

# Modeling Denomination Structures\*

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## Abstract

Previous work on the denomination structure of currency treats as exogenous the distribution of transactions and the denominations held by people. Here, by way of a matching model, both are endogenous. In the model, trades in pairwise meetings alternate in time with the opportunity to freely choose a portfolio of denominations and people trade off the benefits of small-denomination money for transacting against the costliness of carrying a large quantity of small-denomination money. For a given denomination structure, a monetary steady state is shown to exist. The model implies that too small denominations are abandoned.

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## 1 Introduction

The small literature on denomination structures for currency treats as exogenous both the distribution of transactions and the portfolios of denominations held by transactors. Telser [4] starts from a problem in number theory.

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The problem of Bâchet seeks the smallest number of weights capable of weighing any unknown integer. ... The version pertinent to this paper allows the unknown weights to be placed in either pan... The solution is weights that are powers of three, namely, 1, 3, 9, 27 and so on....

[For the currency version,] the unknown amount to be weighed corresponds to a ... transaction. ...Allowing weights to go in either pan corresponds to making change. Finally, that the weights must be capable of weighing any quantity between one and [some] upper bound corresponds to the assumption that a transaction is equally likely to be anywhere between one and some finite upper bound (Telser [4] pages, 425-26).

Bâchet's problem is a one-person problem and its extension to multiple transactors, at least a buyer and a seller, is far from straightforward even given the exogeneity of the distribution of transactions.<sup>1</sup> Van Hove and Heyndels [7] also assume a uniform distribution of transactions. Their criterion is minimization of the average number of monetary items exchanged.<sup>2</sup> They show that a powers-of-two structure, 1, 2, 4, .., can result in a lower average number of monetary objects used in transactions than a powers-of-three structure.

The main limitation of the above work is that too much is taken to be exogenous. As regards portfolios of denominations, people should choose them—presumably, before they know the transaction size. There even seems to be a strategic consideration; if everyone else has small change, then why should I carry small change? As regards the distribution of transactions, it should be allowed to adapt to the denomination structure—through “rounding” or lotteries.

Here, we formulate a model in which both the distribution of transactions and the portfolios of denominations held are endogenous. The model is simple in that it contains no more than the ingredients that we think are necessary

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<sup>1</sup>That is, how are portfolios of denominations determined? One interpretation is that there is a buyer and a seller and that conditional on the transaction to be accomplished, the weights (denominations) that solve Bâchet's problem are distributed (!) between them in a way that allows the transaction to be carried out.

<sup>2</sup>They assume that the portfolios of denominations held by individuals do not limit what can be used in trade. For another discussion of the different criteria used in [4] and [7], see Van Hove [6].

for a coherent model: indivisibility of monetary items, a beneficial role for low-value items in trade, and costs that penalize the holding of many low-value items. We use a matching model of money—a model in which trades of money for goods occur in pairwise meetings, between a buyer and a seller. Those meetings alternate in time with the opportunity to costlessly choose a portfolio of denominations—for example, the opportunity to exchange two \$5’s for a \$10, and vice versa. The two-person trade of money for goods is a natural and simple way to represent the beneficial role of low-value monetary items in trade.

For a given denomination structure, we show that if the costs are sufficiently small, then there exists a *nice* monetary steady state—nice in that the implied value function for (monetary) wealth is strictly increasing and concave.<sup>3</sup> Then we discuss adding fixed societal costs per denomination to the model and what our model says about denominations that are too small.

## 2 The Model

The background environment is borrowed from Shi [3] and Trejos and Wright [5]. Time is discrete. There is a unit measure of each of  $N \geq 3$  types of infinitely lived agents and there are  $N$  distinct produced and perishable types of divisible goods at each date. A type  $n$  agent,  $n \in \{1, 2, \dots, N\}$ , produces type  $n$  good and consumes only type  $n + 1$  good (modulo  $N$ ). Each person maximizes expected discounted utility with discount factor  $\beta \in (0, 1)$ . Trading histories are private so that any kind of borrowing and lending in pairwise meetings is impossible.

