

Information, Trade  
And Incomplete Markets

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**Abstract:** The famous no-trade result of Milgrom and Stokey [5] is often used to argue that in ongoing security markets the arrival of new information cannot generate trade. The usual intuition for this claim is that if traders begin with a Pareto optimal allocation of resources then any trade is for speculative purposes so the willingness of one trader to make a trade indicates to other traders that they should not accept the other side of the trade. This intuition is about a trading game and not about market equilibrium. Nonetheless the no-trade theorem does apply to competitive markets under some assumptions. These assumptions are that traders agree about how to interpret new information and that markets are complete. We show that these assumptions are essentially necessary as well as sufficient for no-trade. In economies in which markets are incomplete and or traders have differing beliefs generically the arrival of new information does generate trade.

# 1 Introduction

A standard interpretation of the No-Trade Theorem in the fundamental paper by Milgrom and Stokey [5] is that the arrival of new private information cannot generate trade between rational traders in an ongoing security market. The usual intuition offered for this theorem is as follows. Suppose that traders' initial security holdings are Pareto optimal and that some traders receive new information. If this new information generates trade then the trade must be for speculative purposes. But any trader who has not received the information would not want to trade as each such trader would know that he is being taken advantage of by the informed traders.

This reasoning is game theoretic; it is not based on the decision making that is usually employed in competitive markets. Under some circumstances the no-trade result applies to rational expectations equilibrium, but the logic above does not apply. There are two reasons for this. First, to make the argument above carefully requires setting up a game and employing a game theoretic equilibrium concept. This requires knowledge assumptions that are irrelevant in competitive equilibrium analysis. In a competitive market no trader reasons about, or needs to reason about, the knowledge, payoffs or rationality of any other trader. In a rational expectations equilibrium every trader makes rational inferences about relevant information from market statistics. But this requires only rationality and knowledge of the equilibrium relationship; it does not require any higher order knowledge assumptions. Second, if one attempts to employ the reasoning above one typically does not get a rational expectations equilibrium even in settings in which traders might trade for risk sharing purposes as well as for speculation. Blume and Easley [1] show that there is no game whose Bayes Nash equilibrium is a rational expectations equilibrium for a broad class of classical economies unless stringent restrictions are placed on the distribution

of information. These restrictions rule out economies in which one trader alone receives valuable private information.

Milgrom and Stokey [5] also provide a non-game theoretic analysis in which they show that if beliefs satisfy a restriction, and if traders begin with a Pareto optimal allocation of state contingent consumptions, then the arrival of information does not generate rational expectations equilibrium trade. This occurs because under the belief restriction they impose the arrival of information does not change marginal rates of substitution. Consumers had a common marginal rate of substitution at the original Pareto optimal allocation. Since the marginal rates of substitution are unchanged by the information the original allocation remains an equilibrium. Thus there is no trade even though beliefs have changed.

This result has two key assumptions: the belief restriction and the Pareto optimality of the original allocation. The belief restriction is that consumers share an understanding of the information generating process. In Bayesian terms a sufficient restriction is that they have a common likelihood function. In a standard statistical problem assuming a correct, and therefore common, likelihood function is natural. It is less natural in a consumer decision problem. Suppose one begins with consumers who have preferences over random consumptions defined on a state space consisting of the payoff relevant states and the signals. Then we typically derive a consumer's beliefs over the space of signals and states as well as his utility function from the expected utility theorem. This approach places no restrictions on beliefs. We show that without the belief restriction the arrival of new information generically generates trade even when consumers begin with a Pareto optimal allocation.

The Pareto optimality of the initial allocation is usually interpreted as arising from a previous round of trade on competitive markets. The first welfare theorem implies that if the markets are complete then any competitive equi-

librium allocation will be Pareto optimal. So in an ongoing complete market, in which there are no new liquidity reasons for trade (no exogenous changes in endowments), the initial allocation at each trading date will be Pareto optimal. Numerous authors, including Milgrom and Stokey [5], note that completeness of markets matters for this claim. We show that if markets are incomplete then generically the arrival of new information does generate trade. This occurs because with incomplete markets the arrival of new information creates new insurance opportunities.

The No-Trade result is often cited as a reason why noise traders are needed to generate trade or why behavioral, not fully rational, traders must be present in markets which experience significant volume of trade. There are good reasons to consider noise or behavioral traders, but our analysis shows that the argument that they must be there because we see more trade than can be accounted for by liquidity reasons is not as simple as it usually made out to be. It requires a restriction on how traders interpret information that goes beyond what is usually meant by an assumption of rationality. It also requires complete markets. We do not believe that either of these assumptions are descriptive.

## 2 The Model

Time is discrete and begins at  $t = 0$ . At time  $t = 1$  signals arrive and at time  $t = 2$  final states are realized. Time, signals and states are represented in a tree that begins with the initial node  $\xi_0$ . At time  $t = 1$  there are intermediate nodes corresponding to signals  $Z = \{\xi_1, \dots, \xi_L\}$  and at time  $t = 2$  there are states of the world  $\Theta = \{1, \dots, S\}$ .

At time 0 individuals trade  $J$  securities that have state contingent payoffs at time  $t = 2$ . At time  $t = 1$  after signals arrive individuals can retrade the securities. At time  $t = 2$  the securities pay off in the single physical good

available in the economy. Payoffs to securities depend only on the  $t = 2$  state. They are described by a  $S \times J$  matrix of security payoffs  $V$  whose  $(s, j)^{th}$  element  $V_s^j$  is the payoff of security  $j$  in state  $s$ . After the securities pay off, individuals consume the sum of their endowment in that state and the payoffs that result from their portfolio choice in periods 0 and 1; consumption occurs only at  $t = 2$ .

The security price vector at time  $t = 0$  is denoted  $q(\xi_0)$ . The security price vector at time  $t = 1$  if signal  $\xi_l$  occurs is denoted  $q(\xi_l)$ .

There are  $I$  individuals indexed by  $h$ . Individuals are described by utility functions, beliefs and endowments. Agent  $h$ 's (von Neumann-Morgenstern) utility of consumption in state  $s$  when signal  $\xi$  occurs is denoted by  $u^h(x_s^h(\xi))$ .

Her beliefs are described by a joint probability distribution over signals and states which is denoted by  $\pi^h : Z \times \Theta \rightarrow (0, 1)$ .

Her unconditional expected utility of consumption is therefore

$$U^h = E^h(u^h(x^h)) = \sum_{s,l>0} \pi^h(\xi_l, s) u^h(x_s^h(\xi_l))$$

And her expected utility of consumption, conditioning on signal  $\xi$  is denoted by

$$U_\xi^h = E^h(u(x^h)|\xi) = \sum_s \pi^h(s|\xi) u^h(x_s^h(\xi)) \text{ where } \xi \in Z$$

The following restriction on agents' beliefs will be needed in the statement of the main theorem.

**Axiom 1** For all  $\xi, h, h', s, s'$ ,

$$\frac{\pi^h(\xi|s)}{\pi^h(\xi|s')} = \frac{\pi^{h'}(\xi|s)}{\pi^{h'}(\xi|s')}.$$

This axiom states that likelihood ratios are equated across agents. If, as in Milgrom and Stokey [5], agents' beliefs are concordant (namely,  $\pi^h(\xi|s) =$

$\pi^{h'}(\xi|s)$  for all  $h, h', \xi, s$ , then this assumption is satisfied. The assumption of belief concordance loosely implies that agents interpret the meaning of signals in similar ways.

