix we show that the first order conditions to this problem imply that

$$r_t^{mj}(s,\pi) = r_t^j(s,\pi)$$

for every $s$ and $\pi$. The bank in this case is able to offer movers and non-movers the same rate of return. Note that the rate of return offered is not random, as it would be if there were not asset markets. It is now evident what role banks, the central bank, and asset markets play in this model. Banks provide liquidity insurance to depositors and the central bank allows for the existence of complete markets over liquidity shocks. Asset markets allow banks to trade credit risk. Note that credit risk is not "intertemporal" but "cross-sectional": that is, risk for which asset markets are used does not concern the intertemporal distribution of resources, but total amount of resources available in a certain period.

We show in the appendix that the rates of return offered to movers and non-movers must be equal to

$$\frac{\eta\left(s\right)}{q_s}\left\{\sum_{s\in S}q_s\left[\gamma_t^j\frac{p_t}{p_{t+1}}+\left(1-\gamma_t^j\right)R_t^j\left(s\right)\right]\right\}. \tag{13}$$

Equation (13) states that the return that a bank promises to movers and non-movers in state $s$ is equal to the total present value of goods received in period $t+1$ weighted by the ratio of the probability of $s$ relative to the price of the Arrow security that pays off in $s$. This is natural given the completeness of Arrow securities markets.

In the appendix we also show that to insure an interior solution for $\gamma_t$ the following condition must hold:

$$\left(q_1+q_2\right)\frac{p_t}{p_{t+1}}=q_1R_t^j\left(s_1\right)+q_2R_t^j\left(s_2\right). \tag{14}$$

Equation (14) is a no-arbitrage condition stating that the return on money (the inverse of the gross inflation rate) must be a weighted average of the promised returns from entrepreneurs of both regions, where the weights depend upon the prices of Arrow securities. Using the normalization adopted so far for the prices of Arrow securities we let

$$q_t\equiv\frac{q_{2t}}{q_{1t}}.$$

Therefore the no-arbitrage condition can be expressed as

$$\frac{p_t}{p_{t+1}}-R_t^j\left(s_1\right)=-q_t\left(\frac{p_t}{p_{t+1}}-R_t^j\left(s_2\right)\right)$$

for every $j$. To analyze the equilibrium of the economy, it is first essential to get the optimal net demand for Arrow securities by each bank type.

We use the constraint $(11)$, and substitute optimal rates of return (13) and the arbitrage condition (14) to obtain

$$\theta_t^j(s) = \left(\frac{\eta(s)}{q_s}\right)\left(q_1 R_t^j(s_1) + q_2 R_t^j(s_2)\right) - \left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]$$

We show in the appendix that the optimal net demand functions for these securities by each bank type are given by

$$\begin{aligned}
\theta_t^1(s_1) &= \left(\frac{p_t}{p_{t+1}}\right)\left[q_t\gamma_t^1 - (1-\eta)(1+q_t)\right] \\
\theta_t^1(s_2) &= \left(\frac{p_t}{q_t p_{t+1}}\right)\left[(1-\eta)(1+q_t) - q_t\gamma_t^1\right]
\end{aligned}$$

$$\begin{aligned}
\theta_t^2(s_1) &= \left(\frac{p_t}{p_{t+1}}\right)\left[\eta(1+q_t) - \gamma_t^2\right] \\
\theta_t^2(s_2) &= \frac{1}{q_t}\left(\frac{p_t}{p_{t+1}}\right)\left[\gamma_t^2 - \eta(1+q_t)\right]
\end{aligned}$$

## 5.1 Equilibrium

In equilibrium $\theta_t^1(s) + \theta_t^2(s) = 0$ for every $s$, so that the asset market clears This condition implies that

$$q_t = \frac{\gamma_t^2 + 1 - 2\eta}{\gamma_t^1 + 2\eta - 1} \tag{15}$$

Thus, the relative price of the Arrow securities, which is the relative cost of transferring resources from one solvency state to the other, must be equal in equilibrium to the ratio of two expressions which depend on the fraction of deposits maintained in cash, $\gamma_t^j$. However, equation (15) can also be rewritten as

$$q_t = \frac{2(1-\eta) - (1-\gamma_t^2)}{2\eta - (1-\gamma_t^1)}$$

This expression shows that the relative cost of transferring goods across states of nature must be related to the probabilities of success for every entrepreneur type and to the fraction of deposits that banks are willing to lend.

Also, note that in equilibrium

$$\left(1+q_t\right)\frac{p_t}{p_{t+1}}=R_t^1\left(s_1\right)+q_tR_t^1\left(s_2\right)=R_t^2\left(s_1\right)+q_tR_t^2\left(s_2\right)$$

holds. Because we assumed that $R_t^1\left(s_2\right)=R_t^2\left(s_1\right)=0$, the last equation is equivalent to

$$R_t^1\left(s_1\right)=q_tR_t^2\left(s_2\right)$$

Hence, the relative cost of transferring goods between states $s_1$ and $s_2$ must be equal to the relative returns that banks obtain from entrepreneur types when their projects are successful.

The money market and the loans markets must also clear. These market clearing conditions remain unchanged:

$$\frac{M}{p_t}=\left(\gamma_t^1+\gamma_t^2\right)x$$

and

$$R_t^j=\frac{\alpha}{x^{1-\alpha}\left(1-\gamma_t^j\right)^{1-\alpha}}$$

## 5.2 Steady state analysis

In the appendix we prove the  that the steady state equilibrium is unique, and in addition we show that there are no other equilibria. Specifically, we show the following:

**Proposition 2** *Under the condition* $\frac{\alpha^{\frac{1}{1-\alpha}}}{x}<\min\left\{\frac{1}{\eta},\frac{1}{1-\eta}\right\}$ *there exists a unique steady state* $\left(\gamma^1,\gamma^2\right)\in\Re_{++}^2$*; the steady state is locally unstable. Hence, it is the unique equilibrium for this economy.*

**Proof.** See the appendix ■

Interestingly, this result implies that the presence of Arrow securities markets in an economy as in Antinolfi, Huybens and Keister (2001) does not allow for the presence of inflationary equilibrium trajectories when the Central Bank acts as a lender of last resort and lends at a zero nominal rate[9].

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[9] Although this is not formally shown, we believe that the way borrowers are modelled is related to this uniqueness result.

It is important to remark that the return that each lender type obtains is equal not only across types but also across regions. To see this, note that in steady state $r^j(\pi, s) = \frac{\eta(s)}{q_s}\ \left\{\sum_{s\in S} q_s\left[\gamma^j + \left(1-\gamma^j\right)R^j(s)\right]\right\}$. Given the definition of $q$ we have that

$$\begin{array}{rcl} r^j(\pi, s_1) & = & \eta\left[\gamma^j + \left(1-\gamma^j\right)R^j(s_1) + q\left(\gamma^j + \left(1-\gamma^j\right)R^j(s_2)\right)\right] \\ r^j(\pi, s_2) & = & (1-\eta)\left[\left(\gamma^j + \left(1-\gamma^j\right)R^j(s_1)\right)\dfrac{1}{q} + \gamma^j + \left(1-\gamma^j\right)R^j(s_2)\right] \end{array}$$

so

$$r^1(\pi, s_1) = \eta\left[\gamma^1(1+q) + \left(1-\gamma^1\right)R^1\right]; \quad r^1(\pi, s_2) = (1-\eta)\left[\left(\left(1-\gamma^1\right)R^1\right)\frac{1}{q} + \gamma^1\left(1+\frac{1}{q}\right)\right]$$

$$r^2(\pi, s_1) = \eta\left[\gamma^2(1+q) + q\left(1-\gamma^2\right)R^2\right]; \quad r^2(\pi, s_2) = (1-\eta)\left[\left(1-\gamma^2\right)R^2 + \gamma^2\left(1+\frac{1}{q}\right)\right]$$

But from the no-arbitrage conditions we know that $(1+q) = R^1 = q\ R^2$, and so $\left(1+\frac{1}{q}\right) = R^2$, $1+q = R^1$, $\frac{R^1}{q} = R^2$ and $qR^2 = R^1$. Thus, replacing these equalities in the expressions for $r^j(s, \pi)$ above:

$$\begin{array}{rcl} r^1(\pi, s_1) & = & \eta\left[\gamma^1 R^1 + \left(1-\gamma^1\right)R^1\right] = \eta R^1 \\ r^1(\pi, s_2) & = & (1-\eta)\left[\left(1-\gamma^1\right)R^2 + \gamma^1 R^2\right] = (1-\eta)R^2 \end{array}$$

$$\begin{array}{rcl} r^2(\pi, s_1) & = & \eta\left[\gamma^2 R^1 + \left(1-\gamma^2\right)R^1\right] = \eta R^1 \\ r^2(\pi, s_2) & = & (1-\eta)\left[\left(1-\gamma^2\right)R^2 + \gamma^2 R^2\right] = (1-\eta)R^2 \end{array}$$

Thus $r^j(\pi, s) = \eta(s)R^s$, for $j = 1, 2$ and $s \in \{s_1, s_2\}$. Thus, Arrow securities and the Central Bank acting together imply an allocation where *all* lenders obtain the same consumption quantity and this only depends on the realization of the real shock $s$.