Let  $s = (s_1, s_2, \dots, s_K)$  be a denomination structure of the currency. Here  $s_k$  is the size of the  $k$ -th smallest denomination. We assume that  $1/s_1$  is a positive integer, that  $s_k/s_1$  is an integer, and that  $s_k > s_{k-1}$ . (For example, in the powers-of-three structure,  $s_k = 3s_{k-1}$ .) The generic symbol for a person’s portfolio is  $y = (y_1, y_2, \dots, y_K)$ , where  $y_k \geq 0$  is the integer quantity of size- $s_k$  money held. The wealth implied by  $y$  is the inner product of  $s$  and  $y$ .

We adopt the following sequence of actions for a period. First, each person gets to choose a portfolio of denominations subject only to a wealth constraint. After the portfolio is chosen, a person meets another person at

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<sup>3</sup>We often apply the term *concave* to functions defined on discrete subsets of  $\mathbb{R}^n$ . Suppose  $X \subset I^n$ , where  $I$  is an interval. We say that  $f : X \rightarrow \mathbb{R}$  is (strictly) concave if there exists  $g : I^n \rightarrow \mathbb{R}$  such that  $f$  is the restriction of  $g$  to  $X$  and  $g$  is (strictly) concave.

random. We assume that trading partners see each other's portfolios and types. There are no-coincidence meetings during which nothing happens. In single-coincidence meetings, we assume that the buyer (consumer) makes a take-it-or-leave-it offer. (This offer can involve a demand for *change*—for some of the monetary items that are part of the portfolio initially held by the seller.) For a type- $n$  person with portfolio  $y$  just prior to pairwise meetings, realized utility in a period is

$$u(q_{n+1}) - q_n - \gamma \sum_k y_k,$$

where  $q_{n+1} \in \mathbb{R}_+$  is consumption of good  $n + 1$  and  $q_n \in \mathbb{R}_+$  is production of good  $n$ , and  $\gamma \geq 0$  is the utility cost of carrying money from the portfolio-choice stage to the pairwise-meeting stage per piece of money. (The linear carrying-cost function could be replaced by a more general class of convex functions of  $\sum_k y_k$ .) The utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is strictly concave, strictly increasing, continuously differentiable and satisfies  $u(0) = 0$  and  $u'(\infty) = 0$ . In addition,  $u'(0)$  is sufficiently large, but need not be infinite.

We formally define and prove existence of a steady state for a version in which people can choose lotteries both at the portfolio choice stage and in meetings. The definition is similar and the existence result holds for a version without lotteries (by allowing randomization over multiple deterministic optima for the individual problems).<sup>4</sup> To save space, the non lottery version is not presented.

### 3 Existence of a Steady State

We begin by defining a symmetric steady state for a given denomination structure,  $s$ —symmetric across the  $N$  types of agents. To achieve compactness, we assume that an individual's wealth  $z \in \{0, s_1, 2s_1, 3s_1, \dots, Z\} \equiv \mathbf{Z}$ , where  $Z$  is an integer multiple of  $s_1$ . We let  $\pi$  denote a probability measure on  $\mathbf{Z}$ , where  $\pi(z)$  is the fraction of each specialization type with wealth  $z$  at the start of a date. As a normalization, we impose  $\sum z\pi(z) = 1$ . (With this normalization,  $1/s_1$  is the ratio of mean wealth to the size of the smallest unit of money and  $Z$  is the ratio of the bound on wealth to mean wealth.) We let  $\mathbf{Y} = \{y = (y_1, y_2, \dots, y_K) \in \mathbb{Z}_+^K : sy \leq Z\}$  be the set of feasible individual

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<sup>4</sup>Lotteries do not eliminate the benefits for trade of small-denomination money because the value function turns out to be strictly concave.

portfolios. (Here,  $\mathbb{Z}_+$  denotes the set of non negative integers.) And we let  $\theta$  denote a probability measure on  $\mathbf{Y}$ , where  $\theta(y)$  is the fraction of each specialization type with portfolio  $y$  just before pairwise meetings. A steady state is a collection of functions  $(w_s, \pi_s, \theta_s)$  that satisfies the conditions described below. We suppress the dependence of a steady state on the denomination structure  $s$  whenever the context is clear.