While agents' endowments are stochastic, we assume that they lie in the strictly positive orthant and that they are not affected by signals. Formally,

**Axiom 2**  $\omega^h \in \mathbb{R}_{++}^{SL}$  and for all  $\xi, \xi', h, s$ :  $\omega_s^h(\xi) = \omega_s^h(\xi')$ .

The space  $\Omega$  of endowments with this property is a  $SI$ -dimensional subspace of  $\mathbb{R}_{++}^{SLI}$ . For simplicity, we write  $\Omega = \mathbb{R}_{++}^{SI}$ . We also assume the following about agents' von Neumann-Morgenstern utility.

**Axiom 3** For all  $h$ ,  $u^h : \mathbb{R}_+ \rightarrow \mathbb{R}$  is continuous on  $\mathbb{R}_+$ , infinitely differentiable on  $\mathbb{R}_{++}$ , strictly increasing and concave.

This technicality will be useful later in the proof of the main theorem. We may now introduce the following

**Definition 4** An Economy  $E$  is a collection of endowments, utilities, beliefs and a matrix of security payoffs  $\{\omega, u, \pi, V\}$ .

### 3 Symmetric Information

In this section, we investigate the case where information is revealed publicly at the intermediate node. Given an economy  $E$ , we assume the following timing of events. Before any information is revealed, economic agents, acting as price takers, trade real securities to achieve the best possible allocation of risk given the (possibly incomplete) security structure  $V$  and given their beliefs about the likelihood of contingencies (elements of states  $S$ ). These beliefs are described by probability distributions  $\pi^h$ . A signal (in the set  $Z$ ) is then revealed publicly

to all agents who use it to update their beliefs about the likelihood of the contingencies they face. Markets re-open, and agents retrade the contracts they originally signed in time period 0. In time period 2, contracts pay off and consumption occurs. This sequence of trades and the resulting equilibrium allocation of goods, securities and equilibrium prices is captured by the usual Arrow-Debreu equilibrium concept. As a matter of notation,  $(x, z, q)$  denotes the vector  $(x^1, \dots, x^I, z^1, \dots, z^I, q_1, \dots, q_J) \in \mathbb{R}^{ISL \times I(1+L)J \times (1+L)J}$ .

**Definition 5** *Given an economy  $E$ , an equilibrium is a collection  $(x, z, q)$  such that*

(i)  $(x^h, z^h) \in \arg \max U^h$  subject to

$$\begin{aligned} q(\xi_0)z^h(\xi_0) &= 0 \\ q(\xi)z^h(\xi) &= q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z \\ x^h(\xi) - \omega(\xi) &= Vz^h(\xi) \text{ for all } \xi \in Z \end{aligned}$$

(ii)  $\sum_{h=1}^I z^h(\xi_0) = 0, \sum_{h=1}^I z^h(\xi) = 0$  for all  $\xi \in Z$

The first line in the budget constraint means that agents in period 0 are constrained to finance the purchase of securities by short-selling other securities. The second line means that agents in period 1 are constrained to finance the purchase of new securities and capital gains payments on securities they short-sold in period 0 by short-selling new securities and by using the capital gains collected on period 0 purchases of securities. The third line means that agents in period 2 are constrained to finance their consumption and dividend payments on securities they short-sold in period 1 by using the dividend payments they receive on securities they purchased in period 1 and their endowment of the physical good. The second condition is a securities market clearing condition.

Assuming that information is symmetric formally means that a signal is publicly revealed in time period 1. We investigate the conditions under which agents have an incentive to change their equilibrium securities allocation from period 0 to period 1. There are  $J$  securities, none of which are redundant so  $J \leq S$  and  $V$  is of full rank. We say that markets are  $\Theta$ -complete if the matrix of security payoffs  $V$  is an  $S \times S$  matrix. The definition of  $\Theta$ -completeness emphasizes the notion that there are no explicit period 1 returns (dividends) on period 0 purchases or sales of securities besides capital gains. Note that  $\Theta$ -completeness of markets implies completeness in the usual sense if  $J \geq L$  and the following matrix has rank  $L$ :

$$\begin{bmatrix} q_1(\xi_1) & \dots & q_J(\xi_1) \\ \dots & \dots & \dots \\ q_1(\xi_L) & \dots & q_J(\xi_L) \end{bmatrix}$$

Evidently, when markets are  $\Theta$ -incomplete, they can never be complete in the usual sense. The first result we present implies that agents have no incentive to retrade in time period 1 the securities they've already traded in time period 0 if and only if their beliefs satisfy assumption A.1. We say that an equilibrium involves no retrade if

$$z^h(\xi_0) = z^h(\xi) \text{ for all } \xi, h$$

We first restate a no trade claim by Milgrom and Stokey [5], recast in a market setting. The claim by Milgrom and Stokey [5] relies on the assumption that agents begin with an ex ante Pareto optimal allocation and trade on  $\Theta$ -complete markets once information arrives. In our definition of an equilibrium, agents have no date zero endowment and acquire securities on  $\Theta$ -complete markets before information arrives.  $\Theta$ -completeness does not guarantee market completeness and therefore the resulting consumption after the first round of

trade may not be ex ante Pareto optimal. As we shall see in this case, because of restrictions on beliefs and endowments,  $\Theta$ -completeness is sufficient to ensure ex ante Pareto optimality.

**Theorem 6** *Suppose axioms 2 and 3 are satisfied. Consider an economy  $E$  where markets are  $\Theta$ -complete. Then axiom 1 is satisfied if and only if there exists a no retrade equilibrium.*

All proofs are given in the appendix.

We now investigate agents' incentive to retrade when markets are incomplete. When the number of signals or states is too few, it is easy to give examples where no retrade occurs, regardless of assumptions made about market completeness or belief concordance. So, in the statement and proof of the next theorem, we impose the following assumption to ensure that the market and information structure is sufficiently rich. All assumptions listed below are used in the proof of the next theorem.

**Axiom 7** (i)  $S \geq 3, L \geq 2, I \geq 2$

(ii) *For all states  $s \in S$ , there exists a security  $k$  so that  $V_s^k > 0$*

(iii) *There exists at least one agent  $h$  so that the matrix  $\{\pi^h(\xi|\cdot)\}_{\xi \in Z}$  is of full rank*

(iv) *There exists a security that pays off in one state only.*

Assumption A.7.(ii) means that agents can transfer income to every state. We ignore states under autarchy since they add nothing to the analysis. Assumption A.7.(iii) means that at least one agent believes she can impute distinct meanings to distinct signals.<sup>1</sup> Assumption A.7.(iv) is a technical condition needed in the proof. We now state the main theorem of this paper.

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<sup>1</sup>In the proof of the main theorem, we use the condition that  $L \geq S - 1$  and A.7.iii to ensure that the following matrix is invertible.

**Theorem 8** *Suppose axioms 2,3 and 7 are satisfied. Consider an economy  $E$  where markets are  $\Theta$ -incomplete and where  $L \geq S - 1$  then there exists a set of full measure  $\Omega^* \subset \Omega$  such that if  $\omega \in \Omega^*$  and if an equilibrium exists then it must involve retrade.*

$J < S$  ensures that markets are  $\Theta$ -incomplete. If no additional information structure is added to our definition of an equilibrium, this theorem states that agents will generically decide to rebalance their portfolios in period 1 after a public signal is observed. Note that this theorem does not use axiom 1. In fact, agents may hold any beliefs and the theorem still holds, provided the other sufficient conditions listed are satisfied. In particular, any belief restriction may hold: agents may have concordant or identical beliefs or their beliefs may satisfy some complicated condition. So, although agents may agree on how to interpret the arrival of information, retrade will generically occur because markets are incomplete. In our model, this means that observing a non zero trade volume after public information is revealed doesn't necessarily mean that agents disagree about the meaning of the information they observe.