It is not difficult to show that this equilibrium is indeed Pareto optimal. Intuitively, this is easy to see from the characteristics of the equilibrium allocation. Borrowers are risk neutral. In addition, because consumption is independent of the state of nature realized, all lenders have the same marginal rate of substitution. Finally, it is easy to show that the marginal rate of substitution between consumption in state $i$ and $j$ is equal to the marginal rate of transformation between the same states of nature. In the appendix, we prove the Pareto optimality of the equilibrium allocation formally by analyzing the problem of a central planner.

# 6 Conclusions

We presented an economy in which a rich financial structure is supported in equilibrium. This financial structure consists of banks, asset markets, and a central bank. Banks arise to provide efficient risk sharing among depositors, who wish to be insured against liquidity shocks. Because of the aggregate nature of liquidity risk, central bank short-term loans are necessary to provide an asset (money) that completes the market against this intertemporal type of risk. Asset markets allow society to diversify real (as opposed to liquidity) risk, that is the uncertainty about the amount of resources available in the economy at a certain future date. The financial structure needs all three actors to be complete, and a complete financial structure produces an equilibrium allocation which is the same that would obtain in an economy with complete markets (in the traditional sense of a sufficient number of contingent claims to span the space generated by the number of states of nature) and a safe asset, and where the liquidity shock is not present.

There are several special features of the model economy that allows banks, asset markets and banks to coexist and deliver an efficient equilibrium allocation. First, as it is common in this literature, we restrict market participation:[10]only intermediaries trade in asset markets. Even though one could think of individual participation costs to justify this assumption, it is a fundamental assumption in generating a separation of roles between asset markets and banks. An alternative model would have to consider on market microstructure, and study the macro consequences of the different incentives and constraints that financial analysts and loan officers face, and render market participation, both from the asset and liability side, endogenous.

Secondly, we assumed that banks cannot perfectly diversify credit risk, which is essentially another form of limited market participation. As noted above, this feature of the model could be made endogenous by modeling the problem of optimal bank size, but in our model there is an additional observation to be made. Letting banks perfectly diversify credit risk would subtract the fundamental rationale for markets in the model, but in general would not generate the same equilibrium allocation obtained with a model with markets. This is a common feature of all model relying on the Diamond and Dybvig (1983) framework, in which the coefficient of relative risk aversion plays an important role.

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[10] See Allen and Gale (2004), and Diamond (1997).

Finally, we built a structure in which a central bank acts as a perfect lender of last resort. The lender of last resort role of the central bank stems from the (aggregate) nature of the liquidity shock. The central bank can allow the economy to have complete risk insurance against liquidity shocks for several reasons. First, money is a perfect vehicle to transfer consumption across periods for an individual, that is it is not a risky asset and it is made freely (that is, at a zero net interest rate) available to the banking system. Second, banks know exactly that the central bank will do always the "right thing" (there is no uncertainty in this respect), and the central bank can always deliver on that expectation, because, and this is very important if one wants to use this framework to think about financial systems, it is always possible to distinguish precisely between real and nominal shocks.

We believe that our model provides a good characterization of a modern financial system. Adding some of the features discussed in these concluding remarks, in particular adding uncertainty over the central bank behavior, and removing the ability to perfectly identify liquidity and real shocks, would make it possible to further study the interaction among asset prices, interest rates, and intermediation.

# A Proofs and Derivations

## A.1 Proof of proposition 1

We need to show the existence of a unique pair of values $(\gamma^1, \gamma^2)$ each of which satisfies

$$\gamma^j = 1 - \eta(s_j) \int_{\frac{\gamma^j}{\gamma^j + \phi(1-\gamma^j)^\alpha}}^{1} F(\pi)\, d\pi$$

which is equivalent to show

$$\frac{1-\gamma^j}{\eta(s_j)} = \int_{\frac{\gamma^j}{\gamma^j + \phi(1-\gamma^j)^\alpha}}^{1} F(\pi)\, d\pi$$

The left hand side is a strictly decreasing line in $\gamma^j$, with slope $-\frac{1}{\eta(s_j)}$. At $\gamma^j = 0$ the left hand side is equal to $\frac{1}{\eta(s_j)} > 1$. Define $\Phi(\gamma^j) \equiv \int_{\frac{\gamma^j}{\gamma^j + \phi(1-\gamma^j)^\alpha}}^{1} F(\pi)\, d\pi$. We then have that $\Phi(0) = \int_0^1 F(\pi)\, d\pi = 1 - E[\pi] < 1$. We also have

$$\Phi'(\gamma^j) = -F\left[\frac{\gamma^j}{\gamma^j + \phi(1-\gamma^j)^\alpha}\right]\left[\frac{\gamma^j + \phi(1-\gamma^j)^\alpha - \gamma^j + \gamma^j \alpha\phi(1-\gamma^j)^{\alpha-1}}{\left[\gamma^j + \phi(1-\gamma^j)^\alpha\right]^2}\right]$$

Recalling that $\pi_j^*(\gamma^j, s_j) = \frac{\gamma^j}{\gamma^j + \phi(1-\gamma^j)^\alpha}$ then it is possible to show that

$$\Phi'(\gamma^j) = -\frac{F\left[\pi_j^*(\gamma^j, s_j)\right]\phi}{\left[\gamma^j + \phi(1-\gamma^j)^\alpha\right]^2\left[1-\gamma^j\right]^{1-\alpha}}\left[1 - \gamma^j(1-\alpha)\right]$$

So, $\gamma^j > 0$ and $(1-\alpha) > 0$ imply $\Phi'(\gamma^j) < 0$. Note that $\lim_{\gamma^j \to 1} \Phi'(\gamma^j) = \infty$, since $\pi_j^*(1, s_j) = 1$,$F(1) = 1$, and $\alpha < 1$. Therefore the curve $\Phi(\gamma^j)$ must intersect the line from above. By continuity, this implies that there exists at least one $\gamma^{j*} \in (0,1)$ such that $\Phi(\gamma^{j*}) = \frac{1-\gamma^{j*}}{\eta(s_j)}$. Now we show that this value is unique. It is enough to show that $|\Phi'(\gamma^j)| < \frac{1}{\eta(s_j)}$ at $\gamma^{j*}$. If this is the case, the existence of a second $\gamma^{j*}$ implies $|\Phi'(\gamma^j)| > 1$, a contradiction. Note that $|\Phi'(\gamma^j)|$ is equal to (after some tedious algebra):

$$\frac{F\left[\pi_j^*(\gamma^j, s_j)\right]\left[1 - \pi_j^*(\gamma^j, s_j)\right]\pi_j^*(\gamma^j, s_j)}{\gamma^j(1-\gamma^j)}\left[1 - \gamma^j + \alpha\gamma^j\right]$$

But since $\alpha < 1$ then this expression is strictly less than $\frac{F\left[\pi_j^*\left(\gamma^j,s_j\right)\right]\left[1-\pi_j^*\left(\gamma^j,s_j\right)\right]\pi_j^*\left(\gamma^j,s_j\right)}{\gamma^j\left(1-\gamma^j\right)}$. Hence it is enough to show that $\frac{F\left[\pi_j^*\left(\gamma^j,s_j\right)\right]\left[1-\pi_j^*\left(\gamma^j,s_j\right)\right]\pi_j^*\left(\gamma^j,s_j\right)}{\gamma^j\left(1-\gamma^j\right)} < \frac{1}{\eta(s_j)}$, or equivalently

$$\eta\left(s_j\right) F\left[\pi_j^*\left(\gamma^j, s_j\right)\right]\left[1-\pi_j^*\left(\gamma^j, s_j\right)\right] \pi_j^*\left(\gamma^j, s_j\right) < \gamma^j\left(1-\gamma^j\right)$$

In equilibrium:

$$\gamma^j\left(1-\gamma^j\right) = \left[1-\eta\left(s_j\right)\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi\right] \eta\left(s_j\right) \int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi$$

Since $F\left(\pi\right)$ is strictly increasing,

$$\gamma^j\left(1-\gamma^j\right) > \left[1-\eta\left(s_j\right)\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi\right] \eta\left(s_j\right) \left(\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) d\pi\right)$$

But $\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) d\pi = \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right)$. Given that $F\left(\pi\right) < 1$ for all $\pi < 1$, then $\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi < 1 - \pi_j^*\left(\gamma^j,s_j\right)$. Since $\eta\left(s_j\right) > 0$ then $\eta\left(s_j\right) \int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi < \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] \Leftrightarrow -\eta\left(s_j\right) \int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi > -\eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] \Leftrightarrow 1 - \eta\left(s_j\right) \int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi > 1 - \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right]$. Therefore

$$\begin{aligned} &\left[1-\eta\left(s_j\right)\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi\right] \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) \\ > \ &\left[1-\eta\left(s_j\right)\left[1-\pi_j^*\left(\gamma^j,s_j\right)\right]\right] \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) \end{aligned}$$