We begin with the choice of a portfolio. We let a person with wealth  $z$  choose any lottery over portfolios in  $\mathbf{Y}$  whose value does not exceed  $z$ . That is, we let  $\Gamma_1(z)$ , a subset of probability measures defined on  $\mathbf{Y}$ , be defined by

$$\Gamma_1(z) = \{\sigma : \sigma(y) = 0 \text{ if } sy > z\}. \quad (1)$$

We let  $h : \mathbf{Y} \rightarrow \mathbb{R}$  denote expected discounted utility after the portfolio is chosen and before pairwise meetings. In terms of  $h$ , the portfolio-choice problem is

$$g(z, h) = \max_{\sigma \in \Gamma_1(z)} \sum \sigma(y)h(y). \quad (2)$$

We denote the set of maximizers in (2) by  $\Delta_1(z, h)$ . That is,  $\Delta_1(z, h)$  is a subset of probability measures on  $\mathbf{Y}$ , where  $\delta \in \Delta_1(z, h)$  is an optimal lottery and  $\delta(y)$  is the implied probability of holding portfolio  $y$ .

We now turn to trade in meetings. Let

$$S(y) = \{v \in \mathbb{Z}_+^K : v_k \leq y_k\}. \quad (3)$$

That is,  $S(y)$  is the set of portfolios that a person with  $y$  could surrender. Now, consider a meeting between a buyer with  $y$  and a seller with  $y'$ . We let

$$X(y, y') = \{x \in [0, Z - sy'] : x = s(v - v'), v \in S(y), v' \in S(y')\}. \quad (4)$$

That is,  $X(y, y')$  is the set of feasible wealth transfers from the buyer to the seller taking into account the denominations held and the possibility of making change. Let  $W$  be an upper bound on  $w$  that is defined in the existence proof. We define  $\Gamma_2(y, y', w)$ , a set of probability measures on  $[0, W] \times X(y, y')$ , by

$$\Gamma_2(y, y', w) = \{\sigma : E_\sigma[-q + \beta w(x + sy')] \geq \beta w(sy')\}, \quad (5)$$

where  $E_\sigma$  denotes the expectation with respect to  $\sigma$  and where the arguments of  $\sigma$  are  $(q, x)$ . Then we let

$$f(y, y', w) = \max_{\sigma \in \Gamma_2(y, y', w)} E_\sigma[u(q) + \beta w(sy - x)]. \quad (6)$$

This description implies the payoffs from trade. That is, the buyer's payoff is  $f(y, y', w)$  and, because the inequality in (5) will be binding, the seller's is  $\beta w(sy')$ .

We denote the set of maximizers in (6) by  $\tilde{\Delta}_2(y, y', w)$ . Because it can be shown that each maximizer is degenerate in  $q$ , in what follows, for a maximizer, we denote the maximizing  $q$  by  $\hat{q}$ .<sup>5</sup> To describe the law of motion, it is convenient to define  $z^{-1}(y, y') = \{x : sy - x = z\}$  for  $z \in \mathbf{Z}$ , the set of asset trades that leave the buyer with wealth  $z$ . Then we define  $\Delta_2(y, y', w)$ , a subset of probability measures on  $\mathbf{Z}$ , by

$$\Delta_2(y, y', w) = \{\delta : \delta(z) = \sum_{z^{-1}(y, y')} \tilde{\delta}(\hat{q}, x), \text{ for } \tilde{\delta} \in \tilde{\Delta}_2(y, y', w)\}. \quad (7)$$

That is, for  $\delta \in \Delta_2(y, y', w)$ ,  $\delta(z)$  is the probability that the buyer with  $y$  leaves with wealth  $z$  after meeting a seller with  $y'$ .