In the Rational Expectations literature, prices reveal all private information as long as the securities are real. Assuming that a result like Radner [6] is applicable here, this theorem would then state that as long as agents have forecast functions relating prices to signals and all private information is revealed through prices then agents will retrade after information is revealed.<sup>2</sup>

$$\begin{bmatrix} \pi^h(\xi_1|1) & \cdots & \pi^h(\xi_1|S-1) \\ \vdots & \ddots & \vdots \\ \pi^h(\xi_{S-1}|1) & \cdots & \pi^h(\xi_{S-1}|S-1) \end{bmatrix}$$

This is true if and only if the following matrix is invertible

$$\begin{bmatrix} \pi^h(1|\xi_1) & \cdots & \pi^h(1|\xi_{S-1}) \\ \vdots & \ddots & \vdots \\ \pi^h(S-1|\xi_1) & \cdots & \pi^h(S-1|\xi_{S-1}) \end{bmatrix}$$

and  $\frac{\Pi \pi^h(\xi_i)}{\Pi \pi^h(s_i)} \neq 0$ .

<sup>2</sup>Radner's result cannot be used directly since he shows that prices generically reveal private information in equilibrium when agents maximize a conditional expected utility, conditional

The key feature in the proof of this theorem remains valid if we assume more generally an incomplete markets economy with real or nominal assets and multiple consumption goods in each state that are traded on spot markets. Indeed, the key feature in the proof when there is a unique consumption good is that when markets are incomplete, agents' marginal rates of substitution generically differ in the space of endowments (the endowment set where this occurs is denoted by  $\Omega^*$ ). This property remains when there are many goods.

## 4 Asymmetric Information

Suppose that each agent receives information that doesn't reveal precisely the true signal at the intermediate node. The timing of events is identical to that described in the previous section except that signals are revealed through agents' information structures. Associate with each agent an information partition  $Z^h$  of the signal space. So,  $Z^h$  is a collection of disjoint subsets of  $Z$  whose union is  $Z$ . Let  $Y$  be the meet of these partitions (this means that for every  $\sigma \in Z^h$ , there is an  $\gamma \in Y$  so that  $\sigma \subset \gamma$  and  $Y$  is the finest such partition). Next, we define an equilibrium where information is asymmetric and where decisions are required to be measurable with respect to individual information partitions.

**Definition 9** *Given an economy  $E$ , an equilibrium with asymmetric information is a collection  $(x, z, q)$  such that*

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on their private information and the information prices convey about others' information. In our case, agents maximize an ex ante expected utility. Information arrives only later.

(i)  $(x^h, z^h) \in \arg \max U^h$  subject to

$$\begin{aligned} q(\xi_0)z^h(\xi_0) &= 0 \\ q(\xi)z^h(\xi) &= q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z \\ z^h(\xi) &= z^h(\xi') \text{ for all } \xi, \xi' \in \sigma \in Z^h \\ x^h(\xi) - \omega^h(\xi) &= Vz^h(\xi) \text{ for all } \xi \in Z \end{aligned}$$

$$(ii) \sum_{h=1}^I z^h(\xi_0) = 0, \sum_{h=1}^I z^h(\xi) = 0, \sum_{h=1}^I x^h(\xi) - \omega^h(\xi) = 0 \text{ for all } \xi \in Z.$$

The assumption of measurability is essential for the problem to be backward solvable. Also, the condition is used in a crucial way in the proof of the theorem below. There does not exist a direct analog to theorem 6 in  $\Theta$ -complete asymmetric information economies. However, one can easily prove the following

**Theorem 10** *Suppose axioms 2 and 3 are satisfied. Consider an economy  $E$  where markets are  $\Theta$ -complete. If axiom 1 is satisfied, there exists a no retrade equilibrium with asymmetric information for any information structure.*

**Proof.** By theorem 6, there exists a no retrade equilibrium  $(x, z, q)$ . It is also an asymmetric equilibrium for any information structure since the optimization problem in the asymmetric case has more constraints and these constraints are satisfied. ■

Note that a converse to this theorem does not exist in general. If, for example, all agents have the coarsest possible information partition, then it's clear that a no retrade equilibrium with asymmetric information does not imply any belief restrictions.

In asymmetric information economies where markets are  $\Theta$ -incomplete, we require an analog to axiom 7. Formally,

**Axiom 11** (i)  $S \geq 3, L \geq 2, I \geq 2$

(ii) For all states  $s \in S$ , there exists a security  $k$  so that  $V_s^k > 0$

(iii) There exists at least one agent  $h$  so that the matrix  $\{\pi^h(\gamma|\cdot)\}_{\gamma \in Y}$  is of full rank

(iv) There exists a security that pays off in one state only.

The notation in part (iii) means

$$\pi^h(\gamma|s) = \sum_{\xi \in \gamma} \pi^h(\xi|s)$$

We now state the following

**Theorem 12** *Suppose axioms 2,3 and 11 are satisfied. Consider an economy  $E$  where markets are  $\Theta$ -incomplete and where  $\#Y \geq S - 1$  then there exists a set of full measure  $\Omega^* \subset \Omega$  such that if  $\omega \in \Omega^*$  and if an equilibrium with asymmetric information exists then it must involve retrade.*

The revelation of information through information partitions can be given a few different interpretations. Different information partitions may be interpreted as the outcome of different information gathering techniques (see ch.2 of Marschak and Radner [4]). Radner [6] calls agents unsophisticated if they cannot extract more information through prices. If however, they are capable of extracting information through equilibrium prices then this theorem states that in the eventuality that all information is not fully revealed through prices,<sup>3</sup> retrade will still occur, as long as the meet of the information partitions is rich enough.

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<sup>3</sup>See for example Rahi [7] and Citanna and Villanacci [2] for examples closely related to the setup of this model. They exploit the "serious" indeterminacy found in nominal economies to show the existence of equilibria with all degrees of revelation.

## 5 More Theorems

### 5.1 Generating an Arbitrary Volume of Trade

Suppose that the assumptions of theorem 8 are satisfied. Choose endowments in a general position. Then for some agent  $h'$ , some security  $j'$  and some signal  $\xi' \in Z$ ,

$$z_{j'}^{h'}(\xi_0) \neq z_{j'}^{h'}(\xi)$$

If  $(x, z, q)$  is the equilibrium in question, there must exist a vector  $\lambda \in \mathbb{R}_{++}^{I(L+1)}$  such that the following conditions hold:

$$\begin{aligned} \lambda^h(\xi_0)q_j(\xi_0) &= \sum_{\xi \in Z} \lambda^h(\xi)q_j(\xi) \text{ for all } j, h \\ \lambda^h(\xi)q_j(\xi) &= \sum_{s \in \Theta} \pi^h(s, \xi)u^{h'}(x_s^h(\xi))V_s^j \text{ for all } j, h, \xi \neq \xi_0 \\ x_s^h(\xi) &= \omega_s^h + V_s z^h(\xi) \text{ for all } j, h, \xi \neq \xi_0 \\ \sum_{h \in I} z_j^h(\xi) &= 0 \text{ for all } j, h, \xi \end{aligned}$$

