ECON 756

Game Theory

Ani Guerdjikova

Fall 2004

1 Bargaining Theory

Two approaches:

- The **strategic (non-cooperative)** approach models the bargaining procedure.
- The **axiomatic (cooperative)** approach characterizes the bargaining outcome by axioms that an outcome is expected to satisfy.

Prototype of a bargaining situation

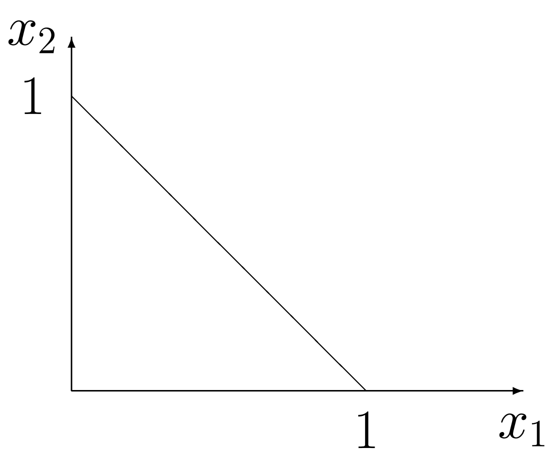

Figure 1. Dividing a cake

Example 0.1 *Wage Bargaining*

w — *wage*

n — *number of workers employed*

- Trade Union:

$$v(w;n) = wn - C(n)$$

- Firm:

$$\pi(w;n) = R(n) - wn$$

- Pareto-optima:

$$\max_{w;n} v(w;n) + \pi(w;n) = R(n) - C(n)$$

$$R'(n) - C'(n) = 0 \Longrightarrow n^*$$

Example 0.2 *Bargaining in the Edgeworth Box*

- Two consumers $i = 1, 2$
- Bargaining over the allocation of two goods x and y
- Initial Allocation:
$$(\bar{x}_i; \bar{y}_i)_{i=1;2}$$
- Preferences:
$$u_i\left(x_i; y_i\right), i = 1; 2$$
  – strictly increasing
  – quasi-concave