Now we can complete the conditions for a steady state. The value function  $w$  must satisfy

$$w(z) = g(z, h), \quad (8)$$

where the function  $h$  is defined by

$$h(y) = -\gamma \sum y_k + \frac{N-1}{N} \beta w(sy) + \frac{1}{N} \sum \theta(y') f(y, y', w). \quad (9)$$

And the measures  $\pi$  and  $\theta$  must satisfy

$$\pi \in T_\pi(w, \theta) \text{ and } \theta \in T_\theta(w, \pi, \theta), \quad (10)$$

where

$$\begin{aligned} T_\pi(w, \theta) &= \{\omega : \omega(z) = \sum_{(y', y'')} \theta(y') \theta(y'') [\delta(z) + \delta(sy' - z + sy'')], \\ &\text{for } \delta \in \Delta_2(y', y'', w)\}, \end{aligned} \quad (11)$$

and

$$T_\theta(w, \pi, \theta) = \{\omega : \omega(y) = \sum_z \pi(z) \delta(y) \text{ for } \delta \in \Delta_1(z, h)\}. \quad (12)$$

(Notice that  $T_\pi(w, \theta)$  is a set of probability measures on  $\mathbf{Z}$  and  $T_\theta(w, \pi, \theta)$  is a set of probability measures on  $\mathbf{Y}$ . Also, notice that the dependence of  $T_\theta(w, \pi, \theta)$  on  $(w, \theta)$  is through the dependence of  $h$  on  $(w, \theta)$ . Finally, in (11), as a convention,  $\delta(x) = 0$  if  $x \notin \mathbf{Z}$ .)

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<sup>5</sup>See Berentsen, Molico, and Wright [1], who were the first to introduce lotteries into matching models of money.

**Definition 1** *Given a denomination structure  $s$ , a steady state is  $(w, \pi, \theta)$  that satisfies (1)-(12).*

Now we can state an existence result.

**Proposition 1** *Given a denomination structure  $s$ , if  $u'(0)$ ,  $1/s_1$ , and  $Z$  are sufficiently large, then there exists  $\gamma_s > 0$  such that for all  $\gamma \in [0, \gamma_s]$  there exists a steady state  $(w, \pi, \theta)$  with  $w$  strictly increasing and strictly concave and with  $\pi$  having full support.*

**Proof.** See the Appendix. ■

As is well known, the challenge in models like the one above is to show that there is a reasonable monetary steady state. This challenge is met by the conclusion that  $w$  is strictly increasing and concave. The proof shows that there is a neighborhood of  $\gamma = 0$  with such a steady state. Notice that the full support property implies that the smallest denomination is held in such a steady state. It says nothing about higher denominations.<sup>6</sup>

Given earlier results in [9] and [11], the proof is simple. The proof establishes properties of the mapping used to define a steady state. First, it uses the fact—previously established in [11], which, in turn, rests on the results in [9]—that for  $\gamma = 0$  the mapping has fixed-point index 1. The result in [11] can be applied because if  $\gamma = 0$ , then portfolios that consist entirely of the smallest denomination money are equilibrium portfolios. But given such portfolios, the model is the same as the money-only case of the model in [11]. (For such portfolios, the non lottery version is the model in [9].) Second, the mapping is shown to be upper hemicontinuous in  $\gamma$  at  $\gamma = 0$ . Those two facts give the result.

Notice, by the way, that the mapping is not continuous in  $\gamma$  at  $\gamma = 0$ ; that is, a portfolio that is an equilibrium portfolio when  $\gamma = 0$  is not, in general, close to those that are equilibrium portfolios when  $\gamma > 0$  and small. That is why the model can have rich implications for portfolios even if  $\gamma$  is small.

In accord with such discontinuity, there is a general conclusion about portfolios that is related to Telser's conclusions about an optimal denomination structure; namely, portfolios do not have surplus divisibility.

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<sup>6</sup>Situations in which an available denomination is not held are familiar; the U.S. \$2 bill is one example.

**Lemma 1** Let  $V(y) = \{x \in \mathbb{R}_+ : x = sv, v \in S(y)\}$ , the set of wealth transfers that is feasible for a person with portfolio  $y$ . If  $\delta$  solves the problem in (2) and  $y$  is in the support of  $\delta$ , then  $\nexists \tilde{y}$  satisfying (i)  $s\tilde{y} = sy$ , (ii)  $V(y) \subseteq V(\tilde{y})$ , and (iii)  $\sum_k \tilde{y}_k < \sum_k y_k$ .