Let  $\varepsilon > 0$ . Then set

$$\begin{aligned} \hat{V}_s^j &= \begin{cases} \varepsilon V_s^j & \text{if } j = j' \\ V_s^j & \text{if not} \end{cases} \text{ for all } s \\ \hat{q}_j(\xi) &= \begin{cases} \varepsilon q_j(\xi) & \text{if } j = j' \\ q_j(\xi) & \text{if not} \end{cases} \text{ for all } \xi \\ \hat{z}_j^h(\xi) &= \begin{cases} \frac{1}{\varepsilon} z_j^h(\xi) & \text{if } j = j' \\ z_j^h(\xi) & \text{if not} \end{cases} \text{ for all } \xi, h \\ \hat{x}^h &= x^h \text{ for all } h \end{aligned}$$

Then  $(\hat{x}, \hat{z}, \hat{q})$  is still an equilibrium since all market clearing conditions still

hold and the multipliers  $\lambda$  still satisfy the new optimality conditions. Note that trade from node  $\xi_0$  to  $\xi'$  can be made arbitrarily large. Indeed, as  $\varepsilon \rightarrow 0$ ,  $\left| \hat{z}_{j'}^{h'}(\xi_0) - \hat{z}_{j'}^{h'}(\xi') \right| = \frac{1}{\varepsilon} \left| z_{j'}^{h'}(\xi_0) - z_{j'}^{h'}(\xi') \right| \rightarrow \infty$ . The intuition behind the result is that as the payoff of security  $j'$  tends to zero, a greater volume of trade must be accomplished in order to keep the optimal consumption bundle  $x$  intact. As it's payoff goes to zero, so does it's price.

## 5.2 Weakening the Assumptions of the Main Theorem

Assumptions *A.7.iii*, *A.7.iv* and  $L \geq S - 1$  can be weakened and the retrade result of theorem 8 still obtains. The proof of theorem 8 is essentially a transversality result. There are  $S + 1$  equations ( $S$  for market clearing and one for a condition obtained by assuming that no trade occurs) in  $S$  unknowns (the underlying Arrow Debreu prices). The difficulty lies in showing the rank condition on the  $(S + 1)$ st equation. If all other assumptions are kept, assumption *A.7.iii* can be changed and  $L \geq S - 1$  dropped if we assume there exists an agent  $h$ , and a signal  $\xi \in Z$  such that the vector

$$\psi^h(\{1, \dots, J\}) = \left( \pi^h(\xi|1) \sum_{k \in \{1, \dots, J\}} V_1^k, \dots, \pi^h(\xi|S) \sum_{k \in \{1, \dots, J\}} V_S^k \right)$$

lies outside the span of the matrix  $V$ .<sup>4</sup>

All three conditions can be dropped if we assume that  $J \leq S - 2$  and that there exists an agent  $h$  and a signal  $\xi \in Z$  such that the vectors  $\psi^h(J_1), \psi^h(J_2)$  lie outside the span of  $V$  and are noncolinear for some subsets  $J_1 \neq J_2 \subset \{1, \dots, J\}$ .<sup>5</sup>

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<sup>4</sup>For any subset of states  $S_{J+1} = \{1, \dots, J + 1\} \subset S$ , if conditional beliefs of agent  $h$  ( $\pi^h(\xi|1), \dots, \pi^h(\xi|J + 1)$ ) satisfy the following condition, then the vector  $\psi^h$  lies outside the span of  $V$ .

$$\det M \neq 0$$

Where  $M$  is the  $J + 1$  columns of the  $(J + 1) \times S$  matrix  $\begin{bmatrix} V \\ \psi^h(\{1, \dots, J\}) \end{bmatrix}$  corresponding to elements of  $S_{J+1}$ .

<sup>5</sup>This can be checked if we assume that there exist two distinct subsets of securities (call

These conditions would ensure local control over the  $(S + 1)$ st equation<sup>6</sup>:

$$\frac{\sum_s \pi^h(\xi|s) \pi_s^h \left( \sum_{k \in A} V_s^k \right)}{\sum_s \pi^h(\xi|s) \pi_s^h \left( \sum_{k \in B} V_s^k \right)} = \frac{\sum_s \pi^h(\xi|s) \pi_s^1 \left( \sum_{k \in A} V_s^k \right)}{\sum_s \pi^h(\xi|s) \pi_s^1 \left( \sum_{k \in B} V_s^k \right)}$$

## 6 Concluding Comments

Note the following immediate implications of our theorems.

a) Information is valuable when markets are incomplete. At least one agent who would be willing to pay for the arrival of information even if that information were immediately made available to others. This means that an agent may benefit from the costly acquisition of information even if she has no "monopoly" over the information.

b) Information is never valuable when markets are complete. In this case, no agent would be willing to pay for the arrival of information in the intermediate node, since intermediate markets will never involve a rebalancing of portfolios for purely informational reasons.

c) The arrival of public or private information revealed through rational market mechanisms may generate arbitrarily large retrade when markets are them  $A$  and  $B$  whose aggregate payoff is non-zero on a subset of  $J + 2$  states  $S_{J+2}$ , and an agent  $h$  whose conditional beliefs  $\pi^h(\xi|\cdot)$  satisfy the following condition.

$$\det M \neq 0$$

where  $M$  is the  $J + 2$  columns of the  $(J + 2) \times S$  matrix  $\begin{bmatrix} V \\ \psi^h(A) \\ \psi^h(B) \end{bmatrix}$  corresponding to the set  $S_{J+2}$ .

<sup>6</sup>Controlling this equation in the transversality argument means that we can find a  $d\pi^h \in \langle V \rangle^\perp$  so that

$$\sum_s \pi^h(\xi|s) d\pi_s^h \left( \sum_{k \in A} V_s^k \right) \neq \frac{\sum_s \pi^h(\xi|s) \pi_s^1 \left( \sum_{k \in A} V_s^k \right)}{\sum_s \pi^h(\xi|s) \pi_s^1 \left( \sum_{k \in B} V_s^k \right)} \sum_s \pi^h(\xi|s) d\pi_s^h \left( \sum_{k \in B} V_s^k \right)$$

incomplete.

When markets are complete, agents' likelihood ratios are equated if and only if a no retrade equilibrium exists. This result recasts Milgrom and Stokey's result in a market setting. While they show that no retrade occurs in the second of two rounds of Pareto optimal trading, we show the same result in a market setting with  $\Theta$ -complete markets. When markets are  $\Theta$ -incomplete, retrade generically occurs regardless of any belief restriction. This implies that although agents may agree on the meaning of signals, they will still want to retrade when new information arrives because new insurance opportunities arise. So, the theory suggests that observing frequent trade or large trade volume need not imply that investors interpret information differently or that some investors are irrational.

## 7 Appendix

In the course of the proofs of the theorems, we'll need the following definitions and lemmas. As a matter of notation,  $x(\xi_0)$  is a vector of consumption that lies in  $\mathbb{R}^{IS}$ . Let

$$U_0^h = \sum_{s=1}^S \pi^h(s) u^h(x_s^h)$$

**Definition 13** *Given an economy  $E$ , an Sequential Equilibrium is collection  $(x(\xi_0), x, z, q)$  where  $x = (x(\xi))_{\xi \in Z} \in \mathbb{R}^{ISL}$  such that for all  $h \in \{1, \dots, I\}$  :*

(i)  $(x^h(\xi_0), z^h(\xi_0)) \in \arg \max U_0^h$  subject to

$$\begin{aligned} q(\xi_0) z^h(\xi_0) &= 0 \\ x^h(\xi_0) - \omega^h(\xi_0) &= Vz^h(\xi_0) \end{aligned}$$

(ii) For all  $\xi \in Z$ ,  $((x^h(\xi), z^h(\xi)) \in \arg \max U_\xi^h$  subject to

$$\begin{aligned} q(\xi) z^h(\xi) &= q(\xi) z^h(\xi_0) \\ x^h(\xi) - \omega^h(\xi) &= Vz^h(\xi) \end{aligned}$$

(iii)  $\sum_{h=1}^I z^h(\xi) = 0$ ,  $\sum_{h=1}^I x^h(\xi) - \omega^h(\xi) = 0$  for all  $\xi \in Z \cup \{\xi_0\}$ .