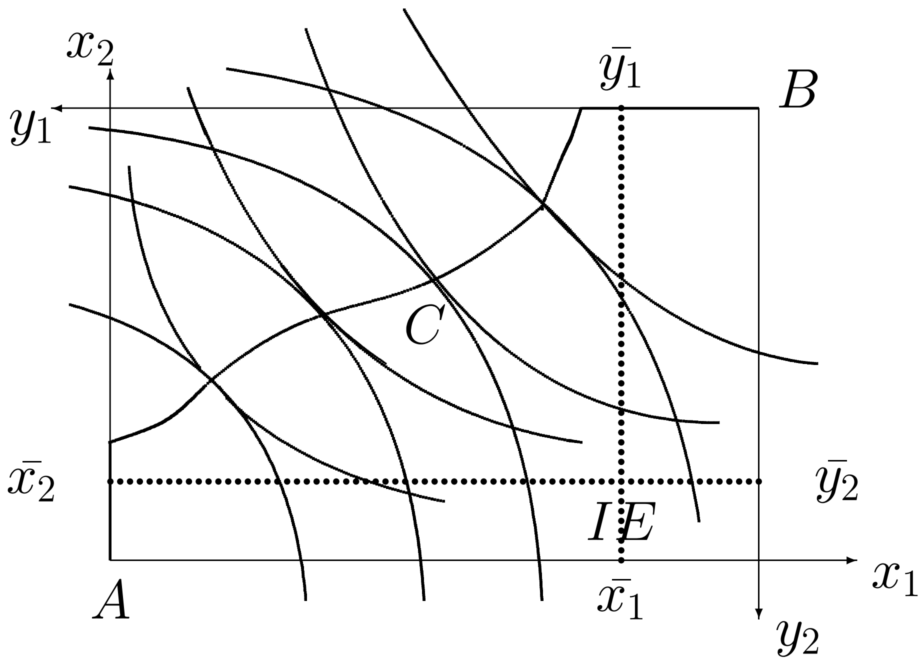

Figure 2. Bargaining in the Edgeworth Box

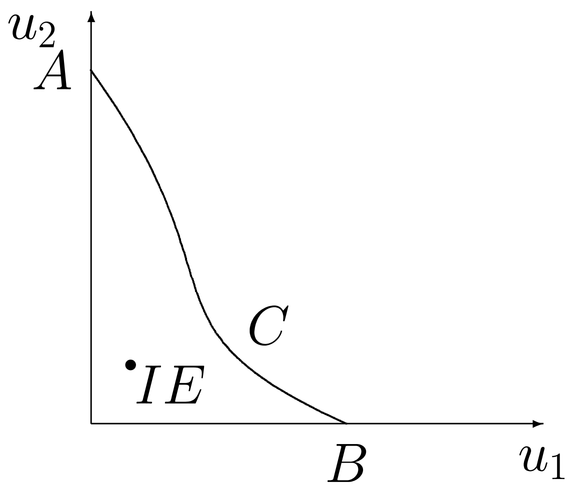

Figure 3. Utility possibility frontier

Edgeworth's Argument

(1) *Individual Rationality:*
Rational players do not accept allocations that make them worse off than their initial endowment.

(2) *Pareto-optimality:*
Rational players will exhaust all potential gains from trade.

Definition:

A bargaining problem $(X; d)$ specifies:

- a set X of *feasible payoff combinations*
  $$X = \left\{(x_1; x_2) \in \mathbb{R}^2 \mid x_2 \leq \psi(x_1)\right\}$$
  and

- a payoff combination $d = (d_1; d_2) \in X$ that obtains in the case of break-down of negotiations — the *disagreement point*.

Assumption:
The set of individually rational payoff combinations
$$\hat{X} = \{(x_1; x_2) \in X \mid x_1 \geq d_1 \text{ and } x_2 \geq d_2\}$$
is compact and convex.

The assumption is satisfied if:

- ψ is *continuous* ($\Rightarrow$ $\hat{X}$ is compact) and

- ψ is *concave* ($\Rightarrow$ $\hat{X}$ is convex).

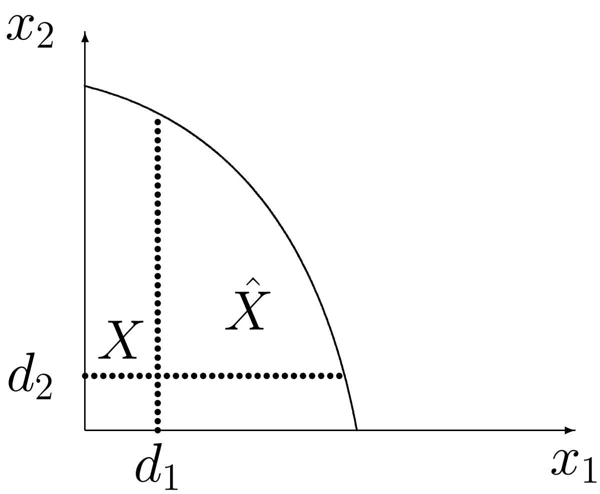

Figure 4. Bargaining problem

1.1 Axiomatic Solutions

Main Idea:

Find a rule which assigns to each bargaining problem $(X; d)$ a *"sensible" bargaining outcome*, i.e.a *solution*.

Notation:

The set of all bargaining problems with a convex set of feasible outcomes is denoted by $\mathcal{X}$.

Definition:

A bargaining solution is a function

$$f : \mathcal{X} \to \mathbb{R}^2$$

which assigns to each bargaining problem $(X; d) \in \mathcal{X}$ a bargaining outcome (solution)

$$(x_1^*; x_2^*) = f(X; d).$$

AXIOMS

Axiom 1: *Strict individual rationality*
The solution

$$(x_1^*; x_2^*) = f\,(X; d)$$

satisfies for all $X \in \mathcal{X}$:

$$x_1^* > d_1 \text{ and } x_2^* > d_2.$$

Axiom 2: *Pareto-optimality*
The solution

$$(x_1^*; x_2^*) = f\,(X; d)$$

satisfies for all $X \in \mathcal{X}$: if $(x_1; x_2)$ and $(x_1'; x_2') \in X$ and

$$x_1' > x_1 \text{ and } x_2' > x_2\text{, then}$$

$$f\,(X; d) \neq (x_1; x_2)\,.$$

Definition:
$(X'; d')$ is obtained from $(X; d)$ by the transformation:

$$\beta_i (x_i) \rightarrow A_i x_i + B_i$$

for $i = 1, 2$ if

$$d'_i = A_i d_i + B_i, \, i = 1, 2$$

and
$X' = \left\{ (A_1 x_1 + B_1; A_2 x_2 + B_2) \in \mathbb{R}^2 : (x_1; x_2) \in X \right\}$.