**Proof.** Suppose, by contradiction, that  $\tilde{y}$  exists. It suffices to show that  $h(\tilde{y}) > h(y)$ . By (9) and properties (i) and (iii),  $h(\tilde{y}) > h(y)$  if  $f(\tilde{y}, y', w) \geq f(y, y', w)$ . To establish the latter, suppose that  $x \in X(y, y')$  (see (4)). That is, there exist  $v \in S(y)$  and  $v' \in S(y')$  with  $x = sv - sv'$ . But by property (ii), there exists  $\tilde{v} \in S(\tilde{y})$  with  $s\tilde{v} = sv$ . Therefore  $x \in X(\tilde{y}, y')$ , which implies that any offer that is feasible when the buyer has  $y$  is also feasible when the buyer has  $\tilde{y}$ . This and property (i) imply that  $f(\tilde{y}, y', w) \geq f(y, y', w)$ . ■

To see that this lemma rules out some portfolios, suppose that  $s$  is the powers-of-two structure and consider  $y = (3, 0, \dots, 0)$  and  $\tilde{y} = (1, 1, 0, \dots, 0)$ . If  $\delta$  solves the problem in (2), then  $y$  cannot be in the support of  $\delta$  because  $\tilde{y}$  satisfies the three conditions in the lemma.

And, using strict concavity of  $w$ , we can say a bit more about offers in meetings. The following lemma shows how to use the best trade when the buyer's portfolio consists entirely of the smallest denomination to construct the best trade for all portfolios with the same wealth. The best trade is described in terms of a lottery, possibly degenerate, over the amount of wealth transferred. The corresponding implied (deterministic) output is that which satisfies the producer's participation constraint in (5) with equality.

**Lemma 2** Consider a buyer with  $y$  and a seller with  $y'$ . (i) If the buyer's portfolio is  $(sy/s_1, 0, 0, \dots, 0)$ , then the solution for the wealth transfer from the buyer is a lottery over  $\{m, m + s_1\}$  for some  $m \in \mathbf{Z}$ . (ii) Let  $m$  be as described in part (i). In general, the solution for the wealth transfer from the buyer is a lottery over

$$\{m_-, m_+\} = \left\{ \max_{x \leq m} X(y, y'), \min_{x \geq m+s_1} X(y, y') \right\},$$

where  $X(y, y')$  is defined in (4) and where the lottery is degenerate on  $m_-$  if  $m_+$  does not exist.

**Proof.** Both (i) and (ii) are obvious consequences of strict concavity of  $w$ . ■

## 4 “Too Many” Denominations

If  $(w_s, \pi_s, \theta_s)$  is a steady state associated with the denomination structure  $s$ , then we view the inner product  $w_s \pi_s$  as the welfare associated with that steady state. That is,  $w_s \pi_s$  is expected utility prior to the assignment of initial wealth, where the assignment is made in accord with the steady-state distribution.<sup>7</sup> As the model stands and from the point of view of that welfare criterion, nothing seems to penalize the availability of many denominations. After all, people are not forced to hold all the denominations offered. Thus, it may well be that for a given  $s_1$ ,  $s = (s_1, 2s_1, 3s_1, \dots, Z)$  is a best structure, and that  $s_1$  should be “small.”

One way to penalize denomination structures with “too many” available denominations is to add a fixed cost that depends on the number of available denominations. For example, there could be a design cost for the government of making a distinct denomination available, a cost that does not depend on the stock of that denomination outstanding. If we assume that any such cost is financed by equal per capita lump-sum taxes, then adding it to the model does not change behavior and does not necessitate any substantive change in the definition of a steady state. In particular, proposition 1 is unaffected.

In order to obtain detailed results on good denomination structures in our model or in variants of it, we suspect that numerical searches have to be carried out. From what we have just said, that should be done for a version with fixed costs per denomination. To find a proposition-1 steady state for a given denomination structure, one iterates on the mapping used to define a steady state—the standard procedure for finding steady states in heterogeneous-agent models. Doing that is feasible provided that the set of possible individual wealth levels is not too large.

