2.3 The Max-Min Expected Utility

Interpretation of capacities as convex sets of probability distributions is problematic:

- not all capacities correspond to a non-empty set;
- not all convex sets can be represented by a capacity.

Instead:

- there is uncertainty about the probability distribution governing the random process;
- there is a convex set $\mathcal{A}$ of possible probability distributions;
- the preferences of a decision-maker can be described by:

$$f \succsim g \text{ iff}$$
$$\min_{\pi \in \mathcal{A}} \int_{s \in S} u(f(s))\, d\pi \geq \min_{\pi \in \mathcal{A}} \int_{s \in S} u(g(s))\, d\pi$$

Examples

(1) Ellsberg Paradox:

$$S = \{R; B; Y\}$$
$$\mathcal{A} = \left\{\pi \in \mathcal{P}(S) \mid \pi(R) = \frac{1}{3}\right\}$$

(2) $S = \{s_1; s_2; s_3\}$

$$\mathcal{A} = \left\{\pi \in \mathcal{P}(S) \mid \pi(s_1) \geq \frac{1}{3}, \pi(s_2) \leq \frac{1}{2}\right\}$$

(3) $S = [0; 1]$

$$\mathcal{A} = \left\{ \begin{array}{c} g \mid g(s) = \frac{s+\varepsilon-0{,}5}{2\varepsilon}, s \in [0{,}5-\varepsilon; 0{,}5+\varepsilon], \\ \varepsilon \in [0{,}1; 0{,}5] \end{array} \right\}$$

(4) $S = \{s_1; s_2; s_3\}$

$$\mathcal{A} = \{\pi \in \mathcal{P}(S) \mid \pi(s_1) = 2\pi(s_2)\}$$

Axioms in the Anscombe-Aumann Setting — MEU

$A1$ *Completeness*:

$$f \succsim g \text{ or } g \succsim f.$$

$A2$ *Transitivity*:

$$f \succsim g \text{ and } g \succsim h \text{ imply } f \succsim h.$$

$A3$ *Certainty Independence*:

For all h, h' and $g \in \mathcal{H}$ such that $g_s = g_{s'}$ for all s, $s' \in S$ and for all $\alpha \in (0;1)$

$$h \succsim h' \Longleftrightarrow h_\alpha g \succsim h'_\alpha g.$$

$A4$ *Archimedian Axiom*:

For all h, h' and $h'' \in \mathcal{H}$ such that $h \succ h' \succ h''$ it is possible to find numbers α, $\beta \in (0;1)$ such that:

$$h_\alpha h'' \succ h' \succ h_\beta h''.$$

$A5$ *Non-Triviality of Preferences*:

$$h \succ h'$$

holds for some h and $h' \in \mathcal{H}$.

$A6$ *Monotonicity:*

For all h, $h' \in \mathcal{H}$ such that

$$h(s) \succsim h'(s) \text{ for all } s \in S$$

$$h \succsim h'$$

holds.

$A7$ *Uncertainty Aversion:*

For all h, $h' \in \mathcal{H}$ and all $\alpha \in (0;1)$

$$h \sim h' \ \Rightarrow \ \alpha h + (1-\alpha) h' \succsim h'.$$

Theorem (Gilboa and Schmeidler 1989):
The following two statements are equivalent:

(i) The preference relation $\succsim$ on $\mathcal{H}$ satisfies $A1$—$A7$.

(ii) There exist

– a unique (up to a positive linear-affine transformation) von Neumann Morgenstern utility function $u : X \rightarrow \mathbb{R}$ and

– a unique compact and convex set $\mathcal{A}$ of probability distributions on S

such that

$$h \succsim g \ \Leftrightarrow$$

$$\min_{\pi \in \mathcal{A}} \sum_{s \in S} \pi\left(s\right) \left[\sum_{x \in supp(h_s)} h_s\left(x\right) u\left(x\right)\right]$$

$$\geq \ \min_{\pi \in \mathcal{A}} \sum_{s \in S} \pi\left(s\right) \left[\sum_{x \in supp(h_s)} g_s\left(x\right) u\left(x\right)\right].$$

