

Noncash payment methods in a cashless economy

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Abstract

Several non-cash record-keeping systems are formulated in a random-matching model. The crucial informational friction is that a seller's endowment, which may be a random variable, and the transfer of goods to a buyer can only be observed by the traders in a pairwise meeting. With this friction monitoring of the payment is distinguished from monitoring of the complete transaction. For each implementation notion we apply from monetary matching models, we show that the noncash systems can implement allocations implementable by the cash system, and for some parameters the noncash system can implement better allocations not implementable by the cash system.

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1 Introduction

In a modern economy, the use of noncash payment methods such as checks and cards (credit and debit) dominates the use of cash in payments.¹ Non-cash transactions seem to differ from cash transactions. Under the standard

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¹Humphrey [10] estimates that in the U.S. the share of legal consumer payments executed with cash is 20% in 2000.

view, cash transactions involve frictions which are difficult to overcome without cash. But in noncash transactions, do those frictions, or some of them, still exist? This question is critical to understand the economic consequences and policy implications of the widespread use of noncash payment methods. For, if noncash transactions occur under no frictions, then monetary policy that affects the allocations in cash transactions will have no direct effect on the allocations in noncash transactions. However, if noncash transactions occur under the same frictions, then the policy may have a direct effect.² Despite its importance, there is no explicit answer to this question in the literature. To address this question, one needs a formal model which makes clear the frictions that cash deals with and the functions that noncash payment methods play. Here we provide such a model.

We embed cash and noncash payment methods in an off-the-shelf matching model. Agents randomly become buyers or sellers, buyers are not endowed, sellers are randomly endowed and buyers are randomly paired with sellers to form meetings. In each pairwise meeting each agent's identity and type—buyer or seller—are public information, but the seller's endowment and the transfer of goods in the meeting are only observed by the agents in the meeting. That outsiders cannot observe the endowment and transfer provides a role for a record-keeping system.

We formulate three (non-coexisting) record-keeping systems: a cash system, a checks system and a reports system. With suitable interpretation, each system carries some form of *reported* information on transfers of goods (which need not be *actual* transfers). Cash in the cash system and checking-account balances in the checks system are the same physical object called money. In the cash system each agent's money holdings and the transfer of money are only observed by the pair in the meeting, in the checks system these quantities are observed by society. In the reports system there is no money but the two agents in each meeting can issue a type of jointly agreed report to society which reports the size of the seller's endowment and the amount of the good transferred. Having publicly observed reports, but only pairwise-observed transfers of goods, allows for public monitoring of payments but not transfers of goods.

The information content of the individual trading histories increases from

²For further relevance of the question, also consider the decline in the demand for money (measured by the M1-to-GDP ratio), which has developed with the spreading use of noncash payment methods and the related settlement arrangements.

the cash system to the checks system to the reports system. In the cash system each agent’s trading history is a scalar which represents his current money holdings, in the checks system each agent’s history is a finite-length vector which includes his money holdings at each date up to the current date. Because a report has sufficient dimensions to carry all relevant information in a meeting not observed by outsiders, the report system provides the richest information content of the individual trading histories.

To study how increasing the information content of trading histories affects risk sharing between buyers and sellers, we adopt a mechanism-design approach. For each implementation used in monetary matching models, we find that a system with more information content in the reported trading histories implements all allocations of a system that has less information content, and that there are parameters such that the reports system (and likely the checks system) dominates the cash system—the former implements the first best but the latter does not. Furthermore, for some implementation notions, we show that the non-cash systems behave like a cash system: roughly speaking, a buyer pays a seller by agreeing to issue a “good” report; and a good report gives the seller a higher expected continuation value, and the buyer a lower expected continuation value, than a bad report.

Of course, all systems in our model are idealizations. Departing from the idealizations should lead to a more realistic world where cash and non-cash payment methods coexist and some noncash transactions involve credit arrangements. In such a world, the same friction—that outsiders cannot observe transfers of goods in meetings—still exists in noncash transactions and cash transactions, and the individual payment histories made available by noncash payment methods should still be utilized by society to support a better allocation.

2 The model

Time is discrete, dated $t \geq 0$. There is a nonatomic measure of ex-ante identical infinitely lived agents index by I . There is one perishable good per date. Following the start of each date, the sequence of events at each date is described as follows. First, with equal probability, the type of an agent is a *buyer* or a *seller*. The probability of the type realizations is independent across time and population. After the type realization agents are randomly matched into buyer-seller pairs. After two agents are matched the buyer is endowed with 0 units of the good and the seller is endowed with $e \in$

$\{0, 1\}$, where $e = 1$ with probability $\rho \in (0, 1]$ and $e = 0$ probability $1 - \rho$.³ The probability of the sellers' endowment realizations is independent across time and population. After the seller's endowment realization, there are two phases of actions. In phase I, the seller chooses to consume $c \in [0, e]$. In phase II, the buyer and seller share the remaining resource in a way described below.⁴

The period utility of consuming q units of the good for the buyer and seller, respectively, is $u_b(q)$ and $u_s(q)$, where $u_b(0) = u_s(0) = 0$, $u'_b, u'_s > 0$, $u''_b < 0, u''_s \leq 0$, and $q^* = \arg \max u_b(q) + u_s(1 - q)$ is unique and positive. (If the seller consumes c_1 at phase I and c_2 at phase II, his period utility is $u_s(c_1 + c_2)$.) Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. Notice that at the start of date 0, the *ex-ante* optimal risk-sharing plan is such that in each future meeting the seller endowed with 1 transfers q^* to the buyer.

Now we turn to the information structure. In this model, there is no private information—information known only to an agent himself, for every action or realization by nature pertaining to the agent is either known by all agents in the economy or observed by his meeting partner. As is standard, information known by all agents in the economy is referred to as *public information*. Also, we refer to information only observed by the pair of agents in a meeting as *pairwise public information*. For each pairwise meeting, the identity of each agent and his type as a buyer or a seller become public information after the meeting is over (they are pairwise information during the meeting).⁵ The seller's endowment realization when $\rho < 1$ and the transfer of goods are directly observed only by the two parties in the meeting, so they are pairwise public information. At the start of each date a publicly observed random variable, uniformly distributed over $[0, 1]$, is realized; this public random device is used in some of the proofs.

Record keeping

Given the information structure described above and the continuum of agents, some record keeping is necessary to induce any risk sharing between

³We can formulate the economy as a version of the specialized-production economy as in Kiyotaki and Wright [14], which does not change anything in substance.

⁴Conceptually, it is natural to allow the seller to freely dispose his own endowment before sharing with the buyer. For our results below, only Lemma 2 depends on this specification (see footnote 16).

⁵Alternatively, we can assume that they are public information at the start of the meeting, which does not change anything in substance.

the buyer and seller in any meeting. The economy is equipped with one of the following three record-keeping systems: a cash system, a checks system and a reports system.

The cash and checks systems use money, a durable and intrinsically useless object, as the record keeping device. The set of individual money holdings is \mathbb{R}_+ ; money can be freely disposed. In the cash system, each agent's money holdings and the transfer of money in each meeting can only be observed by his meeting partner and hence are pairwise public information. In the checks system, the individual holdings are public information, and the transfer becomes public information after the meeting is over (it is pairwise public information during the meeting). The initial distribution of money across agents is public information in both systems; only in the checks system is each person's initial money holding public information.