We do, however, have one general result about best denomination structures. It is based on the following conjecture: the linear interpolation of  $w$  is uniformly bounded above by a continuous function  $v$  with  $v(0) = 0$ .<sup>8</sup> If the

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<sup>7</sup>As this suggests, steady states are not unique. Autarky is a steady state and there are other steady states with trade that have step-function value functions, some of which are described below. Finally, for all we know, there may be multiple proposition-1 steady states.

<sup>8</sup>Under the assumption that  $u'(0)$  is bounded, the conclusion is proved in Zhu [10] for a one-denomination model with  $\gamma = 0$ . We suspect that the result holds for any denomination structure and any fixed  $\gamma$ . It is not known whether boundedness of  $u'(0)$  is necessary for the result.

conjecture holds, then for a fixed carrying cost  $\gamma$ , sufficiently small denominations are not valued or held because the value in trade of a sufficiently small denomination cannot exceed a fixed  $\gamma$ . Therefore, for any positive fixed costs per denomination, sufficiently small denominations are not part of an optimal denomination structure.

Suppose, however, that such small denominations are offered. If the conjecture holds, then, in accord with the discussion above, there is no steady state that satisfies the conclusions of proposition 1. (As noted above, the smallest denomination is held in a proposition-1 steady state.) The monetary steady states that do exist in such cases are best described in the context of a discussion of neutrality.

Let  $A = \{M, s, Z\}$  denote the set of exogenous nominal objects, where  $M$  is average money holdings and  $s = \{s_1, s_2, \dots, s_K\}$ . Now fix  $\gamma$  such that there is a proposition-1 steady state for  $A$ . If  $\lambda$  is a positive integer and  $\lambda A = \{\lambda M, \lambda s, \lambda Z\}$ , where  $\lambda s = \{\lambda s_1, \lambda s_2, \dots, \lambda s_K\}$ , then  $A$  and  $\lambda A$  support the same allocations. This is neutrality. But suppose we compare not  $A$  and  $\lambda A$ , but  $A$  and  $A' = \{\lambda M, s', \lambda Z\}$ , where  $s' = \lambda s \cup \{s_1\}$ . (That is,  $A'$  is  $\lambda A$  but with the addition of the smallest denomination from  $A$ .) For the fixed  $\gamma$  and for sufficiently large  $\lambda$ , although there may be no steady state that satisfies the conclusions of proposition 1, other steady states with trade do exist. In particular, any steady state for  $\lambda A$  (including a proposition-1 steady state) extended to the larger domain corresponding to  $s'$  through step-function value functions and distributions is a steady state for  $A'$ . Let  $\mathbf{Z}_{s'} = \{0, s_1, 2s_1, \dots, \lambda Z\}$  with generic element  $z'$  and let  $\mathbf{Z}_{\lambda s} = \{0, \lambda s_1, 2\lambda s_1, \dots, \lambda Z\}$  with generic element  $z$ . That is, if  $(w_{\lambda s}, \pi_{\lambda s}, \theta_{\lambda s})$  is a steady state for  $\gamma$  and  $\lambda A$ , then the following is a steady state for  $\gamma$  and  $A'$ :  $w_{s'}(z') = w_{\lambda s}(z)$  if  $z' \in \{z, z + s_1, z + 2s_1, \dots, z + (\lambda - 1)s_1\}$  where  $z \in \mathbf{Z}_{\lambda s}$ ;  $\pi_{s'}(z') = \pi_{\lambda s}(z)$  if  $z' = z \in \mathbf{Z}_{\lambda s}$  and  $\pi_{s'}(z') = 0$  otherwise; and analogously for  $\theta_{s'}$ . In such a steady state for  $A'$ , no one holds the denomination  $s_1$ .

To get people to actually discard units of the denomination  $s_1$ , non steady-state initial conditions have to be considered. For example, if there is a zero-measure departure from the above steady state for  $A'$  in which one person begins a date with one unit of the denomination  $s_1$ , then this person wants to discard that unit. It cannot be used at the portfolio-choice stage to get a larger unit and its value in trade is 0, and, hence, does not overcome the cost of carrying it into trade. This is the model's depiction of the public's

abandonment of the penny.<sup>9</sup> While the abandonment could arise as a steady state without the carrying cost, here—if the conjecture holds—it is dictated by the carrying cost.