**Lemma 14** *Suppose that  $(x(\xi_0), x, z, q)$  is a no retrade sequential equilibrium.*

*Then,  $(x, z, q)$  a no retrade equilibrium.*

**Proof.** If  $(x(\xi_0), x, z, q)$  is a no retrade sequential equilibrium, then the following equalities must hold.

$$\begin{aligned}
z^h(\xi_0) &= z^h(\xi_l) \text{ for all } l > 0 \\
q(\xi_0)z^h(\xi_0) &= 0 \\
q(\xi_l)z^h(\xi_l) &= q(\xi_l)z^h(\xi_0) \text{ for all } l > 0 \\
\sum_h z^h(\xi_l) &= 0 \text{ for all } l
\end{aligned}$$

And therefore,  $x(\xi_0) = x(\xi)$  for all  $\xi \in Z$ . Also, there must exist a vector  $\lambda^h = (\lambda^h(\xi_0), \dots, \lambda^h(\xi_L), \lambda^h(1), \dots, \lambda^h(S))$ ,  $\lambda^h(1, \xi_1), \dots, \lambda^h(S, \xi_L) \in \mathbb{R}_{++}^{(S+1)(L+1)}$  so that

$$\begin{aligned}
\lambda^h(\xi_0)q_k(\xi_0) &= \sum_s \lambda^h(s)V_s^k \\
\lambda^h(s) &= \pi^h(s)u^{h'}(x_s^h) \\
\lambda^h(\xi_l)q_k(\xi_l) &= \sum_s \lambda^h(s, \xi_l)V_s^k \\
\lambda^h(s, \xi_l) &= \pi^h(s|\xi_l)u^{h'}(x_s^h)
\end{aligned}$$

Where

$$x_s^h = \omega_s^h + V_s z^h(\xi_0)$$

So:

$$\begin{aligned}
\lambda^h(\xi_l)q_k(\xi_l) &= \sum_s \pi^h(s|\xi_l)u^{h'}(x_s^h)V_s^k \\
\pi^h(\xi_l)\lambda^h(\xi_l)q_k(\xi_l) &= \sum_s \pi^h(s, \xi_l)u^{h'}(x_s^h)V_s^k
\end{aligned}$$

And:

$$\lambda^h(\xi_0)q_k(\xi_0) = \sum_s \pi^h(s)u^{h'}(x_s^h)V_s^k$$

So that:

$$\begin{aligned}
\lambda^h(\xi_0)q_k(\xi_0) &= \sum_s \left( \sum_{l>0} \pi^h(s, \xi_l) \right) u^{h'}(x_s^h) V_s^k \\
&= \sum_{l>0} \left( \sum_s \pi^h(s, \xi_l) u^{h'}(x_s^h) V_s^k \right) \\
&= \sum_{l>0} \pi^h(\xi_l) \lambda^h(\xi_l) q_k(\xi_l)
\end{aligned}$$

Setting

$$\begin{aligned}
\mu^h(\xi_l) &= \pi^h(\xi_l) \lambda^h(\xi_l) \\
\mu^h(\xi_0) &= \lambda^h(\xi_0)
\end{aligned}$$

We obtain the FOCs for the optimization problem of each agent  $h$

$$\begin{aligned}
\mu^h(\xi_0)q_k(\xi_0) &= \sum_{l>0} \mu^h(\xi_l)q_k(\xi_l) \\
\mu^h(\xi_l)q_k(\xi_l) &= \sum_s \pi^h(s, \xi_l) u^{h'}(x_s^h) V_s^k
\end{aligned}$$

The other FOCs (the budget constraints) and securities market clearing hold.

■

**Lemma 15** *Suppose that  $(x, z, q)$  is a no retrade equilibrium. Then, there exist  $(x(\xi_0))$  so that  $(x(\xi_0), x, z, q)$  is a no retrade sequential equilibrium.*

**Proof.** If  $(x, z, q)$  is a no retrade equilibrium, then the following equalities must hold.

$$\begin{aligned}
z^h(\xi_0) &= z^h(\xi_l) \text{ for all } l > 0 \\
q(\xi_0) \cdot z^h(\xi_0) &= 0 \\
q(\xi_l) \cdot z^h(\xi_0) &= q(\xi_l) \cdot z^h(\xi_l) \text{ for all } l > 0 \\
\sum_h z^h(\xi_l) &= 0 \text{ for all } l
\end{aligned}$$

Also, there must exist a vector  $\mu^h = (\mu^h(\xi_0), \dots, \mu^h(\xi_L)) \in \mathbb{R}_{++}^{L+1}$  such that

$$\begin{aligned}
\mu^h(\xi_0)q_k(\xi_0) &= \sum_{l>0} \mu^h(\xi_l)q_k(\xi_l) \\
\mu^h(\xi_l)q_k(\xi_l) &= \sum_s \pi^h(s, \xi_l)u^{h'}(x_s^h)V_s^k
\end{aligned}$$

Where  $x_s^h = x_s^h(\xi)$  for all  $\xi \in Z$ . Set  $x^h(\xi_0) = x^h(\xi)$ , for any  $\xi \in Z$ . So

$$\begin{aligned}
\mu^h(\xi_0)q_k(\xi_0) &= \sum_{l>0} \left( \sum_s \pi^h(s, \xi_l)u^{h'}(x_s^h)V_s^k \right) \\
&= \sum_s \pi^h(s)u^{h'}(x_s^h)V_s^k
\end{aligned}$$

Set

$$\begin{aligned}
\lambda^h(s) &= \pi^h(s)u^{h'}(x_s^h) \\
\lambda^h(\xi_0) &= \mu^h(\xi_0)
\end{aligned}$$

Then the FOCs to the first part of the sequential equilibrium are satisfied:

$$\begin{aligned}
\lambda^h(\xi_0)q_k(\xi_0) &= \sum_s \lambda^h(s)V_s^k \\
\lambda^h(s) &= \pi^h(s)u^{h'}(x_s^h)
\end{aligned}$$

Also:

$$\mu^h(\xi_l)q_k(\xi_l) = \sum_s \pi^h(s|\xi_l)\pi^h(\xi_l)u^{h'}(x_s^h)V_s^k$$

Set

$$\begin{aligned}\lambda^h(\xi_l) &= \frac{\mu^h(\xi_l)}{\pi^h(\xi_l)} \\ \lambda^h(s, \xi_l) &= \pi^h(s|\xi_l)u^{h'}(x_s^h)\end{aligned}$$

Then the FOCs to the second part of the sequential equilibrium are satisfied:

$$\begin{aligned}\lambda^h(\xi_l)q_k(\xi_l) &= \sum_s \lambda^h(s, \xi_l)V_s^k \\ \lambda^h(s, \xi_l) &= \pi^h(s|\xi_l)u^{h'}(x_s^h)\end{aligned}$$

■

Proof of theorem 6.