In the reports system, the buyer brings an empty card to each meeting. There is an assigned place in the card for each agent to fill in some content. The buyer can enter some $(r^q, r^e) \in [0, 1] \times \{0, 1\}$ in his card space. (The number r^q pertains to the transfer of good and the number r^e pertains to the seller's endowment realization.) The seller can enter YES in his card space; if he does not enter YES, then NO automatically appears in the space at the end of the meeting. A filled card, denoted $r = (r^q, r^e, r^s)$ with $r^s \in \{\text{YES}, \text{NO}\}$, is a report. The buyer keeps possession of the card throughout the meeting.⁶ The filled card becomes a report, and its content becomes public information at the end of the current date. Under the reports system, each agent is assigned a one-dimensional *initial index* at the start of date 0, and this index is public information.

Trade and autarky

In the monetary systems a *deterministic trade*, denoted (q, l) , consists of a feasible transfer of q units of the good from the seller to the buyer, and a feasible transfer of l units of money from the buyer to the seller. Feasibility means that a transfer of an object does not exceed the existing resources in the meeting. Autarky means zero transfers of goods and money. In the reports system a deterministic trade, denoted (q, r) , consists of a feasible transfer of q units of the good and a report r . Autarky means zero transfer

⁶We can switch the roles of the buyer and the seller in filling and keeping the card. Also, we can assume that if the buyer does not enter any numbers in his space, some symbol such as NA appears automatically. Neither alternative affects anything in substance. When $\rho = 1$, we can assume that the buyer only enters a number in $[0, 1]$.

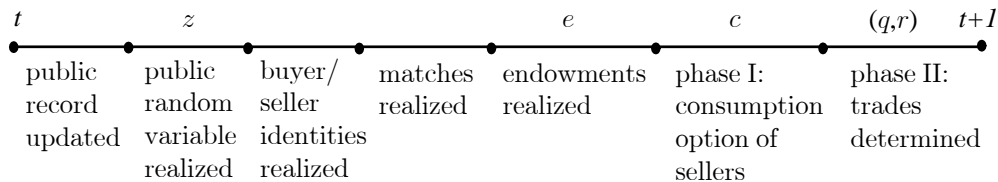


Figure 1: Sequence of Events.

of good, NO in the seller’s space of the card, and the buyer’s option to enter any numbers into his space.

Trading mechanisms and phase II

Conceptually, phase II in a meeting is the trading phase in which the buyer and the seller exchange goods for money or a report. For our purposes, to compare how different record-keeping systems support risk sharing, we formulate phase II in a way that permits us to apply mechanism design, a method that has long been applied to dynamic risk-sharing models including the monetary matching models. Specifically, we formulate a pairwise trading process in phase II as a trading mechanism, the definition of which closely follows the one in Kocherlakota [15].

In a record-keeping system, a generic *trading mechanism*, denoted by T , is a multistage extensive game form which assigns to each terminal node an outcome of T , which is either autarky or a trade. A trade can be a *lottery* over the set of all deterministic trades.⁷ As in [15], the number of stages in T is finite. For us, this restriction serves as a simple way to assure that, independent of the strategy profile, the game between a pair of agents is over before the next matching process starts. None of our results depend on the backwards deduction permitted by this restriction.⁸ A mechanism T is a *no-commitment mechanism* if in each meeting each agent has a sequence of actions leading to autarky, independent of any sequence of actions chosen by his meeting partner. (Simultaneous move voting games and ultimatum games are no-commitment mechanisms.)

⁷Lotteries are first introduced into monetary search models by Berenstein et al [4].

⁸Even if T has an infinite number of stages, there is a formulation to end the game before the date is over. The key is to let the duration of a stage shrink. Our results seem to apply to such a setup.

3 Equilibrium and implementable allocations

The matching process, the endowment process, a record-keeping system, the action in phase I of each meeting, a mechanism T and the basic assumptions of the model define a game. Our equilibrium concept is a version of Perfect Public Equilibrium (PPE). In each date- t meeting, $t \geq 0$, an agent's public strategy only conditions on the public information available at the start of t , and the pairwise public information of the *current* meeting. Therefore, if at date t agent i has two information sets that differ only in his pairwise public information pertaining to his *previous* meetings, then his strategy σ_i specifies the same action at these information sets. A profile of strategies $\sigma^* = \{\sigma_i^*\}_{i \in I}$ is an equilibrium if and only if $\{\sigma_i^*\}_{i \in I}$ are public strategies and $\{\sigma_i^*\}_{i \in I}$ evaluated at any history of the game determines a Nash equilibrium. Following standard practice in monetary matching models, in the cash system we require that in an equilibrium, at the start of each date $t > 0$, each agent has a belief of the distribution of money that is consistent with the equilibrium strategy profile σ and the initial distribution of money.⁹

To be more specific, we introduce some notation used throughout. Fix $i \in I$. Let $\theta_{i,t} = 0$ if i is a buyer at t , and $\theta_{i,t} = 1$ if i is a seller. Let $\gamma_{i,t}$ be agent i 's meeting partner at t . Let $\omega_{i,t}$ be the seller's endowment realization in the meeting between i and $\gamma_{i,t}$ at t . In the monetary systems, let $m_{i,t}$ be i 's start-of- t money holding. In the reports system, let $r_{i,0}$ be i 's initial index, and for $t > 0$, let $r_{i,t} = (r_{i,t}^q, r_{i,t}^e, r_{i,t}^s)$ be the most recent report associated with i available at the start of t , i.e., the report from the meeting between i and $\gamma_{i,t-1}$ at date $t-1$. For $x \in \{\theta, \gamma, \omega, m, r\}$, let $x_i^t = (x_{i,0}, \dots, x_{i,t})$. Let $\eta_i^0 = \emptyset$, and for $t > 0$, let $\eta_i^t = (\theta_i^{t-1}, \gamma_i^{t-1})$. Also, let z_t be the realization of the public random variable at the start of t , and let $z^t = (z_0, \dots, z_t)$. For the date- t meeting in concern, the public information is $\{z^t\} \cup \{\eta_i^t\}_{i \in I}$ in the cash system, $\{z^t\} \cup \{(\eta_i^t, m_i^t)\}_{i \in I}$ in the checks system, and $\{z^t\} \cup \{(\eta_i^t, r_i^t)\}_{i \in I}$ in the reports system. In the cash system, the belief on the distribution of money indicated above is the distribution of the individual money holdings across the set of $\{\eta_i^t\}_{i \in I}$.

An *allocation*, denoted $\{\alpha_t\}_{t=0}^\infty$, specifies that when buyer i meets seller j with endowment e at t , the transfer of goods is $\alpha_t(i, j, e; \Gamma_t)$, where $\Gamma_0 = \emptyset$

⁹Our equilibrium concept is consistent with the one adopted in [15]. In [15], information only known to agents in the present match is termed as public information. Information known to agents in previous matches but unknown to other agents' in the present match is termed as private information.

and $\Gamma_t = \{z^t\} \cup \{(\theta_i^{t-1}, \gamma_i^{t-1}, \omega_i^{t-1})\}_{i \in I}$ is the history of the realizations of the public random variable, the types, the matching outcomes, and the seller's endowments in the economy up to the start of t .