## 5 Concluding Remarks

We have set out a very simple model. With our model as a base, different complications could be added. Replacement costs for different kinds of currency items could be added. The role of denomination structures in encouraging or discouraging illegal transactions could be considered. And idiosyncratic taste shocks could be added—shocks which, perhaps, would bring the model closer to the specification of exogenous transaction distributions. Given a specification that is “sufficiently realistic,” numerical searches could be carried out to obtain conclusions about the best denomination structure.

## 6 Appendix

As a prelude to the proof, we begin with some notation and assumptions. Let  $\gamma \in [0, 1]$ . Let  $R \equiv [N - (N - 1)\beta]^{-1} < 1$ . Also, let  $D$  be the unique solution to  $u'(D) = [2/(R\beta)]^2$ . Notice that  $D > 0$  provided that  $u'(0) > [2/(R\beta)]^2$ , one of our assumptions. Let  $W$  be the unique solution of  $N(1 - \beta)W = u(\beta W)$ . We assume that  $1/s_1 \geq \beta W/D$  and that  $Z \geq 4$ .

Let  $\mathbf{W}$  be the set of non-decreasing and concave functions  $w : \mathbf{Z} \rightarrow [0, W]$  with  $w(4) \geq D/\beta$ . Let  $\mathbf{K} \supset \mathbf{W}$  be the set of non-decreasing functions from  $\mathbf{Z}$  to  $[0, W]$ . Notice that the interior of  $\mathbf{W}$  (relative to  $\mathbf{K}$ ) is non-empty and that any element of the interior is strictly increasing, strictly concave, and satisfies  $w(4) > D/\beta$ . Let  $\mathbf{\Pi}$  be the set of probability measures  $\pi$  defined on  $\mathbf{Z}$  satisfying  $\sum \pi(z)z = 1$ . Let  $\mathbf{\Theta}$  be the set of probability measures  $\theta$  on  $\mathbf{Y}$  satisfying  $\sum \theta(y)sy = 1$ .

Now we can formally define the mapping to be studied. Let the mapping  $T_w : \mathbf{W} \times \mathbf{\Theta} \times [0, 1] \rightarrow \mathbf{K}$  be defined by

$$T_w(w, \theta, \gamma)(z) = g(z, h), \tag{13}$$

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<sup>9</sup>Matters would be more complicated if the model contained different technologies for storing pennies—in a pocket versus in a jar at home.

where  $g(z, h)$  is given in (2) and  $h$  is given by (9). (Here, the dependence of  $T_w(w, \theta, \gamma)$  on  $(w, \theta, \gamma)$  is through the dependence of  $h$  on  $(w, \theta, \gamma)$  which is given by (9) with  $f$  given by (6).) We let  $T : \mathbf{W} \times \mathbf{\Pi} \times \mathbf{\Theta} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{\Pi} \times \mathbf{\Theta}$  be defined by

$$T(w, \pi, \theta, \gamma) = (T_w(w, \theta, \gamma), T_\pi(w, \theta), T_\theta(w, \pi, \theta)),$$

where  $T_w(w, \theta, \gamma)$  is given by (13),  $T_\pi(w, \theta)$  is given by (11), and  $T_\theta(w, \pi, \theta)$  is given by (12).

**Lemma 3** *A fixed point of  $T$  is a steady state.*

**Proof.** Obvious. ■

The next four lemmas constitute a proof of proposition 1 through appeal to results based on fixed-point index theory.

**Lemma 4**  *$T$  is upper hemicontinuous (u.h.c.), compact-valued, and convex-valued.*

**Proof.** This follows from the Theorem of Maximum and convexification by lotteries. (For the non lottery version, one allows for all possible randomizations over the elements of the sets corresponding to  $\Delta_1$  and  $\tilde{\Delta}_2$ .)