But since $\eta\left(s_j\right) < 1$, then $1 - \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] > 1 - \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] = \pi_j^*\left(\gamma^j,s_j\right)$. Thus

$$\begin{aligned} &\left[1-\eta\left(s_j\right)\left[1-\pi_j^*\left(\gamma^j,s_j\right)\right]\right] \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) \\ > \ &\pi_j^*\left(\gamma^j,s_j\right) \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) \end{aligned}$$

We thus have shown that

$$\begin{aligned} \gamma^j\left(1-\gamma^j\right) &= \left[1-\eta\left(s_j\right)\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi\right] \eta\left(s_j\right) \left(\int_{\pi_j^*\left(\gamma^j,s_j\right)}^1 F\left(\pi\right) d\pi\right) \\ &> \pi_j^*\left(\gamma^j,s_j\right) \eta\left(s_j\right) \left[1-\pi_j^*\left(\gamma^j,s_j\right)\right] F\left(\pi_j^*\left(\gamma^j,s_j\right)\right) \end{aligned}$$

which is what we wanted to demonstrate. ■

## A.2 Solving the rest of the bank problem for the case of Central Bank and Arrow Securities

The problem of the bank can be written as

$$\sum_{s\in S}\eta\left(s\right)\left[\int_{0}^{1}\left(\pi\ln\left(r_{t}^{mj}\left(s,\pi\right)\right)+\left(1-\pi\right)\ln\left(r_{t}^{j}\left(s,\pi\right)\right)\right)f\left(\pi\right)d\pi\right]$$

subject to

$$\sum_{s\in S}q_{s}\left[\pi r_{t}^{mj}\left(s,\pi\right)+\left(1-\pi\right)r_{t}^{j}\left(s,\pi\right)\right]=\sum_{s\in S}q_{s}\left[\gamma_{t}^{j}\frac{p_{t}}{p_{t+1}}+\left(1-\gamma_{t}^{j}\right)R_{t}^{j}\left(s\right)\right]$$

Let $\phi\left(\pi\right)$ be the multiplier of this constraint. We solve this backwards. We first take $\gamma_{t}^{j}$ as given and also we take every possible realization of $\pi$ as given. The Lagrangian for this problem is

$$\begin{aligned} L\left(\pi\right) &= \sum_{s\in S}\eta\left(s\right)\left(\pi\ln\left(r_{t}^{mj}\left(s,\pi\right)\right)+\left(1-\pi\right)\ln\left(r_{t}^{j}\left(s,\pi\right)\right)\right)\\ &\quad +\phi\left(\pi\right)\sum_{s\in S}q_{s}\left[\gamma_{t}^{j}\frac{p_{t}}{p_{t+1}}+\left(1-\gamma_{t}^{j}\right)R_{t}^{j}\left(s\right)-\left[\pi r_{t}^{mj}\left(s,\pi\right)+\left(1-\pi\right)r_{t}^{j}\left(s,\pi\right)\right]\right]\end{aligned}$$

The first order conditions are are

$$\begin{aligned}\frac{\partial L\left(\pi\right)}{\partial r_{t}^{jm}\left(s,\pi\right)} &= 0\Leftrightarrow\frac{\eta\left(s\right)}{r_{t}^{mj}\left(s,\pi\right)}=\phi\left(\pi\right)q_{s}\\ \frac{\partial L\left(\pi\right)}{\partial r_{t}^{j}\left(s,\pi\right)} &= 0\Leftrightarrow\frac{\eta\left(s\right)}{r_{t}^{j}\left(s,\pi\right)}=\phi\left(\pi\right)q_{s}\end{aligned}$$

Therefore $r_{t}^{mj}\left(s,\pi\right)=r_{t}^{j}\left(s,\pi\right)$. Replacing this on the left hand side of the constraint yields

$$\sum_{s\in S}q_{s}\left[\pi r_{t}^{mj}\left(s,\pi\right)+\left(1-\pi\right)r_{t}^{j}\left(s,\pi\right)\right]=\frac{\pi}{\phi\left(\pi\right)}+\frac{\left(1-\pi\right)}{\phi\left(\pi\right)}=\frac{1}{\phi\left(\pi\right)}$$

Then, the constraint can be rewritten as

$$\frac{1}{\phi\left(\pi\right)}=\sum_{s\in S}q_{s}\left[\gamma_{t}^{j}\frac{p_{t}}{p_{t+1}}+\left(1-\gamma_{t}^{j}\right)R_{t}^{j}\left(s\right)\right]$$

and so, we had from before that $q_s\ r_t^{mj}\ (s,\pi) = \frac{\eta(s)}{\phi(\pi)} \Longrightarrow$

$$r_t^{mj}(s,\pi) = \frac{\eta(s)}{q_s}\left\{\sum_{s\in S} q_s\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]\right\}$$

but we also had $r_t^{mj}(s,\pi) = r_t^j(s,\pi)$. Therefore, replacing this in the objective function, for every $\pi$ and $s$ we get that $\pi\ \ln\left(r_t^{mj}(s,\pi)\right) + (1-\pi)$ $\ln\left(r_t^j(s,\pi)\right)$ is equal to

$$\ln\left\{\frac{\eta(s)}{q_s}\left\{\sum_{s\in S} q_s\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]\right\}\right\}$$

Therefore the ex-ante utility is

$$\begin{aligned} & \sum_{s\in S}\eta(s)\left[\int_0^1 \left(\pi\ln\left(r_t^{mj}(s,\pi)\right) + (1-\pi)\ln\left(r_t^j(s,\pi)\right)\right) f(\pi)\,d\pi\right] \\ = & \sum_{s\in S}\eta(s)\ln\left(\frac{\eta(s)}{q_s}\right) + \ln\left[\sum_{s\in S} q_s\left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]\right] \end{aligned}$$

So then, choosing $\gamma_t^j$ to maximize $\sum_{s\in S}\eta(s)\ln\left(\frac{\eta(s)}{q_s}\right) + \ln\left[\sum_{s\in S} q_s\left[\gamma_t^j\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s)\right]\right]$ is the same as choosing $\gamma_t^j$ that maximizes $\ln\left[\sum_{s\in S} q_s\left[\gamma_t^j\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s)\right]\right]$, equal to $\sum_{s\in S} q_s\left[\gamma_t^j\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s)\right]$, subject to $\gamma_t^j \in [0,1]$. Therefore, we will have that $\gamma_t^j \in (0,1)$ if

$$(q_1+q_2)\frac{p_t}{p_{t+1}} = q_1 R_t^j(s_1) + q_2 R_t^j(s_2)$$

for every $j$. Otherwise we have a corner solution. Now, if this is the case, we have now that

$$\sum_{s\in S} q_s\left[\gamma_t^j\frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right)R_t^j(s)\right] = q_1 R_t^j(s_1) + q_2 R_t^j(s_2)$$

since $\left[\frac{p_t}{p_{t+1}}(q_1+q_2) - \left(q_1 R_t^j(s_1) + q_2 R_t^j(s_2)\right)\right] = 0$. Let $q_t \equiv \frac{q_{2t}}{q_{1t}}$. Therefore the no-arbitrage condition can be expressed as $(1+q_t)\ \frac{p_t}{p_{t+1}} = R_t^j(s_1) + q_t$ $R_t^j(s_2)$. Let us see the problem for each region.

- **Region 1 Bank**

According to our derivation above, in this case we have

$$\sum_{s\in S} q_s \left[\gamma_t^1 \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^1\right) R_t^1(s)\right] = q_1 R_t^1 + 0$$

Recall that for every $j$, $\theta_t^j(s) = \pi\, r_t^{mj}(s,\pi) + (1-\pi)\, r_t^j(s,\pi) - \left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]$ and also recall that $r_t^{mj}(s,\pi) = r_t^j(s,\pi)$ and:

$$r_t^{mj}(s,\pi) = \frac{\eta(s)}{q_s} \left\{\sum_{s\in S} q_s \left[\gamma_t^j \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^j\right) R_t^j(s)\right]\right\}$$

so for $j=1$ :

$$\theta_t^1(s) = \left[\frac{\eta(s)}{q_s}\right] q_1 R_t^1 - \left[\gamma_t^1 \frac{p_t}{p_{t+1}} + \left(1-\gamma_t^1\right) R_t^1(s)\right]$$

So $\theta_t^1(s_1) = \eta R_t^1 - \gamma_t^1 \frac{p_t}{p_{t+1}} - (1-\gamma_t^1)\, R_t^1$. But from the no-arbitrage condition $(1+q_t)\ \frac{p_t}{p_{t+1}} = R_t^1(s_1) + q_t\ R_t^1(s_2) = R_t^1$, so $R_t^1 = (1+q_t)\ \frac{p_t}{p_{t+1}}$. Replacing this in the above expression for $\theta_t^1(s_1)$ we get:

$$\theta_t^1(s_1) = \left(\frac{p_t}{p_{t+1}}\right) \left[q_t \gamma_t^1 - (1-\eta)(1+q_t)\right]$$