1 Connection between MEU and CEU

Proposition 3.1 *(Schmeidler 1989) Suppose that the preference order $\succsim$ on $\mathcal{H}$ can be represented by*

$$V(h) = \int_S \left[\sum_{x \in supp(h_s)} h_s(x)\, u(x) \right] d\nu$$

The following three statements are equivalent:

- the capacity ν is convex;
- the preference order $\succsim$ on $\mathcal{H}$ exhibits uncertainty aversion;
- the preference order $\succsim$ on $\mathcal{H}$ can be represented by

  $$\tilde{V}(h) = \min_{\pi \in C(\nu)} \sum_{s \in S} \pi(s) \left[\sum_{x \in supp(h_s)} h_s(x)\, u(x) \right],$$

  where $C(\nu)$ is the core of the capacity ν.

2 Generalization of the Multiple Prior Approach

Possible interpretations of MEU:

- For each act, expect the worst possible outcome to occur with the highest possible probability;
- Extreme pessimism;
- Malevolent nature

Alternative: Max-Max-Expected Utility:

$$V(f) = \max_{\pi \in \mathcal{A}} \sum_{s \in S} \pi(s) \left[\sum_{x \in supp(h_s)} h_s(x) \, u(x) \right]$$

- For each act, expect the best possible outcome to occur with the highest possible probability;
- Extreme optimism;
- Benevolent nature

Hurwitz Criterium:

A linear combination of Max-Min and Max-Max:

$$V(f) = \alpha \max_{\pi \in \mathcal{A}} \sum_{s \in S} \pi(s) \left[\sum_{x \in supp(h_s)} h_s(x)\, u(x) \right] + (1-\alpha) \min_{\pi \in \mathcal{A}} \sum_{s \in S} \pi(s) \left[\sum_{x \in supp(h_s)} h_s(x)\, u(x) \right]$$

- α is the weight assigned to the best expected utility of each act;
- α can be interpreted as the degree of optimism;
- $(1-\alpha)$ — degree of pessimism;
- set $\mathcal{A}$ measures the degree of ambiguity.

Connection between the Hurwitz Criterium and the CEU

Recall: a *neo-additive capacity* is given by:

$$\nu(E) = (1-\gamma-\lambda)\pi(E) + \gamma\nu^0(E) + \lambda\nu^1(E) = \\ = \left\{ \begin{array}{ll} 0 & \text{if } E = \emptyset \\ (1-\gamma-\lambda)\pi(E) + \gamma & \text{if } E \subsetneq S \\ 1 & \text{if } E = S \end{array} \right\}$$

Let f be a simple act with values $x_1 \geq ... \geq x_n$

The CEU of f under the neo-additive capacity v is:

$$V(f) = \int u \circ f dv = \gamma u(x_n) + \lambda u(x_1) + \\ + (1-\gamma-\lambda)\sum_{i=1}^{n} u(x_i)\pi(s \mid f(s) = x_i)$$

Define $\mathcal{A}$ as:
$\mathcal{A} = \{\tilde{\pi} \in \mathcal{P}(S) \mid \tilde{\pi}(E) \geq (1-\gamma-\lambda)\,\pi(E) \text{ for all } E \subset S\}$.

- $\mathcal{A}$ is convex and compact;
- $\pi \in \mathcal{A}$.

Chateauneuf, Eichberger and Grant, 2005
For f, ν and $\mathcal{A}$ so defined, we have:

$$V(f) = \int u \circ f dv = \frac{\lambda}{\gamma+\lambda} \max_{\tilde{\pi}\in\mathcal{A}} \sum_{i=1}^{n} u(x_i)\,\tilde{\pi}(s \mid f(s) = x_i) + \frac{\gamma}{\gamma+\lambda} \min_{\tilde{\pi}\in\mathcal{A}} \sum_{i=1}^{n} u(x_i)\,\tilde{\pi}(s \mid f(s) = x_i)$$

- $\frac{\lambda}{\gamma+\lambda}$ measures the optimism of the decision-maker;
- $\frac{\gamma}{\gamma+\lambda}$ measures the pessimism of the decision-maker;
- $\gamma+\lambda$ determines the degree of ambiguity.