■

**Lemma 5** *Let  $\mathbf{\Lambda} \equiv \mathbf{\Pi} \times \mathbf{\Theta}$  and let  $\partial\mathbf{W}$  denote the boundary of  $\mathbf{W}$  (with respect to  $\mathbf{K}$ ). Let  $\mathcal{G}$  denote the set of u.h.c., compact-valued, and convex-valued mappings  $g : \mathbf{W} \times \mathbf{\Lambda} \rightarrow \mathbf{K} \times \mathbf{\Lambda}$  satisfying  $(w, \lambda) \notin g(w, \lambda)$  for all  $(w, \lambda) \in \partial\mathbf{W} \times \mathbf{\Lambda}$ . There exists a fixed-point index defined on  $\mathcal{G}$ , denoted  $ind$ , satisfying: (A1) If  $g$  is constant on  $\mathbf{W} \times \mathbf{\Lambda}$  with the value  $(w_0, \lambda_0)$  where  $w_0 \in \mathbf{W} - \partial\mathbf{W}$ , then  $ind(g) = 1$ ; (A2) if  $ind(g) \neq 0$ , then  $g$  has a fixed point  $(w, \lambda)$  with  $w \in \mathbf{W} - \partial\mathbf{W}$ ; and (A3) if  $g_0, g_1 \in \mathcal{G}$  are homotopic on  $\partial\mathbf{W} \times \mathbf{\Lambda}$  (that is, there exists an u.h.c., compact-valued, and convex-valued  $G : \mathbf{W} \times \mathbf{\Lambda} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{\Lambda}$  such that  $(w, \lambda, \alpha) \notin G(w, \lambda, \alpha)$  for  $(w, \lambda, \alpha) \in \partial\mathbf{W} \times \mathbf{\Lambda} \times [0, 1]$  and  $G(\cdot, \alpha) = g_\alpha$  for  $\alpha = 0, 1$ ), then  $ind(g_0) = ind(g_1)$ .*

**Proof.** For continuous (single-valued) mappings from the closure of an open set  $A \subset \mathbb{R}^n$  to  $\mathbb{R}^n$ , existence of a fixed-point index with the relevant properties can be found in [8, Theorem 12.A, p. 535]. If  $B \subset \mathbb{R}^n$  is closed and

convex and  $A$  is open in  $B$ , then by [8, 13.6a, p. 604] that existence result can be generalized to continuous (single-valued) mappings from the closure of  $A$  to  $B$ . And, by [2, Theorem 36.1, p. 218], a further generalization can be made to u.h.c., compact-valued, and convex-valued mappings. In particular, the hypotheses required by [2, Theorem 36.1, p. 218] are satisfied by  $\mathcal{G}$  and the homotopy  $G$ . ■

**Lemma 6** Denote  $T(., \gamma)$  by  $T_\gamma(.)$ . Then  $T_0 \in \mathcal{G}$  and  $ind(T_0) = 1$ .

**Proof.** Notice that  $T_0$  is identical to the mapping  $\Phi$  studied in [9] and is also identical to the mapping  $\Phi_1$  studied in [11]. Then, by the exact logic used to show that  $\Phi_1 \in \mathcal{G}$  and  $ind(\Phi_1) = 1$  in [11, Lemma 2], we have  $T_0 \in \mathcal{G}$  and  $ind(T_0) = 1$ . ■

**Lemma 7** There exists  $\gamma_s > 0$  such that if  $\gamma \leq \gamma_s$ , then  $T_\gamma$  has a fixed point.

**Proof.** By property (A2) of lemma 5 and by lemma 6,  $T_0$  has a fixed point  $(w, \lambda)$  with  $w \in \mathbf{W} - \partial\mathbf{W}$ . This and lemma 4 imply that there exists  $\gamma_s > 0$  such that if  $\gamma \leq \gamma_s$ , then  $T_\gamma$  does not have a fixed point  $(w, \lambda)$  with  $w \in \partial\mathbf{W}$ . Then, by property (3) of lemma 5 and by lemma 6,  $ind(T_\gamma) = 1$  for  $\gamma \leq \gamma_s$ . Finally, by property (A2) of lemma 5,  $T_\gamma$  has a fixed point. ■

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