**Proof.** Note that existence of an equilibrium is not an issue in this theorem since markets are complete.

(Sufficiency) Suppose that  $(x, z(\xi_0), q(\xi_0))$  solves the following problem. For all  $h$ ,

$$\max_{x^h, z^h} \sum_s \pi^h(s) u^h(x_s^h)$$

Subject to

$$\begin{aligned} q(\xi_0) z^h(\xi_0) &= 0 \\ x^h - \omega^h &= V z^h(\xi_0) \end{aligned}$$

And

$$\sum_h z^h(\xi_0) = 0$$

Then, for all  $h$ , there exists a vector of multipliers  $\lambda^h(\xi_0) \in \mathbb{R}_{++}$  so that

$$-q_k(\xi_0) \lambda^h(\xi_0) + \sum_s \pi^h(s) u^{h'}(x_s^h) V_s^k = 0, \text{ for } k = 1, \dots, J$$

Since markets are complete there exists a unique  $\pi \in \mathbb{R}_{++}^S$  so that  $q(\xi_0) = \pi V$ . This implies that

$$\pi^h(s) = \frac{\pi_s \lambda^h(\xi_0)}{u^{h'}(x_s^h)}$$

After information is revealed, agents solve the following problem:

$$\max_{x_s^h, \xi_l, z^h(\xi_l)} \sum_s \pi^h(s|\xi_l) u^h(x_s^h(\xi_l))$$

Subject to

$$\begin{aligned} q(\xi_l)z^h(\xi_l) &= q(\xi_l)z^h(\xi_0) \\ x_s^h(\xi_l) - \omega_s^h &= V_s z^h(\xi_l) \end{aligned}$$

We now construct an equilibrium where the solution to this problem is

$$\begin{aligned} z^h(\xi_l) &= z^h(\xi_0) \\ x_s^h &= x_s^h(\xi_l) \end{aligned}$$

The assumption of the theorem implies the existence of a function  $g$  so that

$$\frac{\pi^1(\xi_l|s)}{\pi^h(\xi_l|s)} = g(h, \xi_l)$$

But

$$\pi^h(\xi_l|s) = \pi^h(s|\xi_l) \frac{\pi^h(\xi_l)}{\pi^h(s)}$$

So that

$$\pi^1(s|\xi_l) \frac{\pi^1(\xi_l)}{\pi^1(s)} = g(h, \xi_l) \pi^h(s|\xi_l) \frac{\pi^h(\xi_l)}{\pi^h(s)}$$

But

$$\pi^h(s) = \frac{\pi_s \lambda^h(\xi_0)}{u^{h'}(x_s^h)}$$

So that

$$\begin{aligned} \pi^1(s|\xi_l) \pi^1(\xi_l) \frac{u^{1'}(x_s^1)}{\pi_s \lambda^1(\xi_0)} &= g(h, \xi_l) \pi^h(s|\xi_l) \pi^h(\xi_l) \frac{u^{h'}(x_s^h)}{\pi_s \lambda^h(\xi_0)} \\ \pi^1(s|\xi_l) u^{1'}(x_s^1) \frac{\pi^1(\xi_l)}{\pi_s \lambda^1(\xi_0)} &= \pi^h(s|\xi_l) u^{h'}(x_s^h) \frac{g(h, \xi_l) \pi^h(\xi_l)}{\pi_s \lambda^h(\xi_0)} \\ \pi^1(s|\xi_l) u^{1'}(x_s^1) \left( \frac{\pi^1(\xi_l)}{\lambda^1(\xi_0)} \right) &= \pi^h(s|\xi_l) u^{h'}(x_s^h) \left( \frac{g(h, \xi_l) \pi^h(\xi_l)}{\lambda^h(\xi_0)} \right) \end{aligned}$$

Set

$$\begin{aligned}\lambda^1(\xi_l) &= \left( \frac{\pi^1(\xi_l)}{\lambda^1(\xi_0)} \right)^{-1} \\ \lambda^h(\xi_l) &= \left( \frac{g(h, \xi_l) \pi^h(\xi_l)}{\lambda^h(\xi_0)} \right)^{-1} \text{ for } h = 2, \dots, I\end{aligned}$$

So

$$\frac{\pi^1(s|\xi_l) u^{h'}(x_s^1)}{\lambda^1(\xi_l)} = \frac{\pi^h(s|\xi_l) u^{h'}(x_s^h)}{\lambda^h(\xi_l)}$$

Set

$$q_k(\xi_l) = \sum_s \frac{\pi^h(s|\xi_l) u^{h'}(x_s^h)}{\lambda^h(\xi_l)} V_s^k \text{ for } k = 1, \dots, J$$

These are the FOCs necessary and sufficient for the optimization problem of each agent, with  $x_s^h(\xi_l) = x_s^h$ . Since this implies that  $z^h(\xi_l) = z^h(\xi_0)$ , the budget constraints and equilibrium conditions hold trivially. We've just constructed a no retrade sequential equilibrium. Lemma 14 completes the proof.

(Necessity) Since markets are  $\Theta$ -complete, we think of  $V$  as an  $S \times S$  identity matrix, without loss of generality. By lemma 15, a no retrade equilibrium is a no retrade sequential equilibrium. The key FOCs from agents' optimization problems in a sequential equilibrium are:

$$\pi^h(s) u^{h'}(x_s^h) = q_s(\xi_0) \lambda^h(\xi_0) \text{ for all } s, h$$

$$\pi^h(s|\xi) u^{h'}(x_s^h) = q_s(\xi) \lambda^h(\xi) \text{ for all } s, h, \xi \in Z$$

So

$$\frac{\pi^h(s) u^{h'}(x_s^h)}{\lambda^h(\xi_0)} = \frac{\pi^j(s) u^{j'}(x_s^j)}{\lambda^j(\xi_0)} \text{ for all } h, j, s$$

$$\frac{\pi^h(s|\xi) u^{h'}(x_s^h)}{\lambda^h(\xi)} = \frac{\pi^j(s|\xi) u^{j'}(x_s^j)}{\lambda^j(\xi)} \text{ for all } h, j, s, \xi \in Z$$

Dividing these expressions implies that

$$\frac{\pi^h(s|\xi)\lambda^h(\xi_0)}{\pi^h(s)\lambda^h(\xi)} = \frac{\pi^j(s|\xi)\lambda^j(\xi_0)}{\pi^j(s)\lambda^j(\xi)} \text{ for all } h, j, s, \xi \in Z$$

Recall that

$$\frac{\pi^h(s|\xi)}{\pi^h(s)} = \frac{\pi^h(\xi|s)}{\pi^h(\xi)}$$

So that

$$\frac{\pi^h(\xi|s)\lambda^h(\xi_0)}{\pi^h(\xi)\lambda^h(\xi)} = \frac{\pi^j(\xi|s)\lambda^j(\xi_0)}{\pi^j(\xi)\lambda^j(\xi)} \text{ for all } h, j, s, \xi \in Z$$

Dividing by the corresponding expression for  $s'$  concludes the proof. ■

Proof of theorem 8.