Of interest is whether an allocation may be implemented by a record-keeping system. We consider three strengthened notions of implementation, termed as SIR, PE, and RP implementation, which are ordered in their restrictiveness. Here SIR, PE, and RP stand for sequential individual rationality, Pareto efficiency, and renegotiation proof, respectively. Each notion can be found in the literature on monetary matching models.¹⁰ We begin with SIR implementation, the least restrictive notion.

Definition 1 *A record-keeping system SIR implements an allocation $\{\alpha_t\}$ by a trading mechanism T and an equilibrium strategy profile σ if $\{\alpha_t\}$ can be generated by on-the-path plays specified by σ and if T is a no-commitment mechanism.*

In Definition 1, the no-commitment requirement on T plays a weaker role than it plays in [15], because at phase I a seller can freely dispose his endowment.¹¹ Nevertheless, this requirement still has a role in ensuring individual sequential rationality; for instance, it ensures that one agent cannot be forced to give up his money.

The concept of PE implementation is motivated by the Coase Theorem. Specifically, in each meeting, following each feasible phase-I consumption by the seller, the buyer and seller enter into a pairwise trading process, the one that is specified by T , and by the Coase Theorem, they ought to reach a pairwise Pareto efficient outcome.

Definition 2 *A record-keeping system PE implements an allocation $\{\alpha_t\}$ by a trading mechanism T and an equilibrium strategy profile σ if it SIR implements $\{\alpha_t\}$ by T and σ , and if in each meeting following each feasible*

¹⁰To name a few references, SIR implementation is adopted by Kocherlakota [15], Kocherlakota and Wallace [17], and Kazman et al. [13], and PE implementation is adopted by Hu et al. [9] and Zhu and Wallace [29]. Most monetary matching models adopt some form of generalized Nash bargaining. In our RP implementation, the trade in a meeting is generated by a suitable form of generalized Nash bargaining.

¹¹For a match in [15], there is no counterpart of phase I, so that any allocation can be implemented by a trivial mechanism. In the trivial implementation, choosing autarky would make a seller better off than any positive transfer specified by the allocation, but since autarky is not a feasible outcome under the mechanism it cannot be chosen.

phase-I consumption by the seller, any outcome of T implied by σ is pairwise Pareto efficient.

In Definition 2, the Pareto-efficient restriction applies regardless of whether some agent in the meeting has deviated from σ in the previous meetings, or whether the seller has deviated from σ in phase I.

The concept of RP implementation is motivated by the assumption that a pair of agents cannot bilaterally commit to select a Pareto dominated trade in a trading process—they would renegotiate away from it to a Pareto efficient one. (With PE implementation, the pair of agents could end up at a Pareto dominated outcome due to some off-equilibrium play in phase II.) Specifically, in RP implementation, in each meeting, following each feasible phase-I consumption by the seller, on any path in which the buyer and seller reach an outcome of T that is Pareto dominated, they will renegotiate to a pairwise efficient outcome. Following widespread convention, we assume that the buyers's bargaining power is $\lambda \in (0, 1]$ in any such renegotiation step, where the bargaining set consists of all feasible trades in the stage from which the inefficient outcome of T was reached.

Definition 3 *A record-keeping system RP implements an allocation $\{\alpha_t\}$ by a trading mechanism T and an equilibrium strategy profile σ if it PE implements $\{\alpha_t\}$ by T and σ , and if in each meeting no agent can improve from any outcome implied by σ by any deviation which results in renegotiation.*

In Definition 3, the role of renegotiation is analogous to the role of renegotiation in models in the literature on contracting and Nash implementation. Indeed, if a contract specifies a Pareto dominated outcome following some agent's off-equilibrium play, then renegotiation occurs. Also, our treatment of the renegotiation process is the same as the standard treatment in the contracting literature, i.e., agents are assigned exogenous bargaining powers.¹²

4 Inclusion results

For each notion of implementation, there are some general inclusion relationships among the record-keeping systems. To see this consider a mechanism T

¹²In the contracting literature, Maskin and Moore [21] study a more generalized treatment, i.e., a renegotiation process is a mapping that maps the set of alternatives into the set of Pareto efficient alternatives. If we allow any mapping of this sort, then our RP-implementation concept would imply no more restrictions than PE-implementation.

and an equilibrium σ under the cash system. We can duplicate σ and obtain an equilibrium strategy profile σ' for the same T under the checks system by letting the dependence of the agents' actions on m_i^t in σ' reduce to the dependence on $m_{i,t}$ in σ . Also, we can map m_i^t one-to-one into r_i^t , which, again, enables us to duplicate an equilibrium σ under the checks system and obtain an equilibrium strategy profile σ' under the reports system. In other words, from cash to checks, and from checks to reports, the equilibrium information content of the public record increases, permitting a strategy profile in the more information-rich system to duplicate an equilibrium strategy profile in the previous system. This argument leads to the following result.

Proposition 1 *Fix a notion of implementation in Definitions 1-3. We say that record-keeping system A includes system B if A implements any allocation B implements.*

- (i) *The checks system includes the cash system.*
- (ii) *The reports system includes the checks system.*

Proof. Here we provide a proof for part (ii). The proof for part (i) is similar so it is delegated to the Appendix. For part (ii), first consider the checks system and let $m_{i,0}$ be agent i 's initial money holding. Suppose that the mechanism T and the equilibrium strategy profile σ implement the allocation $\{\alpha_t\}$. Next consider the reports system and let agent i 's initial index be $f(m_{i,0})$, where f is an arbitrary strictly increasing function \mathbb{R}_+ to $[0, 1]$. Now we construct the mechanism T' and the strategy profile σ' as follows. Fix t and $\{z^t\} \cup \{(\eta_k^t, r_k^t)\}_{k \in I}$. For $k \in I$, let $m_{k,0} = f^{-1}(r_{k,0})$ and $m_{k,\tau} = g(r_{k,\tau})f^{-1}(r_{k,\tau}^q)$ for $\tau > 0$, where f^{-1} is the inverse of f , $g(r_{k,\tau}) = 0$ if $r_{k,\tau}^s = \text{NO}$, and $g(r_{k,\tau}) = 1$ if $r_{k,\tau}^s = \text{YES}$.

Fix a date- t meeting between buyer i and seller j . In this meeting, for each agent, T' specifies the same sets of actions as T . If a sequence of actions leads to autarky under T , then under T' it also leads to autarky. If sequence of actions leads to a trade under T , then under T' it leads to the same trade but with l being replaced by $r = (f(l), 1, \text{YES})$. At phases I and II of the meeting, σ' specifies the same actions that σ specifies under the checks system when the public information is $\{z^t\} \cup \{(\eta_k^t, m_k^t)\}_{k \in I}$. Moreover, when autarky is reached, σ' specifies the buyer to enter 0 in the card. ■

Referring to the mapping f in the proof of Proposition 1, one may interpret money as a special form of report—special in that it supports a more

restrictive information record—and, therefore, interpret each monetary system as a special reports system.