We can now solve for $\theta_t^1(s_2)$ using identical arguments. Recalling that $R_t^1 = (1+q_t)\frac{p_t}{p_{t+1}}$ and using the equation for $\theta_t^1(s)$ at $s=s_2$ then:

$$\theta_t^1(s_2) = \left(\frac{p_t}{q_t p_{t+1}}\right) \left[(1-\eta)(1+q_t) - q_t \gamma_t^1\right]$$

- **Region 2 Bank**

Following the same steps as above we obtain:

$$\theta_t^2(s_1) = \left(\frac{p_t}{p_{t+1}}\right) \left[\eta(1+q_t) - \gamma_t^2\right]$$

and so

$$\theta_t^2(s_2) = \frac{1}{q_t} \left(\frac{p_t}{p_{t+1}}\right) \left[\gamma_t^2 - \eta(1+q_t)\right]$$

## A.3 Proof of proposition 2

We first show that there exists a unique $R^1$, $R^2$ satisfying the stationary equilibrium equations. Note that the first condition can be written as $\frac{R^1+R^2}{R^2} = R^1$, which simplifies to

$$R^2 = \frac{R^1}{R^1 - 1}$$

The second equation can be rewritten as

$$2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}} = 2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}}$$

Hence we have two equations in two unknowns. The first equation defines a curve on the plane $(R^1, R^2)$ which is strictly decreasing with asymptotes in $(1, 1)$. The second equation also defines implicitly a curve on the plane $(R^1, R^2)$. To get the derivative we apply the implicit function theorem to the map:

$$\Phi\left(R^1, R^2\right) \equiv -\left(2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}}\right) + 2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}}$$

at the point where $\Phi\left(R^1, R^2\right) = 0$. In general we have that $\frac{dR^2}{dR^1} = -\frac{\Phi_{R^1}\left(R^1,R^2\right)}{\Phi_{R^2}\left(R^1,R^2\right)}$. In this case we have

$$\Phi_{R^1}\left(R^1, R^2\right) = 2\eta + \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\left(\frac{\alpha}{1-\alpha}\right)\left(R^1\right)^{\frac{-\alpha}{1-\alpha}-1}$$

$$\Phi_{R^2}\left(R^1, R^2\right) = -\left(2\left(1-\eta\right) + \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\frac{\alpha}{1-\alpha}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}-1}\right)$$

so

$$\frac{dR^2}{dR^1} = \frac{2\eta + \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\left(\frac{\alpha}{1-\alpha}\right)\left(R^1\right)^{\frac{-\alpha}{1-\alpha}-1}}{\left(2\left(1-\eta\right) + \frac{\alpha^{\frac{1}{1-\alpha}}}{d}\frac{\alpha}{1-\alpha}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}-1}\right)} > 0$$

It remains to show that for low $R^1$ the first curve is above the second one and for large $R^1$ the reverse is true. Clearly, according to the first curve, $R^2$ approaches infinity when $R^1 \downarrow 1$. According to this curve it is also true that as $R^1 \uparrow \infty$ then $R^2 \downarrow 1$. For the second curve behavior it is convenient to write down the condition

$$2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}} = 2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}}$$

Suppose first that $R^1 \downarrow 0$. Therefore $2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}} \downarrow -\infty$ since $\frac{-\alpha}{1-\alpha} < 0$. Hence $2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}} \downarrow -\infty$ whenever $R^1 \downarrow 0$. Therefore it must happen that $R^2 \downarrow 0$. Suppose this is not the case. Then we could have $R^2 \downarrow R^*$ finite and positive. But then $\left(2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}}\right)$ has a finite limit and so for a sufficient small $R^1$ the equality $\left(2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}}\right) = 2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}}$ is not satisfied. Since the curve is strictly increasing, $R^2$ cannot go to $+\infty$ when $R^1$ goes down to 0. Hence when $R^1 \downarrow 0$ then $R^2 \downarrow 0$ and so the curve tends towards the origin. Hence also at $R^1 = 1$ the value of $R^2$ is finite. Therefore at $R^1 = 1 + \varepsilon$ with $\varepsilon > 0$ and small the first curve is strictly above the second curve. Also, when $R^1 \uparrow +\infty$ we will have that $R^2 \uparrow +\infty$ . Otherwise, if $R^2 \uparrow R^{**} < +\infty$, the expression $2\left(1-\eta\right)R^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}}$ goes towards a finite number, but the expression $2\eta R^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}} \uparrow +\infty$ when $R^1$ does, therefore the equality should not hold for sufficiently big $R^1$. Because the curve is strictly increasing, it cannot happen that while $R^1 \uparrow +\infty$ then $R^2 \downarrow -\infty$ . Therefore, for sufficiently large $R^1$ the value of $R^2$ must also be very large. Therefore the second curve is strictly above the first curve for $R^1$ big. Because of the monotonicity properties and continuity, there exists a unique $R^1, R^2$ where the two curves intersect. Because of the properties of the first curve, both $R^1$ and $R^2$ must be strictly greater than one.

Now we show that this pair of interest rates is a stationary equilibrium. To do this we show that there is an equivalence between this pair of interest rates and the pair $\left(\gamma^1, \gamma^2\right) \in \left(0,1\right)^2$ that solves the steady state of the dynamical system in $\gamma_t^j$. Suppose $R^1, R^2$ is a steady state for the system

$$\frac{2 - \left(\frac{\alpha^{\frac{1}{1-\alpha}}}{x}\right)\left(\frac{1}{\left(R_{t+1}^1\right)^{\frac{1}{1-\alpha}}} + \frac{1}{\left(R_{t+1}^2\right)^{\frac{1}{1-\alpha}}}\right)}{2 - \left(\frac{\alpha^{\frac{1}{1-\alpha}}}{x}\right)\left(\frac{1}{\left(R_t^1\right)^{\frac{1}{1-\alpha}}} + \frac{1}{\left(R_t^2\right)^{\frac{1}{1-\alpha}}}\right)} = \frac{R_t^1 R_t^2}{R_t^1 + R_t^2}; \quad \frac{R_t^1}{R_t^2} = \frac{2\left(1-\eta\right) - \frac{\alpha^{\frac{1}{1-\alpha}}}{\left(R_t^2\right)^{\frac{1}{1-\alpha}} x}}{2\eta - \frac{\alpha^{\frac{1}{1-\alpha}}}{\left(R_t^1\right)^{\frac{1}{1-\alpha}}(x)}}$$

and that $R^j > 1$. Hence it must be true that $1 = \frac{R^1 R^2}{R^1 + R^2}$ and

$$\frac{R^1}{R^2} = \frac{2(1-\eta) - \frac{\alpha^{\frac{1}{1-\alpha}}}{(R^2)^{\frac{1}{1-\alpha}}(x)}}{2\eta - \frac{\alpha^{\frac{1}{1-\alpha}}}{(R^1)^{\frac{1}{1-\alpha}}(x)}}$$

Hence given that $1 - \gamma^j = \frac{\alpha^{\frac{1}{1-\alpha}}}{(R^j)^{\frac{1}{1-\alpha}}(x)}$ and since $R^j > 1$ it is clear that $\gamma^j < 1$. Let $\phi \equiv \frac{\alpha}{(x)^{1-\alpha}}$. Since we assumed that $\phi < 2^{1-\alpha}\left[\eta^{1-\alpha} + (1-\eta)^{1-\alpha}\right]$ we claim that in this steady state $R^1 > \frac{\phi}{(2\eta)^{1-\alpha}}$ and $R^2 > \frac{\phi}{(2(1-\eta))^{1-\alpha}}$. To show this, it sufficient to show that the point $\widehat{R}^1 \equiv \frac{\phi}{(2\eta)^{1-\alpha}}$, $\widehat{R}^2 \equiv \frac{\phi}{(2(1-\eta))^{1-\alpha}}$ is to the left of the intersection of the two curves. To prove this last fact, note that at $\left(\widehat{R}^1, \widehat{R}^2\right)$ it is true that

$$\begin{aligned}\Phi\left(\widehat{R}^1, \widehat{R}^2\right) &= -\left(2(1-\eta)\widehat{R}^2 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(\widehat{R}^2\right)^{\frac{-\alpha}{1-\alpha}}\right) + 2\eta\widehat{R}^1 - \frac{\alpha^{\frac{1}{1-\alpha}}}{x}\left(\widehat{R}^1\right)^{\frac{-\alpha}{1-\alpha}} \\ &= -\left((2(1-\eta))^{\alpha}\phi - \phi(2(1-\eta))^{\alpha}\right) + \left((2\eta)^{\alpha}\phi - \phi(2\eta)^{\alpha}\right) = 0\end{aligned}$$