**Proof.** Since markets are incomplete, we know there exists a set of full measure  $\Omega^* \subset \Omega$  so that if  $\omega \in \Omega^*$ , then agents' normalized gradients obtained from the first part of the sequential equilibrium problem are different (lemma 22, whose proof is adapted to this setting from the proof of theorem 11.6 in Magill and Quinzii, 1998). By way of contradiction, suppose that  $(x, z, q)$  is a no retrade equilibrium. Lemma 15 implies that  $(x(\xi_0), x, z, q)$  is a no retrade sequential equilibrium. This in turn implies that there exist scalars  $\lambda^h \in \mathbb{R}_{++}$  so that

$$q^k(\xi_0)\lambda^h(\xi_0) = \sum_s \pi^h(s)u^{h'}(x_s^h)V_s^k$$

Also, there exist  $\mu^h(\xi) \in \mathbb{R}_{++}$  for  $\xi \in Z$  so that

$$q^k(\xi)\mu^h(\xi) = \sum_s \pi^h(s|\xi)u^{h'}(x_s^h)V_s^k \quad (1)$$

Define

$$\pi_s^h \equiv \frac{\pi^h(s)u^{h'}(x_s^h)}{\lambda^h(\xi_0)} \quad (2)$$

By 1:

$$\frac{1}{\mu^h(\xi_0)} \sum_s \pi^h(s|\xi)u^{h'}(x_s^h)V_s^k = \frac{1}{\mu^1(\xi_0)} \sum_s \pi^1(s|\xi)u^{1'}(x_s^1)V_s^k$$

Using equation 2 yields:

$$\frac{\lambda^h(\xi_0)}{\mu^h(\xi_0)} \sum_s \frac{\pi^h(s|\xi)}{\pi^h(s)} \pi_s^h V_s^k = \frac{\lambda^1(\xi_0)}{\mu^1(\xi_0)} \sum_s \frac{\pi^1(s|\xi)}{\pi^1(s)} \pi_s^1 V_s^k$$

But

$$\frac{\pi^h(s|\xi)}{\pi^h(s)} = \frac{\pi^h(s, \xi)}{\pi^h(\xi)\pi^h(s)} = \frac{\pi^h(\xi|s)}{\pi^h(\xi)}$$

So that

$$\frac{\lambda^h(\xi_0)}{\mu^h(\xi_0)\pi^h(\xi)} \sum_s \pi^h(\xi|s)\pi_s^h V_s^k = \frac{\lambda^1(\xi_0)}{\mu^1(\xi_0)\pi^1(\xi)} \sum_s \pi^1(\xi|s)\pi_s^1 V_s^k$$

This in turn implies that

$$\frac{\sum_s \pi^h(\xi|s)\pi_s^h V_s^k}{\sum_s \pi^h(\xi|s)\pi_s^h V_s^j} = \frac{\sum_s \pi^1(\xi|s)\pi_s^1 V_s^k}{\sum_s \pi^1(\xi|s)\pi_s^1 V_s^j} \quad (3)$$

From our assumptions, we know there exists a security  $K \in \{1, \dots, J\}$  that pays off in only one state, say  $s_K \in \{1, \dots, S\}$ . Taking  $j = K$  in the equation above implies

$$\frac{\sum_s \pi^h(\xi|s)\pi_s^h V_s^k}{\pi_{s_K}^h} = \frac{\sum_s \pi^1(\xi|s)\pi_s^1 V_s^k}{\pi_{s_K}^1}$$

The price of security  $K$  in the first part of the sequential equilibrium satisfies

$$q_K(\xi_0) = \pi_{s_K}^h = \pi_{s_K}^1$$

So that

$$\sum_s \pi^h(\xi|s)\pi_s^h V_s^k = \sum_s \pi^h(\xi|s)\pi_s^1 V_s^k$$

Summing over securities implies that

$$\sum_s \pi^h(\xi|s)\pi_s^h \left( \sum_k V_s^k \right) = \sum_s \pi^h(\xi|s)\pi_s^1 \left( \sum_k V_s^k \right)$$

Using  $\pi_{s_K}^h = \pi_{s_K}^1$ , and assuming without loss of generality that  $S = s_K$ , we obtain

$$\sum_{s=1}^{S-1} \pi^h(\xi|s)\pi_s^h \left( \sum_k V_s^k \right) = \sum_{s=1}^{S-1} \pi^h(\xi|s)\pi_s^1 \left( \sum_k V_s^k \right)$$

The vector  $\left( \sum_k V_s^k \right)$  lies in  $\mathbb{R}_{++}^S$  by our assumption that at least one security pays off in each state. Let  $y_s^h = \pi_s^h - \pi_s^1$ . We know by assumption that  $L \geq S-1$ .

In matrix form, the first  $S - 1$  equations (by varying  $\xi$ ) and  $S - 1$  unknowns can be written as:

$$\begin{bmatrix} \pi^h(\xi_1|1) \left( \sum_k V_1^k \right) & \cdots & \pi^h(\xi_1|S-1) \left( \sum_k V_1^k \right) \\ \vdots & \ddots & \vdots \\ \pi^h(\xi_{S-1}|1) \left( \sum_k V_1^k \right) & \cdots & \pi^h(\xi_{S-1}|S-1) \left( \sum_k V_1^k \right) \end{bmatrix} \begin{bmatrix} y_1^h \\ \vdots \\ y_{S-1}^h \end{bmatrix} = 0$$

The matrix has full rank since there exists at least one agent such that the matrix  $\{\pi^h(\xi|\cdot)\}_{\xi \in Z}$  is of full rank and multiplying columns by strictly positive scalars will not alter its rank. So, there exists a unique solution to the first  $S - 1$  equations, namely  $y_s^h = 0$ ,  $1 \leq s \leq S - 1$ . So

$$\pi_s^1 = \pi_s^h \text{ for all } s \in \Theta \quad (4)$$

The statement that normalized utility gradients are different if  $\omega \in \Omega^*$  means that

$$\pi^h \neq \pi^{h'} \text{ for all } h, h'$$

This, in turn, implies that equation (4) cannot hold. Therefore, for each  $\omega \in \Omega^*$ , the resulting equilibrium cannot be a no retrade equilibrium. So, either an equilibrium exists involving retrade, or there is no equilibrium. ■

For the proof of theorem 12, we'll need the following

**Definition 16** *Given an economy  $E$ , an equilibrium with symmetric information is a collection  $(x, z, q)$  such that*

(i)  $(x^h, z^h) \in \arg \max U^h$  subject to

$$q(\xi_0)z^h(\xi_0) = 0$$

$$q(\xi)z^h(\xi) = q(\xi)z^h(\xi_0) \text{ for all } \xi \in Z$$

$$z^h(\xi) = z^h(\xi') \text{ for all } \xi, \xi' \in \sigma \in Y$$

$$x^h(\xi) - \omega^h(\xi) = Vz^h(\xi) \text{ for all } \xi \in Z$$

(ii)  $\sum_{h=1}^I z^h(\xi_0) = 0, \sum_{h=1}^I z^h(\xi) = 0, \sum_{h=1}^I x^h(\xi) - \omega^h(\xi) = 0$  for all  $\xi \in Z$ .

Proof of theorem 12.