Some of our results below for the checks and report systems can be conveniently described in a system called the *unilateral-reports system*, which is a weakened reports system. Specifically, this system is the same as the reports system except that a unilateral report only consists of a number in $[0, 1]$ entered by the buyer. For our purposes, it suffices to let the unilateral report of the buyer be a number which represents the transfer of the good.¹³

When the context is clear, we refer to a unilateral report as a report, and adopt the same notation for a unilateral report as for a report. That is, we denote by r a unilateral report, by $r_{i,0}$ agent i 's initial index in the unilateral report system, and by $r_{i,t} \in [0, 1]$ the unilateral report from the meeting between i and $\gamma_{i,t-1}$ at $t - 1 \geq 0$.

The unilateral-reports system is useful because of the next inclusion relationship, which follows from the same argument leading to Proposition 1.

Lemma 1 *Fix a notion of implementation in Definitions 1-3. The checks system includes the unilateral-reports system.*

Proof. See the Appendix. ■

To strengthen the inclusion results, in the next three sections we investigate whether more information-rich record keeping systems can support better allocations. We organize the investigation of this question around implementing the *first best* allocation—the buyer consumes q^* whenever the seller's endowment realization is 1—and by the three implementation concepts. Notice that if

$$\beta \rho u_b(q^*) > (2 + \rho\beta - 2\beta)[u_s(1) - u_s(1 - q^*)], \quad (1)$$

then the first best can be implemented under perfect information.¹⁴

¹³A unilateral report can be interpreted as IOU issued by the buyer, while it is never been redeemed by the buyer directly. Also, when $\rho = 1$ a report can be interpreted as IOU issued by the buyer and confirmed by the seller.

¹⁴The only concern is that the seller may deviate in the current meeting. With perfect monitoring, the seller can be punished by permanent autarky for any deviation. So the most the seller can obtain from deviation is $u_s(1) + 0.5\beta\rho u_s(1)/(1 - \beta)$. The seller obtains $u_s(1 - q^*) + 0.5\beta\rho[u_s(1 - q^*) + u_b(q^*)]/(1 - \beta)$ if he does not deviate.

5 SIR implementation

Here we start with results for the cash system. Because each agent's type is public information, when $\rho = 1$ it is easy to construct a “right” holding at each t for each agent by conditioning on his history of type realizations. An agent with a “wrong” money holding will be punished by autarky from t on. Given (1), this assures the cash system will SIR implement the first best. When $\rho < 1$ it becomes impossible to implement the first best under the cash system. The reason is that for implementation it is necessary for a buyer (with a right holding) to transfer some positive amount of money to a seller (with a right holding), regardless of the seller's endowment e being 0 or 1. But then at some meeting for some such buyer, if he deviates to autarky when $e = 0$, he can benefit from the extra money he has saved.

Proposition 2 *Let (1) hold. Then the cash system can SIR implement the first best if and only if $\rho = 1$.*

Proof. For the “if” part, let $\rho = 1$. Let $d_{i,0} = 1$ and define by induction: $d_{i,t+1} = d_{i,t} + 10^{-(t+1)^2}$ if $\theta_{i,t} = 1$, and $d_{i,t} = d_{i,t} - 10^{-(t+1)^2}$ if $\theta_{i,t} = 0$. Let $m_{i,0} = 1$, all i . Now consider the following trading mechanism. When buyer k meets and seller j at t , the two agents simultaneously announce a number in $\{1, 0\}$. If the seller consumes $c = 0$ at phase I and both say 1, then $(q, l) = (q^*, 10^{-(t+1)^2})$; otherwise, $(q, l) = (0, 0)$. Under this T , consider the following strategy profile. For the meeting in concern, at phase I, the seller consumes $c = 0$. At phase II, if $m_{i,t} = d_{i,t}$ for $i = k, j$, then each agent says 1; otherwise, each agent says 0. When (1) holds, this strategy profile is an equilibrium and implements the first best. The proof for the “only-if” part is delegated to the Appendix. ■

In the proof of the “if” part of Proposition 2, $\theta_{i,t}$ (agent i 's type) being public information and $m_{i,t}$ (agent i 's current money holding) being pairwise public information both play a role. As it turns out, this part of the proposition does not hold if $\theta_{i,t}$ is weakened to pairwise public information, but still holds if $m_{i,t}$ is weakened to private information. Each of the two weakened assumptions gives rise to a natural alternative formulation of the cash system. Compared to those alternatives, the formulation we adopt preserves a unified treatment across record keeping systems and permits us to

concentrate on one dimension, i.e., the role of record keeping regarding the transfer of the good.¹⁵

For the “only-if” part of Proposition 2, that $m_{i,t}$ is only pairwise public information is critical to have some buyer gain from deviating to autarky when meeting a zero-endowed seller. In the proof we show that the buyer can later dispose of the extra money due to the deviation in a way that both escapes detection and correctly adjusts money holdings after choosing an autarkic consumption when he becomes an endowed seller. In fact, the same argument implies the following.

Corollary 1 *Let $\rho < 1$. Suppose the cash system PE implements some $\{\alpha_t\}$. Then it is impossible that $\{\alpha_t\}$ specifies the same positive transfer of the good in all meetings when sellers are endowed.*

In the checks system $m_{i,t}$ becomes public information, which assures SIR-implementation of the first best.

Proposition 3 *Let (1) hold. Then the checks system SIR implements the first best.*

Proof. Let $m_{i,0} = 1$, all i . Consider the following trading mechanism. When buyer k meets seller j at t , after the seller consumes $c \in [0, e]$ in phase I, the two agents simultaneously announce a number in $\{1, 0\}$. If $c = 0$, $e = 1$ and both say 1, then $l = 10^{-(t+1)^2}$ and $q = q^*$; otherwise $(q, l) = (0, 0)$. Under this T , consider the following strategy profile. For the meeting in concern, at phase I, the seller consumes $c = 0$. At phase II, if $m_i^t = d_i^t$ for $i = k, j$ and if the measure of agents in I with $m_i^t \neq d_i^t$ is zero, then each agent says 1, where $d_i^t = (d_{i,0}, \dots, d_{i,t})$ with $d_{i,t}$ being defined in the proof of Proposition 2. Otherwise, each agent says 0. When (1) holds, such defined strategy profile is an equilibrium and implements the first best. ■

In the proofs of Proposition 2 and Proposition 3, neither implementation makes a distinction between the following two situations in which a seller has an off-equilibrium money holding. In one, the seller has defected in the past. In another, the seller has not defected in the past, but has the off-equilibrium holding after meeting a buyer who has defected in the past.

¹⁵For $\rho \leq 1$, if money cannot be freely disposed, the “if” part of Proposition 2 holds under the first weakened assumption. See Kocherlakota [16].

In this case, even though the seller is not a previous defector, he will be punished as a by-product of punishing the defector buyer. In particular, the non-defector seller and defector buyer both obtain the autarky payoff in the future. The mutual-autarky payoff, however, makes the meeting outcome pairwise Pareto dominated.¹⁶ For this reason neither implementation in the two proofs is a PE-implementation. We note that Proposition 3 can also be proved by constructing an equilibrium in which a single agent’s detectable deviation triggers global autarky—all agents in the economy get the autarky payoff in the future, but this is not a PE implementation either.