That is, the point $\left(\widehat{R}^1, \widehat{R}^2\right)$ lies on the curve defined by $\Phi(R^1, R^2) = 0$. Now we compare $\widehat{R}^2$ with $\frac{\widehat{R}^1}{\widehat{R}^1 - 1}$ to see whether this point lies above or below the second, downward sloping curve. We know that $\widehat{R}^2 = \frac{\phi}{(2(1-\eta))^{1-\alpha}}$ and

$$\frac{\widehat{R}^1}{\widehat{R}^1 - 1} = \frac{\phi}{(2\eta)^{1-\alpha}\left(\frac{\phi}{(2\eta)^{1-\alpha}} - 1\right)} = \frac{\phi}{\phi - (2\eta)^{1-\alpha}}$$

Since $\phi < (2\eta)^{1-\alpha} + (2(1-\eta))^{1-\alpha}$ then $\phi - (2\eta)^{1-\alpha} < (2(1-\eta))^{1-\alpha}$. Thus, $\frac{\phi}{(2(1-\eta))^{1-\alpha}} < \frac{\phi}{\phi - (2\eta)^{1-\alpha}}$. Therefore $\widehat{R}^2 < \frac{\widehat{R}^1}{\widehat{R}^1 - 1}$ and so the point $\left(\widehat{R}^1, \widehat{R}^2\right)$ is below the downward sloping curve. But then the pair $(R^1, R^2)$ where both curves intersect must imply $R^1 > \widehat{R}^1$ and $R^2 > \widehat{R}^2$. This shows the claim. As a consequence, recalling that $1 - \gamma^j = \frac{\alpha^{\frac{1}{1-\alpha}}}{(R^j)^{\frac{1}{1-\alpha}} x} = \frac{\phi^{\frac{1}{1-\alpha}}}{(R^j)^{\frac{1}{1-\alpha}}}$ and given the last claim, it is clear that $\frac{1}{(R^1)^{\frac{1}{1-\alpha}}} < \frac{1}{\left(\frac{\phi^{\frac{1}{1-\alpha}}}{2\eta}\right)}$ and $\frac{1}{(R^2)^{\frac{1}{1-\alpha}}} < \frac{1}{\left(\frac{\phi^{\frac{1}{1-\alpha}}}{2(1-\eta)}\right)}$. Thus:

$$1 - \gamma^1 = \frac{\phi^{\frac{1}{1-\alpha}}}{(R^1)^{\frac{1}{1-\alpha}}} < 2\eta \quad \text{and} \quad 1 - \gamma^2 = \frac{\phi^{\frac{1}{1-\alpha}}}{(R^2)^{\frac{1}{1-\alpha}}} < 2(1-\eta)$$

which implies that $\gamma^1 > 1 - 2\eta$ and $\gamma^2 > 2\eta - 1$, and therefore $\gamma^j > 0$ for both $j$. With this in mind, recall then that

$$\frac{R^1}{R^2} = \frac{2\left(1-\eta\right) - \frac{\alpha^{\frac{1}{1-\alpha}}}{\left(R^2\right)^{\frac{1}{1-\alpha}}\left(\frac{\beta}{2+\beta}x\right)}}{2\eta - \frac{\alpha^{\frac{1}{1-\alpha}}}{\left(R^1\right)^{\frac{1}{1-\alpha}}\left(\frac{\beta}{2+\beta}x\right)}} = \frac{2\left(1-\eta\right) - \left(1-\gamma^2\right)}{2\eta - \left(1-\gamma^1\right)}$$

Clearly $\frac{R^1}{R^2} > 0$. Therefore either $2\ (1-\eta) - (1-\gamma^2) > 0$ and $2\eta - (1-\gamma^1) > 0$ or $2\ (1-\eta) - (1-\gamma^2) < 0$ and $2\eta - (1-\gamma^1) < 0$. If the first two set of inequalities hold, then we have at the same time that $\gamma^2 > 2\eta - 1$ and $\gamma^1 > 1 - 2\eta$. If the first two set of inequalities hold then we have at the same time $\gamma^2 < 2\eta - 1$ and $\gamma^1 < 1 - 2\eta$, but this second case must be ruled out since this implies that at least one $\gamma^j$ is strictly negative. So the first set of inequalities holds. Given the definition of $\gamma^j$ we have that $R^j = \frac{\alpha}{\left(1-\gamma^j\right)^{1-\alpha}\left(\frac{\beta}{2+\beta}x\right)^{1-\alpha}}$ for every $j$. Hence $\frac{R^1}{R^2} = \frac{\left(1-\gamma^2\right)^{1-\alpha}}{\left(1-\gamma^1\right)^{1-\alpha}}$ and so:

$$\frac{\left(1-\gamma^2\right)^{1-\alpha}}{\left(1-\gamma^1\right)^{1-\alpha}} = \frac{2\left(1-\eta\right) - \left(1-\gamma^2\right)}{2\eta - \left(1-\gamma^1\right)} = \frac{1 - 2\eta + \gamma^2}{2\eta - 1 + \gamma^1}$$

or

$$\left(1-\gamma^2\right)^{1-\alpha}\left(2\eta - 1 + \gamma^1\right) = \left(1 - 2\eta + \gamma^2\right)\left(1-\gamma^1\right)^{1-\alpha}$$

which is one of the two equations of the dynamical system in $\gamma_t^j$ (in steady state).

On the other hand, we also had $R^1 + R^2 = R^1\ R^2$, equivalent to $\frac{R^1}{R^2} + 1 = R^1$. Replacing we have

$$1 + \frac{\left(1-\gamma^2\right)^{1-\alpha}}{\left(1-\gamma^1\right)^{1-\alpha}} = \frac{\alpha}{\left(1-\gamma^1\right)^{1-\alpha}\left(\frac{\beta}{2+\beta}x\right)^{1-\alpha}}$$

But $\frac{\left(1-\gamma^2\right)^{1-\alpha}}{\left(1-\gamma^1\right)^{1-\alpha}} = \frac{2(1-\eta)-\left(1-\gamma^2\right)}{2\eta-\left(1-\gamma^1\right)} = \frac{1-2\eta+\gamma^2}{2\eta-1+\gamma^1}$ so after some algebra we get:

$$\gamma^1 + \gamma^2 = \frac{\alpha\left(2\eta - 1 + \gamma^1\right)}{\left(1-\gamma^1\right)^{1-\alpha}\left(\frac{\beta}{2+\beta}x\right)^{1-\alpha}}$$

Hence the values of $\gamma^j$ defined above satisfies the two equations that must hold at the steady state equilibrium for the system with $(\gamma_t^1, \gamma_t^2)$. Since $R^j$ determines uniquely $\gamma^j$ and by the uniqueness of $(R^1, R^2)$ then the pair of $(\gamma^1, \gamma^2)$ that satisfies both equations are also unique.

The converse is also straightforward to show. Suppose there exists a unique pair of $\gamma^1, \gamma^2$ where $\gamma^1 > 1 - 2\eta$ and $\gamma^2 > 2\eta - 1$ and satisfying

$$\gamma^1 + \gamma^2 = \frac{\alpha\left(2\eta - 1 + \gamma^1\right)}{\left(1-\gamma^1\right)^{1-\alpha}(x)^{1-\alpha}} = \frac{\left(2\eta - 1 + \gamma^1\right)\phi}{\left(1-\gamma^1\right)^{1-\alpha}}$$

and

$$\left(1-\gamma^2\right)^{1-\alpha}\left(2\eta - 1 + \gamma^1\right) = \left(1 - 2\eta + \gamma^2\right)\left(1-\gamma^1\right)^{1-\alpha}$$

Therefore, define $R^j \equiv \frac{\phi}{\left(1-\gamma^j\right)^{1-\alpha}}$. We need to show that the pair $(R^1, R^2)$ satisfies both equations

$$1 = \frac{R^1 R^2}{R^1 + R^2}$$

and

$$2(1-\eta)R^2 - \phi^{\frac{1}{1-\alpha}}\left(R^2\right)^{\frac{-\alpha}{1-\alpha}} = 2\eta R^1 - \phi^{\frac{1}{1-\alpha}}\left(R^1\right)^{\frac{-\alpha}{1-\alpha}}$$

equivalent to

$$R^2\left[2(1-\eta) - \left(\frac{\phi}{R^2}\right)^{\frac{1}{1-\alpha}}\right] = R^1\left[2\eta - \left(\frac{\phi}{R^1}\right)^{\frac{1}{1-\alpha}}\right]$$