Axiom 3: *Invariance to equivalent utility representations*
Let $(X'; d')$ be obtained from $(X; d)$ by the transformation

$$\beta_i (x_i) \rightarrow A_i x_i + B_i$$

for $i = 1, 2$. Then

$$f_i (X'; d') = A_i f_i (X; d) + B_i, \, i = 1, 2.$$

Remark

Consider $(X; d)$ such that:

$$X = \left\{(x_1; x_2) \in \mathbb{R}^2 \mid x_2 \leq a - bx_1\right\}.$$

There is a linear transformation that maps the set

$$\Delta_X = (x_1; x_2) \in \hat{X}$$

onto

$\Delta = \left\{(x_1; x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0 \text{ and } x_1 + x_2 \leq 1\right\}.$

The transformation is given by:

$$\begin{aligned} A_1 &= \frac{b}{a - d_2 - bd_1} \\ B_1 &= -\frac{bd_1}{a - d_2 - bd_1} \\ A_2 &= \frac{1}{a - d_2 - bd_1} \\ B_2 &= -\frac{d_2}{a - d_2 - bd_1}. \end{aligned}$$

Axiom 4: *Independence of Irrelevant Alternatives*
If $(X;d)$ and $(X';d) \in \mathcal{X}$ with $X' \subset X$ and

$$f(X;d) \in X',$$

then

$$f(X';d') = f(X;d).$$

Definition:
A bargaining problem $(X;d)$ is *symmetric*, if

$$d_1 = d_2$$

and for each

$$(x_1;x_2) \in X,$$
$$(x_2;x_1) \in X.$$

Axiom 5: *Symmetry*
If the bargaining problem $(X;d)$ is symmetric, then

$$x_1^* = x_2^*.$$

Nash's Theorem:

Theorem 1.1 *If for each* $(X; d) \in \mathcal{X}$ *there is a*

$$(x_1; x_2) \gg (d_1; d_2),$$

then there is a unique bargaining solution

$$f : \mathcal{X} \to \mathbb{R}^2$$

that satisfies axioms 1-5. It is given by:

$$f(X; d) = \arg \max_{(x_1; x_2) \in X} (x_1 - d_1)(x_2 - d_2).$$

Asymmetric Nash Solutions

Theorem 1.2 *If for each* $(X; d) \in \mathcal{X}$ *there is a*

$$(x_1; x_2) \gg (d_1; d_2),$$

then the following two statements are equivalent:

(1) The bargaining solution f satisfies axioms 1-4.

(2) There exists an $\alpha \in (0; 1)$ such that

$$f(X; d) = \arg \max_{(x_1; x_2) \in X} (x_1 - d_1)^{\alpha} (x_2 - d_2)^{1-\alpha}.$$

The Kalai-Smorodinsky Solution

Drop axiom 4, Invariance of irrelevant alternatives and assume:

Axiom 6: *Monotonicity*

Let $X' \subset X$ and

$$\max\{x_1 \mid (x_1; x_2) \in X\} = \max\{x_1 \mid (x_1; x_2) \in X'\}$$
$$\max\{x_2 \mid (x_1; x_2) \in X\} = \max\{x_2 \mid (x_1; x_2) \in X'\}.$$

Then

$$f_1(X; d) \geq f_1(X'; d)$$
$$f_2(X; d) \geq f_2(X'; d).$$

Utopia point:

$$(x_1^u; x_2^u)$$
$$x_i^u = \max\{x_i \mid x \in X, x_j \geq d_j \text{ for } j \neq i\}$$

The solution:

$$f^{KS}(X; d) = d + \lambda^* (x^u - d)$$
$$\lambda^* = \max\{\lambda \in \mathbb{R} \mid d + \lambda (x^u - d) \in X\}$$

1.2 The Strategic Approach

Nash-Program

Any *bargaining solution*

should be supplemented by a *bargaining procedure*

that implements this outcome as a *strategic equilibrium*.