6 PE implementation

Here we first give a general characterization about PE implementation. We state the result for the reports system, and by Proposition 1, it can be adapted to other systems with obvious modifications. Here and below, we denote by \tilde{r} a generic lottery over the set of reports.

Lemma 2 *Suppose the reports system PE implements some $\{\alpha_t\}$ by some T and σ . Fix a meeting in which the seller’s endowment is $e = 1$. Let (q, \tilde{r}) with $q > 0$ be an outcome of T implied by σ , and let (q', \tilde{r}') be an outcome of T implied by σ after the seller consumes $c' > 1 - q$ at phase I. Let v_b (v'_b) and v_s (v'_s) be the expected continuation values of the buyer and seller, respectively, implied by \tilde{r} (\tilde{r}'). If $q' < q$, then $v'_b > v_b$ and $v'_s < v_s$.*

Proof. First, we note that in each of the two outcomes, (q, \tilde{r}) and (q', \tilde{r}') , the report component, as indicated by the convention of the notation, can be stochastic, but by strict concavity of u_b and pairwise Pareto efficiency, the transfer component must be deterministic. Now let $q' < q$. First suppose $v'_s \geq v_s$. But then the seller can improve his payoff from (q, \tilde{r}) by consuming c' at phase I, which leads to (q', \tilde{r}') . Next suppose $v'_s < v_s$. But given $v'_s < v_s$, (q', \tilde{r}') is Pareto dominated by (q', \tilde{r}) . ■

By Lemma 2, in order to induce the buyer to truthfully report a deviation by a seller, it is necessary for PE-implementation to give the buyer an incentive (regardless of the discount factor). In other words, a rise in the buyer’s

¹⁶For example, consider the mechanism in the proof of Proposition 2 (ii). At $t = 1$, when buyer k with $d_{k,1} \neq m_{k,1} \geq 10^{-4}$ meets seller j with $m_{j,1} = d_{i,1}$, then $(q, l) = (q^*, 10^{-4})$ Pareto dominates $(q', l') = (0, 0)$, the trade implied by the mechanism.

off-equilibrium continuation value (from the in-equilibrium level) is associated with a fall in the seller's (from the in-equilibrium level).¹⁷ This means that in the cashless economies traders' continuation values have properties that qualitatively correspond to well-known properties of cash economies. Furthermore, Lemma 2 implies that it is impossible to PE implement the first best by the global-autarky trigger strategy indicated in the last section.

Lemma 2, however, does not say that it cannot be an equilibrium outcome that the buyer and seller leave the meeting with the same continuation value, provided that they have the same continuation value in the end of the last date. For such an implementation, the key is to let each agent in each meeting use all available public information related to his meeting partner. To be more specific, we introduce some new terminology.

Under the checks system, we call $h_i^t = (\eta_i^t, m_i^t)$ agent i 's *direct public history* available at the start of t . Under the reports system and the unilateral-reports systems, this history is $h_i^t = (\eta_i^t, r_i^t)$. For $t > 0$ let $\psi_{i,t}^{t-1} = \{\gamma_{i,t-1}\}$, and when $t > 1$ define by induction

$$\psi_{i,t}^\tau = \{\gamma_{i,\tau}\} \cup \{\gamma_{k,\tau} : k \in \psi_{i,t}^\varsigma, \varsigma \geq \tau + 1\}, \quad 0 \leq \tau \leq t - 2.$$

That is, if $j \in \psi_{i,t}^\tau$, then at date τ agent j meets either agent i or some agent k who will directly or indirectly meets agent i before t . The significance of j to i is that if j has deviated before τ , then at τ his previous deviation may affect i 's direct public history at the start of t (either by meeting i or someone who directly or indirectly meets i before t).¹⁸ Now let $H_i^0 = \emptyset$, and for $t > 0$ let

$$H_i^t = \{h_k^\tau : k \in \psi_{i,t}^\tau, 0 \leq \tau \leq t - 1\}. \quad (2)$$

¹⁷To see that Phase I is critical for the result consider eliminating phase I and not including a corresponding consumption stage in phase II. Then the following mechanism PE implements the first best under the unilateral-report system when $\rho = 1$ and (1) holds. In a meeting, two agents simultaneously announce a number in $\{0, 1\}$. If both agents have a non-defector public history and both say 1, and if 1 unit of good is still available, then the trade is $(q, r) = (1, 1)$; otherwise, $(q, r) = (0, 0)$. Here a fall in the buyer's off-equilibrium continuation value is associated with a fall in the seller's.

¹⁸This point can be seen from the equilibrium in the proof of Proposition 3. There, at each date- t meeting, each agent conditions his actions in the current meeting only on the direct public history of his meeting partner. But if agent i has met a defector before t , even though i himself has never defected, the component of money in his direct public history differs from the one if he has not met a defector.

We call H_i^t as agent i 's *indirect public history* available at the start of t .¹⁹ We call (h_i^t, H_i^t) as agent i 's *public history* available at the start of t .

By the next proposition, when each agent conditions his actions in each meeting on his partner's public history, the reports system PE implements the first best when (1) holds.

Proposition 4 *Let (1) hold. Then the reports system can PE-implement the first best.*

Proof. To begin with, we define a finite set of one-dimensional statistics, denoted \mathbf{V} , which turns out to be the set of the continuation values admitted in the equilibrium σ we construct to implement the first best. Let v^* and \underline{v} satisfy $2(1 - \beta)v^* = \rho[u_b(q^*) + u_s(1 - q^*)]$ and $2(1 - \beta)\underline{v} = \rho u_s(1)$ (i.e., v^* is the first best value and \underline{v} is the autarky value). Let $\delta_s = \rho[u_s(1) - u_s(1 - q^*)]$, and let $v_+ = v^* + 0.5\delta_s$ and $v_- = v^* - \beta^{-1}\delta_s$. By (1), $v_- > \underline{v}$. Let $\mathbf{V} = \{v^*, v_-, v_+, \underline{v}\}$. (While \mathbf{V} is finite, lotteries permit that the expected continuation value of the trade implied by σ varies by a continuous way when the seller's phase I consumption varies.)

First we consider the case $\rho = 1$. By Proposition 1 and Lemma 1, it suffices to show that the unilateral-reports system PE-implements the first best. In this system, each agent is assigned the initial index v^* . We use the following mechanism T for the implementation: the buyer makes a take-it-or-leave-it offer to the seller following each phase-I consumption $c \in [0, 1]$ of the seller.

To define the candidate equilibrium σ , we first define a mapping, denoted $g \equiv (g_b, g_s)$, which permits us to encode an agent's public history into a statistics in \mathbf{V} . The mapping g represents an artificial dynamic process. In this process, the statistic v^* is initially attached to each agent, and this statistic changes as information is updated, including reported information. To be concrete, consider agent b with statistic $v_b \in \mathbf{V}$ and agent s with statistic $v_s \in \mathbf{V}$ at the start of t . Let z be the realization of the public random variable. Let the matching and type realizations be such that b meets s , b becomes a buyer, and s becomes a seller. Let r be the unilateral report of the meeting. Let $x = (r, v_b, v_s, z)$. The statistics attached to b and

¹⁹Notice that for $j \in \psi_{i,t}^T$, any public information related to him from $\tau+1$ is not included in H_i^t . Also, the related parts in the direct and indirect histories must be consistent. For example, let $t = 2$, $\gamma_{i,1} = j$ and $\theta_{i,1} = 1$. Then it must be that $\gamma_{j,1} = i$ and $\theta_{i,1} = 0$. Also, under the reports system, it must be that $r_{i,1} = r_{j,1}$.

s at the start of $t + 1$ are $g_b(x)$ and $g_s(x)$, respectively, which are defined in the following six exclusive and exhaustive cases.