We basically work backwards relative to the first part of the proof. We know that

$$\gamma^1 + \gamma^2 = \frac{\left(2\eta - 1 + \gamma^1\right)\phi}{\left(1-\gamma^1\right)^{1-\alpha}} \Longrightarrow \frac{\gamma^1 + \gamma^2}{\left(2\eta - 1 + \gamma^1\right)} = \frac{\phi}{\left(1-\gamma^1\right)^{1-\alpha}}$$

but $\frac{\gamma^1+\gamma^2}{2\eta-1+\gamma^1} = \frac{\gamma^1+2\eta-1+1-2\eta+\gamma^2}{2\eta-1+\gamma^1} = 1 + \frac{1-2\eta+\gamma^2}{2\eta-1+\gamma^1}$. Hence $\frac{\phi}{\left(1-\gamma^1\right)^{1-\alpha}} = 1 + \frac{1-2\eta+\gamma^2}{2\eta-1+\gamma^1}$. But from the second equation $\left(1-\gamma^2\right)^{1-\alpha}\left(2\eta - 1 + \gamma^1\right) = \left(1 - 2\eta + \gamma^2\right)\left(1-\gamma^1\right)^{1-\alpha}$, so

$$1 + \frac{1 - 2\eta + \gamma^2}{2\eta - 1 + \gamma^1} = 1 + \frac{\left(1-\gamma^2\right)^{1-\alpha}}{\left(1-\gamma^1\right)^{1-\alpha}}$$

But $1+\frac{\left(1-\gamma^{2}\right)^{1-\alpha}}{\left(1-\gamma^{1}\right)^{1-\alpha}}$ is equal to $1+\frac{\left(\frac{\phi}{\left(1-\gamma^{1}\right)^{1-\alpha}}\right)}{\left(\frac{\phi}{\left(1-\gamma^{2}\right)^{1-\alpha}}\right)}$, which in equilibrium is equal to $1+\frac{R^{1}}{R^{2}}$. Therefore the equality $\frac{\phi}{\left(1-\gamma^{1}\right)^{1-\alpha}}=1+\frac{1-2\eta+\gamma^{2}}{2\eta-1+\gamma^{1}}$ is equivalent to

$$R^{1}=\frac{\phi}{\left(1-\gamma^{1}\right)^{1-\alpha}}=1+\frac{1-2\eta+\gamma^{2}}{2\eta-1+\gamma^{1}}=1+\frac{R^{1}}{R^{2}}=\frac{R^{2}+R^{1}}{R^{2}}$$

so $R^{1}R^{2}=R^{1}+R^{2}$ and the first equation is shown.

$>$From the second condition in the $(\gamma^{1},\gamma^{2})$ system $\left(1-\gamma^{2}\right)^{1-\alpha}\left(2\eta-1+\gamma^{1}\right)$ $=\left(1-2\eta+\gamma^{2}\right)\left(1-\gamma^{1}\right)^{1-\alpha}$ implies after some algebra that $R^{1}\left(2\eta-1+\gamma^{1}\right)$ $=R^{2}\left(1-2\eta+\gamma^{2}\right)$, therefore $R^{1}\left(2\eta-\left(1-\gamma^{1}\right)\right)$ is equal to $R^{2}\left(2\left(1-\eta\right)-\left(1-\gamma^{2}\right)\right).$ But from the definition of $R^{j}$ we have $1-\gamma^{j}=\frac{\phi^{\frac{1}{1-\alpha}}}{R^{j\frac{1}{1-\alpha}}}$. Hence the equality $R^{1}$ $\left(2\eta-\left(1-\gamma^{1}\right)\right)=R^{2}\left(2\left(1-\eta\right)-\left(1-\gamma^{2}\right)\right)$ is equivalent to

$$R^{1}\left(2\eta-\frac{\phi^{\frac{1}{1-\alpha}}}{\left(R^{1}\right)^{\frac{1}{1-\alpha}}}\right)=R^{2}\left(2\left(1-\eta\right)-\frac{\phi^{\frac{1}{1-\alpha}}}{\left(R^{2}\right)^{\frac{1}{1-\alpha}}}\right)$$

which is the second equation we wanted to get. This completes the proof of uniqueness of steady state.

To show that the stationary equilibrium is indeed the unique equilibrium, let $\phi$ be defined as above. Note that the condition $\phi<\min\left\{\frac{1}{\eta},\frac{1}{1-\eta}\right\}$ implies that $\phi<2^{1-\alpha}\left\{\eta^{1-\alpha}+\left(1-\eta\right)^{1-\alpha}\right\}$. To show this, note that $\min\left\{\frac{1}{\eta},\frac{1}{1-\eta}\right\}=\frac{1}{1-\eta}$ when $\eta<\frac{1}{2}$ and $\min\left\{\frac{1}{\eta},\frac{1}{1-\eta}\right\}=\frac{1}{\eta}$ for $\eta>\frac{1}{2}$. For $\eta<\frac{1}{2}$ define

$$\Phi_{1}\left(\eta\right)\equiv 2^{1-\alpha}\left\{\eta^{1-\alpha}+\left(1-\eta\right)^{1-\alpha}\right\}-\frac{1}{1-\eta}$$

Note that $\Phi_{1}\left(0\right)=2^{1-\alpha}-1>0$ (since $2^{1-\alpha}>1^{1-\alpha}=1$). Also $\Phi_{1}\left(\frac{1}{2}\right)=0$ and also we have that

$$\begin{aligned}\Phi_{1}'\left(\eta\right)&=2^{1-\alpha}\left(1-\alpha\right)\left[\eta^{-\alpha}-\left(1-\eta\right)^{-\alpha}\right]-\frac{1}{\left(1-\eta\right)^{2}}\\ \Phi_{1}^{"}\left(\eta\right)&=2^{1-\alpha}\left(1-\alpha\right)\left(-\alpha\right)\left[\eta^{-(1+\alpha)}+\left(1-\eta\right)^{-(1+\alpha)}\right]-\frac{2}{\left(1-\eta\right)^{3}}<0\end{aligned}$$

The first expression is zero for some $\eta_{1}\in\left(0,\frac{1}{2}\right)$. Clearly $\Phi_{1}\left(\eta_{1}\right)\geq 2^{1-\alpha}>0$. Hence, the function $\Phi_{1}\left(\eta\right)$ attains a strictly positive value at its maximum in $\left(0,\frac{1}{2}\right)$ and the function is strictly concave in the whole interval.

This implies that $\Phi_1(\eta) > 0$ for all $\eta \in \left(0, \frac{1}{2}\right)$. Hence for $\eta \in [0, \frac{1}{2})$ then $2^{1-\alpha}\left\{\eta^{1-\alpha} + (1-\eta)^{1-\alpha}\right\} > \frac{1}{1-\eta}$. For $\eta > \frac{1}{2}$ consider

$$\Phi_2(\eta) \equiv 2^{1-\alpha}\left\{\eta^{1-\alpha} + (1-\eta)^{1-\alpha}\right\} - \frac{1}{\eta}$$

Clearly $\Phi_2\left(\frac{1}{2}\right) = 0$ and $\Phi_2(1) = 2^{1-\alpha} - 1 > 0$. Computing the derivatives, it easy to show that $\Phi_2'(\eta) = 0$ for some $\eta = \eta_2 \in \left(\frac{1}{2}, 1\right)$. Since $\Phi_2$ is strictly concave then at $\eta_2$ the function $\Phi_2$ attains a strictly positive value at $\eta_2$ and therefore $\Phi_2(\eta) > 0$ for all $\eta \in (\frac{1}{2}, 1]$. These two arguments state then that $\min\left\{\frac{1}{\eta}, \frac{1}{1-\eta}\right\} \leqq 2^{1-\alpha}\left[\eta^{1-\alpha} + (1-\eta)^{1-\alpha}\right]$, where the equality only holds at $\eta = \frac{1}{2}$. This shows then that $\phi < \min\left\{\frac{1}{\eta}, \frac{1}{1-\eta}\right\}$ implies that the condition in the statement of the proposition is satisfied.

We now undertake the proof of uniqueness of equilibrium in two steps, one corresponding to the case $\eta < \frac{1}{2}$ and the other to the case $\eta > \frac{1}{2}$ (the case $\eta = \frac{1}{2}$ is trivial). $\gamma^1 > 1 - 2\eta$, $\gamma^2 > 2\eta - 1$.

- **Case 1**: $\eta > \frac{1}{2}$.