Rules of the Bargaining Game

- T rounds
- player 1 begins
- in each round:
  - one of the players proposes a payoff allocation $(x_1; x_2)$
  - the opponent responds by either accepting (a) or rejecting (r) the offer
- if the offfer is accepted, $(x_1; x_2)$ is implemented
- if the offer is rejected, a new round starts and the player who has rejected the offer proposes a new allocation
- discount factors: δ_1 and δ_2
- if no agreement is reached at time T, both players receive 0.

Theorem 2.1 *Let*

- ψ be concave and differentiable
- $\delta_i < 1$ for $i = 1, 2$,

then there is a unique subgame perfect equilibrium for the bargaining game with $T = \infty$.
The equilibrium payoff combination is determined by:

$$\psi\left(x_1^*\right) = \delta_2 \psi\left(\delta_1 x_1^*\right)$$

and is obtained in the first round.
The equilibrium strategies are given by:

Player 1: *propose* $\left(x_1^*; \psi\left(x_1^*\right)\right)$
accept $\left(x_1; \psi\left(x_1\right)\right)$ *if* $x_1 \geq \delta_1 x_1^*$
reject $\left(x_1; \psi\left(x_1\right)\right)$ *if* $x_1 < \delta_1 x_1^*$

Player 2: *propose* $\left(\delta_1 x_1^*; \psi\left(\delta_1 x_1^*\right)\right)$
accept $\left(x_1; \psi\left(x_1\right)\right)$ *if* $\psi\left(x_1\right) \geq \delta_2 \psi\left(\delta_1 x_1^*\right)$
reject $\left(x_1; \psi\left(x_1\right)\right)$ *if* $\psi\left(x_1\right) < \delta_2 \psi\left(\delta_1 x_1^*\right)$

1.3 Subgame Perfect Equilibria

Notation:
Given a game in extensive form Γ:
N — set of nodes
$\mathcal{T}(N)$ — set of end nodes
U — set of information sets
$r_i(n)$ — payoff of player i at an end node n

Definition 1.1 *The set of nodes following a node x, N_x constitutes a subgame Γ_x, if*

- $\{x\}$ is an information set;
- The set of all information sets U can be partitioned into two subsets U_x and U_{-x} such that

$$u \in U_x \Rightarrow u \subset N_x$$
$$u \notin U_x \Rightarrow u \subset N \backslash N_x$$

Definition 1.2 *A behavioral strategy of player i specifies the probability of choosing an (available) action a at an information set u:* $b_i^u(a)$.

Notation:
$q_b(y \mid x)$ the probability to reach node y conditional on playing a behavioral strategy combination b and being currently at node x
$R_i(b \mid x)$ the payoff of player i conditional on playing a behavioral strategy b and being currently at node x:

$$R_i(b \mid x) = \sum_{n \in \mathcal{T}(N)} r_i(n)\, q_b(n \mid x).$$

Definition 1.3 *A behavioral strategy combination*

$$b = (b_1 ... b_I)$$

is a subgame perfect equilibrium of Γ if for all subgames Γ_x and for all players $i \in \{1 ... I\}$

$$R_i(b \mid x) \geq R_i(b_i'; b_{-i} \mid x).$$

1.4 The Nash Solution and the Rubinstein Bargaining Game

Notation:

τ — length of period

r_i, $i \in \{1;2\}$ — time preferences of the players:

$$\delta_i = e^{-r_i\tau}$$

Consider the bargaining problem $(\Delta; 0)$ with

$$\begin{aligned} x_2 &= \psi(x_1) = 1 - x_1 \\ d_1 &= d_2 = 0. \end{aligned}$$

Nash Solution:

$$x_1^{N*} = \alpha, x_2^{N*} = 1 - \alpha.$$

Rubinstein Solution:

$$x_1^{R*} = \frac{(1-\delta_2)}{1-\delta_1\delta_2}, x_2^{R*} = \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}.$$

$$\begin{aligned} \lim_{\tau\to 0} x_1^{R*} &= \frac{r_2}{r_1+r_2}. \\ \lim_{\tau\to 0} x_2^{R*} &= \frac{r_1}{r_1+r_2} \end{aligned}$$

If $\alpha = \frac{r_2}{r_1+r_2}$ *then both solutions coincide.*