Case 1: $(v_b, v_s) = (v^*, v^*)$. Then $g(x) = (g_b(x), g_s(x)) = (v^*, v^*)$ for $r \geq q^*$, and $g(x) = (v_+, v_-)$ for $r < q^*$.

Case 2: $(v_b, v_s) = (v^*, v_+)$. Then $g(x) = (v^*, v^*)$.

Case 3: $(v_b, v_s) = (v^*, v_-)$. Then $g(x) = (v^*, v^*)$ for $r \geq q^*$, and $g(x) = (v_+, v_-)$ for $r < q^*$.

Case 4: $(v_b, v_s) = (v_+, v^*)$. Then $g(x) = (v^*, v^*)$ for $r \geq q^*$, and $g(x) = (v_+, v_-)$ for $r < q^*$.

Case 5: $(v_b, v_s) = (v_-, v^*)$. Let $\bar{z} \equiv 1 - 2\delta_s/[u_b(q^*) + \beta\delta_s]$. By (1), $\bar{z} \in (0, 1)$. If $z \leq \bar{z}$ then $g(x) = (v^*, v^*)$ for $r \geq q^*$, and $g(x) = (v_+, v_-)$ for $r < q^*$. If $z > \bar{z}$ then $g(x) = (v_-, v^*)$.

Case 6: For all remaining (v_b, v_s) , let $g(x) = (v_b, v_s)$.

Now we can map an agent's public history into a statistic in \mathbf{V} as follows. For $t > 0$, we say that agent i 's public history (h_i^t, H_i^t) (see (2)) is equivalent to $v \in \mathbf{V}$ if given (h_i^t, H_i^t) and z^t , v is the statistic attached to the agent at the the start of t in the dynamic process represented by g . This equivalence relationship defines a mapping $(h_i^t, H_i^t) \mapsto f_t(h_i^t, H_i^t; z^{t-1}) \in \mathbf{V}$. For $t > 0$, let φ_t be the distribution of the individual agent's public histories at the start of t , and given z^{t-1} define the distribution π_t over \mathbf{V} by $\pi_t[0, w] = \varphi_t\{(h_i^t, H_i^t) : f_t(h_i^t, H_i^t; z^{t-1}) \leq w\}$ (recall that $\pi_0\{v^*\} = 1$).

To define σ , let z be the realization of the public random variable at the start of t , and consider a date- t meeting in which a buyer has a public history equivalent to v_b and a seller has a public history equivalent to v_s . There are two exhaustive and exclusive situations. In the first situation, $\pi_t\{v^*\} \neq 1$. Then at phase I the seller consumes 1. At phase II the buyer offers $(0, 0)$, and the seller accepts an offer if and only if the transfer of the good is zero. In the second situation, $\pi_t\{v^*\} = 1$. Then at phase I the seller consumes 0. Let $U_b(q, \tilde{r}, v_b, v_s, z) = u_b(q) + \beta E g_b(\tilde{r}, v_b, v_s, z)$ and $U_s(q, \tilde{r}, v_b, v_s, z) = u_s(1 - q) + \beta E g_s(\tilde{r}, v_b, v_s, z)$, where \tilde{r} denotes a generic lottery over the set of reports and E is the expectation operator; without loss of generality, we can restrict the support of \tilde{r} to $\{0, q^*\}$. Given the seller's phase-I consumption c , at phase II the buyer offers

$$(q(c), \tilde{r}(c)) = \arg \max [U_b(q, \tilde{r}, v_b, v_s, z) - U_b(0, 0, v_b, v_s, z)]$$

subject to $0 \leq q \leq 1 - c$ and $U_s(q, \tilde{r}, v_b, v_s, z) \geq U_s(0, 0, v_b, v_s, z)$; and the seller accepts an offer (q, \tilde{r}) if and only if $U_s(q, \tilde{r}, v_b, v_s, z) \geq U_s(0, 0, v_b, v_s, z)$. In both situations, the buyer enters 0 in the card whenever autarky occurs.

For each $c \in [0, 1]$, the construction of g implies the following properties of $(q(c), \tilde{r}(c))$. First, $U_s(q(c), \tilde{r}(c), v_b, v_s, z) = U_s(0, 0, v_b, v_s, z)$. Next, if (v_b, v_s, z) is such that the values of g_b and g_s do not depend on r (as in case 2, case 5 with $z > \bar{z}_-$, and case 6) then $q(c) = 0$. If (v_b, v_s, z) is such that the values of g_b and g_s depend on r (as in cases 1-4, and case 5 with $z \leq \bar{z}$) then $q(c) = \min\{q^*, 1 - c\}$ (note $u_b(q^*) \geq u_s(1 - q^*)$), and the support of $\tilde{r}(c)$ is $\{q^*\}$ if and only if $q(c) = q^*$.

To verify that the above σ is an equilibrium, fix i and t and let agent i 's public history at the start of t be equivalent to v . It suffices to show that if all other agents always follow σ and agent i follows σ from $t + 1$ then agent i 's continuation value at the start of t , denoted w , equals v . This follows from the construction of g and the properties of $(q(c), \tilde{r}(c))$ given above. If $v = v^*$ then $w = 0.5[u_b(q^*) + \beta v^*] + 0.5[u_s(1 - q^*) + \beta v^*] = v^*$ (refer to case 1 in the construction of g). If $v = v_+$ then $w = 0.5[u_b(q^*) + \beta v^*] + 0.5[u_s(1) + \beta v^*] = v_+$ (refer to cases 2 and 4). If $v = v_-$ then $w = 0.5\bar{z}[u_b(q^*) + \beta v^*] + 0.5(1 - \bar{z})\beta v_- + 0.5[u_s(1 - q^*) + \beta v^*] = v_-$ (refer to cases 3 and 5). If $v = \underline{v}$ then $w = 0.5\beta \underline{v} + 0.5[u_s(1 - q) + \beta \underline{v}] + 0.5\underline{v} = \underline{v}$ (refer to case 6).

Now we consider $\rho < 1$. As it turns out, the unilateral system cannot PE implement the first best, but the reports system can. Let each agent's initial index be v^* . Consider the following T . Fix a meeting. When $e = 1$ the buyer makes a take-it-or-leave-it offer at phase II. When $e = 0$, both agents simultaneously announces a number in $\{0, 1\}$ at phase II. If both say 1, then the trade is $(0, R^*)$ with $R^* = \{(0, 0), \text{YES}\}$; otherwise the meeting ends up with autarky. The major part of the rest of the proof is an extension of the proof for $\rho = 1$. We leave the details of the proof in the Appendix. ■

To see the role of the agent's public history in the proof of Proposition 4, one can refer to the construction of g in case 5 when $z > \bar{z}$. There, when a non-defector seller (the one with the statistic v^*) meets a defector buyer (the one with the statistic v_-), the seller's future payoff does not depend on the (unilateral) report issued by the buyer.