This implies that $2\eta - 1 > 0$ and so $\gamma^2 > 0$. Hence it remains to show in this case that $\gamma^1 > 0 > 1 - 2\eta$. To do this, recall that the dynamical system in $(\gamma_t^1, \gamma_t^2)$ can be written as:

$$\gamma_{t+1}^1 + \gamma_{t+1}^2 = \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1 - \gamma_t^1\right)^{1-\alpha}}$$

$$\left(1 - \gamma_t^1\right)^{1-\alpha}\left(\gamma_t^2 + 1 - 2\eta\right) = \left(1 - \gamma_t^2\right)^{1-\alpha}\left(\gamma_t^1 + 2\eta - 1\right)$$

The second equation can also be re-expressed as

$$\left(1 - \gamma_{t+1}^1\right)^{1-\alpha}\left(\gamma_{t+1}^2 + 1 - 2\eta\right) = \left(1 - \gamma_{t+1}^2\right)^{1-\alpha}\left(\gamma_{t+1}^1 + 2\eta - 1\right)$$

and from the first equation we get both $\gamma_{t+1}^2 = \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}} - \gamma_{t+1}^1$ and $1 - \gamma_{t+1}^2 = 1 + \gamma_{t+1}^1 - \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}}$. Replacing these two expressions in the last equation we get:

$$\begin{aligned} &\left(1 - \gamma_{t+1}^1\right)^{1-\alpha}\left(\frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1 - \gamma_t^1\right)^{1-\alpha}} - \gamma_{t+1}^1 + 1 - 2\eta\right) \\ = &\left(1 + \gamma_{t+1}^1 - \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1 - \gamma_t^1\right)^{1-\alpha}}\right)\left(\gamma_{t+1}^1 + 2\eta - 1\right) \end{aligned}$$

which is a one - dimensional dynamical system. We already know that this system has two steady states. One in $\gamma_{t+1}^{1} = 1 - 2\eta$ (which is not an equilibrium) and the other one where $\gamma_{t+1}^{1} > 1 - 2\eta$ obtained in the last subsection. The system defines implicitly a curve on the $\left(\gamma_t^1, \gamma_{t+1}^1\right)$ plane. We first show that this curve is strictly increasing on $\Re_{++}^{2}$. We will then show that this curve pass through a point $(\bar{\gamma}, 1)$, where $\bar{\gamma} < 1$. These facts will ensure that the equilibrium steady state is in fact the unique possible equilibrium in this economy.

**Lemma 3** *Let*

$$\begin{aligned} F\left(\gamma_{t+1}^1, \gamma_t^1\right) \equiv & \left(\frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}} - \left(\gamma_{t+1}^1 - 1 + 2\eta\right)\right)\left(1-\gamma_{t+1}^1\right)^{1-\alpha} \\ & - \left(1 + \gamma_{t+1}^1 - \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}}\right)^{1-\alpha}\left(\gamma_{t+1}^1 + 2\eta - 1\right) \end{aligned}$$

*Then the equation* $F\left(\gamma_{t+1}^1, \gamma_t^1\right) = 0$ *defines implicitly a curve* $\gamma_{t+1}^1$ *as a function of* $\gamma_t^1$, *and it can be shown that* $\frac{d\gamma_{t+1}^1}{d\gamma_t^1} > 0$ *for* $\gamma_t^1$ *and* $\gamma_{t+1}^1$ *greater than or equal to* $1 - 2\eta$ *and strictly less than one.*

**Proof of Lemma.** To show that there is an implicit function, by the Implicit Function Theorem it is enough to show that $F_{\gamma_{t+1}^1}\left(\gamma_t^1, \gamma_{t+1}^1\right) \neq 0$ (we will in fact show that this is strictly negative). To do this, we just compute this derivative:

$$\begin{aligned} & F_{\gamma_{t+1}^1}\left(\gamma_{t+1}^1, \gamma_t^1\right) \\ = & -\left(1-\gamma_{t+1}^1\right)^{1-\alpha} - \left(\frac{\alpha\left(\gamma_t^1 + 2\eta - 1\right)}{x^{1-\alpha}\left(1-\gamma_t^1\right)^{1-\alpha}} - \left(\gamma_{t+1}^1 - 1 + 2\eta\right)\right)(1-\alpha)\left(1-\gamma_{t+1}^1\right)^{-\alpha} \\ & -(1-\alpha)\left(1 + \gamma_{t+1}^1 - \frac{\alpha\left(\gamma_t^1 + 2\eta - 1\right)}{x^{1-\alpha}\left(1-\gamma_t^1\right)^{1-\alpha}}\right)^{-\alpha}\left(\gamma_{t+1}^1 + 2\eta - 1\right) \\ & - \left(1 + \gamma_{t+1}^1 - \frac{\alpha\left(\gamma_t^1 + 2\eta - 1\right)}{x^{1-\alpha}\left(1-\gamma_t^1\right)^{1-\alpha}}\right)^{1-\alpha} \cdot 1 \end{aligned}$$

>From this expression it is straightforward to see that for any $\gamma_t^1 \geq 1 - 2\eta$, $\gamma_{t+1}^1 \geq 1 - 2\eta$, and less than one, then $F_{\gamma_{t+1}^1}\left(\gamma_{t+1}^1, \gamma_t^1\right) < 0$. This shows that

the implicit function is well defined. On the other hand, to sign the implicit derivative $\frac{d\gamma_{t+1}^1}{d\gamma_t^1}$ we need to get $F_{\gamma_t^1}\left(\gamma_{t+1}^1,\gamma_t^1\right)$. This is equal to:

$$F_{\gamma_t^1}\left(\gamma_{t+1}^1,\gamma_t^1\right)=\phi\left(\frac{\left(1-\gamma_t^1\right)^{(1-\alpha)}+\left(\gamma_t^1+2\eta-1\right)(1-\alpha)\left(1-\gamma_t^1\right)^{-\alpha}}{\left(1-\gamma_t^1\right)^{2(1-\alpha)}}\right)\cdot$$
$$\left[\left(1-\gamma_{t+1}^1\right)^{1-\alpha}+\left(\gamma_{t+1}^1+2\eta-1\right)(1-\alpha)\left(1+\gamma_{t+1}^1-\phi\frac{\left(\gamma_t^1+2\eta-1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}}\right)^{-\alpha}\right]$$

For any $\gamma_t^1\geq 1-2\eta$, $\gamma_{t+1}^1\geq 1-2\eta$, and less than one, then $F_{\gamma_t^1}\left(\gamma_{t+1}^1,\gamma_t^1\right)>0$. Therefore, by the Implicit Function Theorem it is obvious that

$$\frac{d\gamma_{t+1}^1}{d\gamma_t^1}=-\frac{F_{\gamma_t^1}\left(\gamma_{t+1}^1,\gamma_t^1\right)}{F_{\gamma_{t+1}^1}\left(\gamma_{t+1}^1,\gamma_t^1\right)}>0$$

This ends the proof of this lemma. ■

The second part shows that this map goes through $(\bar{\gamma},1)$ with $\bar{\gamma}<1$.

**Lemma 4** *When* $\gamma_{t+1}^1\to 1$ *then* $\gamma_t^1\to\bar{\gamma}<1$.

**Proof.** >From the equation defining the dynamical system on $\gamma_t^1$ take limits on both sides where $\gamma_{t+1}^1\to 1$ and $\gamma_t^1\to\bar{\gamma}$. We must have that

$$\begin{aligned}&(1-1)^{1-\alpha}\left(\frac{\phi\left(\bar{\gamma}+2\eta-1\right)}{\left(1-\bar{\gamma}\right)^{1-\alpha}}-1+1-2\eta\right)\\&=\left(1+1-\frac{\phi\left(\bar{\gamma}+2\eta-1\right)}{\left(1-\bar{\gamma}\right)^{1-\alpha}}\right)(1+2\eta-1)\end{aligned}$$

Given that the left hand side is zero, this is equivalent to $0=2\eta\left(2-\frac{\phi(\bar{\gamma}+2\eta-1)}{(1-\bar{\gamma})^{1-\alpha}}\right)$, and given that $\eta>0$, what this implies is

$$\frac{\phi\left(\bar{\gamma}+2\eta-1\right)}{\left(1-\bar{\gamma}\right)^{1-\alpha}}=2$$

To get this equality, it is necessary that $\bar{\gamma}>1-2\eta$ (which is true since the map is strictly increasing) and that $(1-\bar{\gamma})^{1-\alpha}>0$, which implies $\bar{\gamma}<1$ as desired. ■

Given that the curve $\gamma_{t+1}^1\left(\gamma_t^1\right)$ pass through the $45^o$ line only through two points, one at $(1-2\eta,1-2\eta)$ and another one above this, the two lemmas imply that the curve must cut the $45^o$ line at the second steady state (the stationary equilibrium) from below. This shows that this steady state is unstable and so, if $\gamma^1 > 0$, then it is the unique equilibrium for this economy, since any other combination of $\left(\gamma_t^1, \gamma_{t+1}^1\right)$ outside the stationary equilibrium leads to either the point $(1-2\eta, 1-2\eta)$ or to some value greater than one. Neither of the two situations cannot be equilibrium cases.