**Proof.** By way of contradiction, given  $\omega \in \Omega^*$ , suppose there exists an equilibrium  $(x, z, q)$  with asymmetric information that involves no retrade. Then it must also be an equilibrium with symmetric information since a symmetric equilibrium has more constraints and these constraints are satisfied by  $(x, z, q)$ . Note that if an equilibrium with symmetric information exists, it must involve retrade. Indeed, the proof of theorem 8 can easily be adapted to the case of symmetric information economies. Simply replace the individual elements of the signal space by the elements of the meet  $Y$ . The condition that  $\#Y \geq S - 1$  and axiom 11.(iii) ensure that the proof of theorem 8 still works. This, in turn, implies that the equilibrium with symmetric information  $(x, z, q)$  must involve retrade. This is a contradiction to our assumption. So, if an equilibrium with asymmetric information exists, it must involve retrade. ■

In the following two lemmas, we focus on the standard two-period finance economy. Lemma 22 shows that generically in the space of endowments, equilibria are not Pareto Optimal. This is formalized by showing that generically, normalized utility gradients differ across agents. This property is proved in theorem 11.6 in Magill and Quinzii [3]. Their result cannot however be used directly since they allow consumption in period 0 and their endowment space is  $\mathbb{R}^{S+1}$ . However, a modification of their proof is offered below that deals with this issue. For the remainder of the appendix, we use the notation

$$U^h = \sum_{s=1}^S \pi^h(s) u^h(x_s^h)$$

**Lemma 17** *Let  $\pi^h \in \mathbb{R}^S$  denote normalized marginal utilities for agent  $h$ . Let  $\bar{x}^h \in \mathbb{R}^S$ . Then the mapping  $D_{x^h} \pi^h(\bar{x}^h) : T_{\bar{x}^h}(I_{\bar{x}^h}^h) \rightarrow \mathbb{R}^S$  is surjective.*

**Proof.** Let  $\lambda^h = \nabla U^h(x^h) \in \mathbb{R}^S$ . We wish to show that for every  $d\pi^h = \frac{d\lambda^h}{\lambda_0^h} \in \mathbb{R}^S$ , there exists vectors  $dx^h \in T_{\bar{x}^h}(I_{\bar{x}^h}^h)$  (the subspace tangent to the indifference surface at  $\bar{x}^h$ ) and  $d\lambda^h \in \mathbb{R}^S$  where

$$d\lambda_s^h = \sum_{\sigma=1}^S \frac{\partial U^h(\bar{x}^h)}{\partial x_s^h \partial x_\sigma^h} dx_\sigma^h$$

The condition that  $dx^h \in T_{\bar{x}^h}(I_{\bar{x}^h}^h)$  is equivalent to imposing that

$$\nabla U^h(\bar{x}^h) dx^h = 0$$

This means the following two equations must hold.

$$\begin{aligned} D^2 U^h(\bar{x}^h) dx^h &= \lambda_0^h d\pi^h \\ \nabla U^h(\bar{x}^h) dx^h &= 0 \end{aligned}$$

So, the following matrix must be invertible.

$$\begin{bmatrix} D^2U^h(\bar{x}^h) & \nabla U^h(\bar{x}^h)^T \\ \nabla U^h(\bar{x}^h) & 0 \end{bmatrix}$$

By way of contradiction, if the mapping isn't surjective, then this matrix is not invertible so there must exist a  $dx^h \neq 0$  so that

$$\begin{aligned} D^2U^h(\bar{x}^h)dx^h &= 0 \\ \nabla U^h(\bar{x}^h)dx^h &= 0 \end{aligned}$$

Premultiplying by  $(dx^h)^T$  the first equation gives

$$(dx^h)^T D^2U^h(\bar{x}^h)dx^h = 0$$

This contradicts our assumption about the shape of the utility function. Indeed, by axiom 3, it is easy to show that for each  $x \in \mathbb{R}_{++}^S$ ,

$$h^T D^2U^h(x)h < 0$$

for any  $h \in \mathbb{R}^S$ ,  $h \neq 0$ , such that  $\nabla U^h(\bar{x}^h)h = 0$ . ■

Define  $G^h : \mathbb{R}_{++}^S \times \mathbb{R}_+ \rightarrow \mathbb{R}_{++}^S$  by

$$G^h(\pi, y) = \arg \max [u^h(x^h) \text{ subject to } \pi x^h = y]$$

Also, define  $F^h : \mathbb{R}_{++}^S \times \mathbb{R}_{++}^S \rightarrow \mathbb{R}_{++}^S$  by

$$F^h(\pi, \omega^h) = \arg \max [u^h(x^h) \text{ subject to } \pi x^h = \pi \omega^h \text{ and } (x^h - \omega^h) \in \langle V \rangle]$$

Finally, define an excess demand function  $Z : \mathbb{R}_{++}^S \times \mathbb{R}_{++}^{SI} \rightarrow \mathbb{R}_{++}^S$  by

$$Z(\pi, \omega) = G^1(\pi, 1) + \sum_{h=2}^I F^h(\pi, \omega^h) - \sum_{h=1}^I \omega^h$$

It is easy to check the following

**Proposition 18** (i)  $Z$  is  $C^\infty$  and

(ii)  $Z(\pi, \omega) = 0$  iff

$$\pi\omega^1 = 1$$

and

$$G^1(\pi, \pi\omega^1) + \sum_{h=2}^I F^h(\pi, \omega^h) - \sum_{h=1}^I \omega^h = 0$$

**Definition 19** A no arbitrage equilibrium is a collection  $(x, \pi)$  so that  $Z(\pi, \omega) = 0$ .

**Definition 20** A financial market equilibrium is a collection  $(x, z, q)$  such that

(i)  $(x^h, z^h) \in \arg \max [u^h(x^h)]$  subject to  $qz^h = 0$  and  $x^h - \omega^h = Vz^h$

(ii)  $\sum_h x^h = \sum_h \omega^h$  and  $\sum_h z^h = 0$

One can easily prove that

**Proposition 21** (i) If  $(x, z, q)$  is a financial market equilibrium then there exists a  $\pi$  so that  $(x, \pi)$  is a no arbitrage equilibrium.

(ii) If  $(x, \pi)$  is a no arbitrage equilibrium then there exist a  $(z, q)$  so that  $(x, z, q)$  is a financial market equilibrium.

**Proof.** For part (i), take

$$\pi = \frac{\pi^1}{\pi^1 \omega^1}$$

where  $\pi^1$  is the vector of normalized marginal utilities of agent 1 at equilibrium  $(x, z, q)$ . For part (ii), take  $q = \pi V$ . The remainder of the proof is obvious and omitted. ■

**Lemma 22** *Given an economy  $E$  parametrized by endowments with at least 2 agents and where markets are incomplete, there exists a set of full measure  $\Omega^* \subset \Omega = \mathbb{R}_{++}^{SI}$  such that normalized marginal utilities are different across agents.*

**Proof.** Since markets are incomplete, there exists a  $s \in \{1, \dots, S\}$  so that  $d\pi_s \neq 0$ , with  $d\pi = (\dots, d\pi_s, \dots) \in \langle V \rangle^\perp$ . Consider the mapping  $h : \mathbb{R}_{++}^S \times \Omega \rightarrow \mathbb{R}^{S+1}$  defined by

$$h(\pi, \omega) = (Z(\pi, \omega), \pi_s^h - \pi_s)$$

where  $h > 1$ . To apply the transversality theorem, we need to show that the derivative mapping is surjective. Clearly,  $D_{\omega^1} Z = -I$  where  $I$  is the identity matrix. So, we need only show that we can locally control the last equation. By our choice of  $s$ , there is a  $d\pi^h \in \langle V \rangle^\perp$  so that  $d\pi_s = 1$ . By lemma 17, there exists a change  $dx^h \in \mathbb{R}_{++}^S$  such that

$$\pi^h(x^h + dx^h) = \pi^h + d\pi^h$$

Set

$$d\pi = 0, d\omega^h = dx^h, d\omega^{h'} = 0, \text{ for } h' \neq h$$

Then the first  $S$  equations remain unchanged (since  $d\omega^h = dx^h$ ) and the FOCs of agent  $h$  are still satisfied.

$$\pi^h(x^h + dx^h)V = (\pi^h + d\pi^h)V = \pi^h V = \pi V$$

Applying a transversality theorem (11.3 in Magill and Quinzii [3])  $h$  times yields the desired result. ■

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