To PE-implement the first best when $\rho < 1$, it is crucial to distinguish the public records of two possible outcomes of a meeting which could potentially be confused with each other. One outcome pertains to the situation when the seller consumes $c = 1$ (when the seller's endowment $e = 1$) in phase I. Another outcome pertains to the situation when $e = 0$. Under the reports system, it is easy to distinguish two such records: in the above proof, one record is $\{(0, 1), \text{YES}\}$, and the other is $\{(0, 0), \text{YES}\}$. But it is impossi-

ble to distinguish two such records under the checks system (as well as the unilateral-reports system), and, therefore, we have the following.

Proposition 5 *Let (1) hold. Then the checks system can PE-implement the first best if and only if $\rho = 1$.*

Proof. The “if” part follows the proof of Proposition 4 for $\rho = 1$. For the “only-if” part, let $\rho < 1$ and assume by contradiction that some T and σ implement the first best. Denote by w^* the same the same post-meeting continuation value, at any t , for all in-equilibrium agents. Consider buyer i and seller j in a meeting at date 0. When $e = 1$, let (q^*, l) be the outcome of T implied by σ after j consumes some $c \leq 1 - q^*$ implied by σ at phase I; it is without loss of generality that this outcome is deterministic. By transferring l , both i and j obtain w^* as the post-meeting continuation value. Let $(0, l')$ be the outcome of T implied by σ after j consumes $c = 1$ at phase I. It is without loss of generality that this outcome is deterministic and that $l' = 0$. Let w_k be the post-meeting continuation value of k , $k \in \{i, j\}$, implied by zero transfer of money. By Lemma 2, $w_i > w^*$ and $w_j < w^*$. When $e = 0$, let $(0, l'')$ be the outcome of T implied by σ ; it is without loss of generality that this outcome is deterministic. By transferring l'' , both i and j obtain w^* as the post-meeting continuation value. But then when $e = 0$, the buyer can benefit by choosing autarky. ■

The next corollary follows from the proof of Proposition 5.

Corollary 2 *Let $\rho < 1$. Let the checks system PE implement some $\{\alpha_t\}$ by some T and σ . If $\{\alpha_t\}$ specifies a positive transfer for some meeting at t , then the equilibrium distribution of the individual end-of- t continuation values cannot be degenerate. In particular, it is impossible that $\{\alpha_t\}$ specifies the same positive transfer of the good for all meetings when sellers are endowed.*

Finally, we remark that we have examined whether the cash system can PE implement the first best when $\rho = 1$, but we have not been able to establish any result along this line. The reader may, however, refer to Proposition 7 in the next section, a partial impossibility result for RP-implementation.

7 RP implementation

Our first result here is for the noncash systems. When $\rho = 1$, both noncash systems can RP implement the first best, regardless of the buyer’s bargaining

power λ being 1 or less than 1. For $\lambda = 1$, we simply apply the above proof for PE implementation (Proposition 4). For $\lambda < 1$, we can still work on the unilateral-reports system, but depending on the value of λ , we may need a value of β higher than the one given by (1). Specifically, we need $\beta > \beta_\lambda$, where β_λ satisfies

$$\frac{(1 - \lambda)[u_b(q^*) - 0.5\beta_\lambda(1 - \beta_\lambda)^{-1}\delta_s]}{\lambda[-\delta_s + 0.5\beta_\lambda(1 - \beta_\lambda)^{-1}u_b(q^*)]} = \frac{u'_b(q^*)}{u'_s(1 - q^*)} \quad (3)$$

with $\delta_s = u_s(1) - u_s(1 - q^*)$. (If $\lambda \geq 1/2$ then it is easy to check that this extra condition is redundant.)

To understand that the general bargaining solution requires the extra condition on β , consider a meeting in which both agents start the date with the equilibrium continuation value v^* . Denote by v_+ and v_- the buyer's and seller's start-of-the-next-date continuation values, respectively, if the buyer sends some r indicating autarky occurred; and let $f_b(q) = u_b(q) - \beta(v_+ - v^*)$ and $f_s(q) = u_s(1 - q) - u_s(1) + \beta(v^* - v_-)$. For the equilibrium transfer q^* to survive renegotiation, it must maximize the Nash product $[f_b(q)]^\lambda [f_s(q)]^{1-\lambda}$ subject to $q \leq 1 - c$, where $c \leq 1 - q^*$ is the seller's phase I consumption.²⁰ It follows that $(1 - \lambda)f_b(q^*)u'_s(1 - q^*) \leq \lambda f_s(q^*)u'_b(q^*)$, but for the seller not to consume $c > 1 - q^*$ at phase I, this must hold at equality. To maintain the equality as the buyer's bargaining power decreases, either the buyer's (on-the-path) gain from trade, $f_b(q^*)$, decreases, or the seller's (on-the-path) gain from trade, $f_s(q^*)$, increases. But that requires $\beta > \beta_\lambda$ because $v_+ - v^*$ in $f_b(q^*)$ and $v^* - v_-$ in $f_s(q^*)$ are bounded above by $0.5(1 - \beta)^{-1}\delta_s$ and $0.5(1 - \beta)^{-1}u_b(q^*)$, respectively.

When $\rho < 1$ even the reports system cannot RP implement the first best. The reason is as follows. In RP implementation, as in PE implementation, the public record resulting from the seller consuming $c = 1$ in phase I (when $e = 1$) in a meeting must be distinguished from the public record when $e = 0$. But in RP implementation, these two outcomes must survive renegotiation. Once renegotiation is triggered, the set of feasible trades related to the first outcome is the same as the set related to the second outcome. That is, renegotiation makes it impossible to establish a distinction between the two outcomes.

²⁰In this Nash product, the meeting report r is restricted to an equilibrium report, say, r^* , so that its value is v^* . Of course, one should derive conditions for $r = r^*$ from the Nash product in which no such restriction is imposed.

Proposition 6 *Let (1) hold and $\beta > \beta_\lambda$ in (??), where $\lambda \in (0, 1]$ is the buyer's bargaining power in renegotiation. Then the checks or reports system RP can RP implement the first best if and only if $\rho = 1$.*

Proof. We omit the proof for the checks system and leave the proof of the “if” part in the Appendix. For the “only-if” part, let $\rho < 1$ and assume by contradiction that some T and σ implement the first best under the reports system. Denote by w^* the post-meeting continuation value, at any t , for all in-equilibrium agents. Consider a meeting between buyer i and seller j at date 0. When $e = 1$, let (q^*, r) be the outcome of T implied by σ after j consumes some $c \leq 1 - q^*$ implied by σ at phase I; it is without loss of generality that this outcome is deterministic. By reporting r , both i and j obtain w^* as the post-meeting continuation value. Let $(0, r')$ be the outcome of T implied by σ after j consumes $c = 1$ at phase I; it is without loss of generality assume that this outcome is deterministic. Let w_k be the post-meeting continuation value of k , $k \in \{i, j\}$, implied by r' . By Lemma 2, $w_i > w^*$ and $w_j < w^*$. When $e = 0$, let $(0, r'')$ be the outcome of T implied by σ ; it is without loss of generality that this outcome is deterministic. By reporting r'' , both i and j obtain w^* as the post-meeting continuation value. With RP implementation, $(0, r'')$ and $(0, r')$ must survive renegotiation. That is, if renegotiation is triggered when $e = 0$, then $(0, r'')$ must be the outcome of renegotiation; and if renegotiation is triggered after the seller consumes $e = 1$, then $(0, r')$ must be the outcome of renegotiation. But for both of those situations, the sets of feasible trades are the same, so it is a contradiction to have the continuation values associated with r'' different from the continuation values associated with r' . (E.g., if $\lambda = 1$ then with $w_i > w^*$ and $w_j < w^*$, the buyer can find a lottery better than $(0, r'')$.) ■

The next corollary follows from the proof of Propositions 6 (ii).