It remains to be shown that $\gamma^1 > 0$. Given the last lemma, it is sufficient to show that when $\gamma_{t+1}^1 = 0$ then $\gamma_t^1 > 0$. To prove this, recall that the system can be reduced to:

$$\begin{aligned} F\left(\gamma_{t+1}^1, \gamma_t^1\right) &\equiv \left(\frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}} - \left(\gamma_{t+1}^1 - 1 + 2\eta\right)\right)\left(1-\gamma_{t+1}^1\right)^{1-\alpha} \\ & \quad - \left(1 + \gamma_{t+1}^1 - \frac{\phi\left(\gamma_t^1 + 2\eta - 1\right)}{\left(1-\gamma_t^1\right)^{1-\alpha}}\right)^{1-\alpha}\left(\gamma_{t+1}^1 + 2\eta - 1\right) = 0 \end{aligned}$$

We evaluate $F$ at $(0,0)$, which gives $F(0,0) = (2\eta-1)\left[\phi - 1 - (1-\phi(2\eta-1))^{1-\alpha}\right]$. We know that $\eta > \frac{1}{2}$ so $(2\eta-1) > 0$. Also we know that $F$ is strictly decreasing in $\gamma_{t+1}^1$ and strictly increasing in $\gamma_t^1$. Therefore, it is sufficient to show that $F(0,0) < 0$. If this is true, then when $\gamma_t^1 = 0$ then the corresponding value of $\gamma_{t+1}^1$ must be strictly negative. But $F(0,0)$ if and only if $\phi - 1 < (1-\phi(2\eta-1))^{1-\alpha}$. To show this we proceed by contradiction. Suppose then that $\phi - 1 \geq (1-\phi(2\eta-1))^{1-\alpha} = (\phi + 1 - 2\phi\eta)^{1-\alpha}$. However, since $\phi < \frac{1}{\eta}$ (recalling that $\min\left\{\frac{1}{\eta}, \frac{1}{1-\eta}\right\} = \frac{1}{\eta}$ for $\eta > \frac{1}{2}$) then $\phi\eta < 1$ and so $-2\phi\eta > -2$, and so $1 - 2\phi\eta > -1$. Since $1-\alpha > 0$ then $(\phi + 1 - 2\phi\eta)^{1-\alpha} > (\phi - 1)^{1-\alpha}$. Putting things together we get that $\phi - 1 \geq (1-\phi(2\eta-1))^{1-\alpha} = (\phi + 1 - 2\phi\eta)^{1-\alpha} > (\phi-1)^{1-\alpha}$. But since $1-\alpha < 1$ then this implies that $\phi - 1 > 1$ or $\phi > 2$. But $\phi < \min\left\{\frac{1}{\eta}, \frac{1}{1-\eta}\right\} \leqq 2$, a contradiction. Thus $\phi - 1 < (1-\phi(2\eta-1))^{1-\alpha}$ as desired. Hence $F(0,0) < 0$ and so, when $\gamma_t^1 = 0$ then $\gamma_{t+1}^1 < 0$.

- **Case 2:** $\eta < \frac{1}{2}$.

The proof here follows similar arguments, so we just sketch part of it. First note that $\gamma^1 > 1 - 2\eta > 0$. Then it remains to show that $\gamma^2 > 0 > 2\eta - 1$. The difference in the procedure is that we will work with $\gamma_t^2$ as the variable instead of $\gamma_t^1$. Recall that the equilibrium conditions were:

$$\frac{p_t}{p_{t+1}} = \frac{\left(\gamma_{t+1}^1 + \gamma_{t+1}^2\right)}{\left(\gamma_t^1 + \gamma_t^2\right)} = \frac{q_t R_t^2}{1+q_t}; \quad q_t = \frac{\gamma_t^2 + 1 - 2\eta}{\gamma_t^1 + 2\eta - 1}$$

and

$$\left(1-\gamma_t^1\right)^{1-\alpha}\left(\gamma_t^2+1-2\eta\right)=\left(1-\gamma_t^2\right)^{1-\alpha}\left(\gamma_t^1+2\eta-1\right)$$

>From the first three equations we get

$$\frac{\left(\gamma_{t+1}^1+\gamma_{t+1}^2\right)}{\left(\gamma_t^1+\gamma_t^2\right)}=\frac{q_t}{1+q_t}\left(\frac{\phi}{\left(1-\gamma_t^2\right)^{1-\alpha}}\right)=\left(\frac{\gamma_t^2+1-2\eta}{\gamma_t^1+\gamma_t^2}\right)\left(\frac{\phi}{\left(1-\gamma_t^2\right)^{1-\alpha}}\right)$$

Therefore the dynamical system describing the equilibrium (which is equivalent to the one presented at the beginning of this proof) is

$$\left(\gamma_{t+1}^1+\gamma_{t+1}^2\right)=\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}$$

$$\left(1-\gamma_t^1\right)^{1-\alpha}\left(\gamma_t^2+1-2\eta\right)=\left(1-\gamma_t^2\right)^{1-\alpha}\left(\gamma_t^1+2\eta-1\right)$$

We proceed as before, reducing this system to a one-dimensional system in $\gamma_t^2$. From the first equation

$$\gamma_{t+1}^1=\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}-\gamma_{t+1}^2;\quad 1-\gamma_{t+1}^1=1+\gamma_{t+1}^2-\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}$$

and replacing in the second equation forwarded one period gives:

$$\begin{aligned}&\left(1+\gamma_{t+1}^2-\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}\right)^{1-\alpha}\left(\gamma_{t+1}^2+1-2\eta\right)\\ =\;&\left(1-\gamma_{t+1}^2\right)^{1-\alpha}\left(\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}-\gamma_{t+1}^2+2\eta-1\right)\end{aligned}$$

We then define

$$\begin{aligned}&G\left(\gamma_t^2,\gamma_{t+1}^2\right)\\ \equiv\;&\left(1-\gamma_{t+1}^2\right)^{1-\alpha}\left(\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}-\gamma_{t+1}^2+2\eta-1\right)\\ &-\left(1+\gamma_{t+1}^2-\frac{\phi\left(\gamma_t^2+1-2\eta\right)}{\left(1-\gamma_t^2\right)^{1-\alpha}}\right)^{1-\alpha}\left(\gamma_{t+1}^2+1-2\eta\right)\end{aligned}$$

Therefore an equilibrium path is characterized by $G\left(\gamma_t^2,\gamma_{t+1}^2\right)=0$, which implicitly defines a function $\gamma_{t+1}^2\left(\gamma_t^2\right)$ provided that the conditions for the

Implicit Function Theorem hold. Following identical arguments as in the lemma before, it can be shown that $G_{\gamma^2_{t+1}}\left(\gamma^2_t, \gamma^2_{t+1}\right) < 0$ and $G_{\gamma^2_t}\left(\gamma^2_t, \gamma^2_{t+1}\right) > 0$ for all $\gamma^2_t$ and $\gamma^2_{t+1}$ less than one and strictly greater than $2\eta - 1$. Hence $\gamma^2_{t+1}\left(\gamma^2_t\right)$ is well defined and $\frac{d\gamma^2_{t+1}}{d\gamma^2_t} > 0$. In a similar fashion as in case 1, it is easy to prove that if $\gamma^2_{t+1} \to 1$ then $\gamma^2_t \to \widehat{\gamma} < 1$, which implies that the steady state $\gamma^2 > 2\eta - 1$ is locally unstable. This concludes the proof of the proposition.■

## A.4 Model with Arrow securities only and the objective function.

In the economy with only Arrow securities the objective function for bank 1 includes an expression equal to

$$\int\limits_{\pi^*_{11}}^{1} \left\{ \pi \log \gamma^1 + (1-\pi) \log \left[ \left(1-\gamma^1\right) R_1 - \frac{q_1}{q_2} \theta^2_1 \right] \right\} f(\pi)\, d\pi$$

which after replacing in $\theta^1_2$, $R_1$ and $\frac{q_1}{q_2}$ by the expressions obtained above it is

$$\int\limits_{\pi^*_{11}}^{1} \left\{ \pi \log \gamma^1 + (1-\pi) \log \left[ \phi \left(1-\gamma^1\right)^{\alpha-1} \left( 1 - \gamma^1 - (1-\eta) \int_{\pi^*_{1,2}}^{1} F(\pi)\, d\pi \right) \right] \right\} f(\pi)\, d\pi$$

But note that when $\pi^*_{1,1} = 1$ it is because $1-\gamma^1 = (1-\eta)\int_{\pi^*_{1,2}}^{1} F(\pi)\, d\pi$. This expression would be equal to an expression which is indeterminate, since $\int\limits_{1}^{1} (1-\pi) f(\pi)\, d\pi = 0$ and $\log[0] = -\infty$. However it can be shown that when $\gamma^1$ converges (from above) to $1 - (1-\eta)\int_{\pi^*_{1,2}}^{1} F(\pi)\, d\pi$ then the expression

$$\log \left[ 1 - \gamma^1 - (1-\eta) \int_{\pi^*_{1,2}}^{1} F(\pi)\, d\pi \right] \int\limits_{\pi^*_{11}}^{1} (1-\pi) f(\pi)\, d\pi$$

converges to 0 for $\gamma^1$ sufficiently close to $1 - (1-\eta)\int_{\pi^*_{1,2}}^{1} F(\pi)\, d\pi$. Hence the equilibrium with $\pi^*_{jj} = 1$ exists as long as we define the equilibrium value of the expression in the objective function mentioned above equal to its limit (equal to 0).

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