Corollary 3 *Let $\rho < 1$. Suppose the checks or reports system RP implement some $\{\alpha_t\}$ by some T and σ . If $\{\alpha_t\}$ specifies a positive transfer for some meeting at t , then the equilibrium distribution of the individual end-of- t continuation values cannot be degenerate. In particular, it is impossible that $\{\alpha_t\}$ specifies the same positive transfer of the good for all meetings when sellers are endowed.*

Next we examine whether the cash system RP-implements the first best when $\rho = 1$. We find a partial answer. That is, the cash system cannot RP

implement the first best if $\lambda = 1$ and

$$(1 - \beta)^{-1}[u_s(1) - u_s(1 - q^*)] > u_b(q^*). \quad (4)$$

(Notice that (4) does not conflict with (1).)

The proof is by way of contradiction and starts from the following observation: In order for a seller to have an incentive to transfer $q > 0$ at any t , the sum of his money spent in any continuation path between $t + 1$ and $t + \hat{t}$ in which he is a buyer at least once must exceed his start-of- t holding, where \hat{t} is sufficiently large and does not depend on t . This observation relies on $\lambda = 1$. As implied by the observation, some in-equilibrium agents will have arbitrarily large holdings at the start of t if t is arbitrarily large. Also, by the observation, a seller with a large start-of- t holding will spend a large amount of money at some $t + \tau$, with $1 \leq \tau \leq \hat{t}$, when he is a buyer.

Now consider this date- t seller. When he becomes a buyer at $t + \tau$, if he deviates to autarky, then obviously the right side of (4) is his current loss. But with the money saved by choosing autarky, a lower bound on his future gains to the deviation is approximated by the left side of (4). To see this, notice that given $\lambda = 1$, he obtains q^* as a buyer as long as he pays the seller the same amount he would pay as a non-defector. On the other hand, since the holdings of most buyers are far less than the large amount of money saved by choosing autarky at $t + \tau$, he can finance his expenses with the savings even if he always chooses autarky when he is a seller (i.e., even if he has no inflow of money) until some $t + \tau + \bar{\tau}$, where $\bar{\tau}$ is large and does not depend on t and τ .

Proposition 7 *Let (4) hold and $\rho = 1$. Let $\lambda = 1$. Then the cash system cannot RP implement the first best.*

Proof. See the appendix. ■

Next we demonstrate that when $\rho < 1$ the noncash systems can RP implement some allocation that the cash system cannot RP implement. To this end, we first consider the *modified cash system* in which money is indivisible, each agent can hold up to $Z \geq 1$ units of money, and the average money holding is $0 < M < Z$. For this system with $Z = 1$, if $u'_b(0) = \infty$ then for any $\lambda > 0$ and any $\rho > 0$ there exists a stationary monetary equilibrium. This result simply follows the standard argument in the literature on monetary matching models.

Lemma 3 *Let $u'_b(0) = \infty$. Let $Z = 1$ in the modified cash system. Then this system RP implements a non-autarky allocation.*

The modified cash system is used to prove the following.

Proposition 8 *Let $u'_b(0) = \infty$. Then there exists a continuum of allocations that can be RP-implemented by the checks and reports systems but not by the cash system.*

Proof. Fix $M < 1$ and denote by $\{\alpha_t\}$ and σ the corresponding Lemma-3 allocation and equilibrium strategy profile. By the argument in the proof of Proposition 1, σ can be duplicated in the the checks system (i.e. with *divisible* money and without a bound on money holdings) and the reports system. Hence both noncash systems RP implement $\{\alpha_t\}$. But σ cannot be duplicated, and hence the allocation cannot be implemented, in the original cash system (i.e. with divisible money and without a bound on money holdings). For in the original cash system, when an endowed seller and a buyer both hold one unit of money, they will trade money for goods, a trade that is ruled out by the upper bound on money holdings in the modified cash system. Since for each $M \in (0, 1)$ the associated $\{\alpha_t\}$ is distinct, it follows that there exists a continuum of allocations that can be RP-implemented by the checks and reports systems but not by the cash system. ■

Of course, it is of greater interest to know whether the noncash systems RP implement some allocation that is better than any allocation RP implementable in the cash system when $\rho < 1$. To answer this question, the main difficulty comes from a fact implied by Corollary 3. That is, under any record-keeping system, cash or noncash, in any equilibrium that RP implements a non-autarky allocation, the distribution of the individual continuation values after each round of pairwise meetings is not degenerate. As is well known, such distributions make monetary matching models hard to analyze, and this is true for our cash system as well. For example, we do not even know whether there exists a monetary equilibrium in the cash system.

Given this difficulty and given the way that we prove Proposition 8, we resolve a less ambitious question: Whether the noncash systems RP implement some allocation better than any allocation the modified cash system RP implements when $\rho < 1$. We can find such an allocation when $\lambda = 1$, measuring welfare by the expected continuation value at the start of date 0.

The equilibrium in the noncash systems is derived from the known equilibrium in the modified cash system. To understand the construction, imagine that agents now can hold up to 2 units of money in a special way. When an agent with 1 unit of money meets an agent with 1 or 0, they obey the one-unit bound on the money holding as in the known equilibrium. When an agent with 2 meets another agent, they obey the two-unit bound. Moreover, when a buyer with 2 meets a seller with 1 or 0, if the buyer's post meeting holding m is positive, then 1 unit of money is taken away from him (so his end-of-date holding is $m - 1$).

It is clear that this new equilibrium relies on the ability of the noncash systems to monitor each agent's history of money holdings. In the new equilibrium, agents with 0 or 1 have the same continuation values as in the known equilibrium, but agents with 2 have a higher value than anyone in the known equilibrium. In fact, such an equilibrium construction can be generalized to any arbitrary bound on the holdings of indivisible money in the modified cash system. For such a system, we can establish a stationary non-autarky equilibrium by the method in Zhu [28] and show that the checks system (with divisible money) and the reports system support a better equilibrium. This is summarized in the following proposition.

Proposition 9 *Let $u'_b(0) = \infty$. Let $\lambda = 1$. Then for any allocation that the modified cash system RP implements, there exists an allocation that is RP implemented by the checks and reports systems and has a higher expected continuation value at the start of date 0.*

Proof. See the appendix. ■

8 Discussion

We set up our model as a risk-sharing model in which buyers and seller share the risk resulting from the type and endowment realizations. Our model differs in two aspects from the standard risk-sharing models (see, e.g., Green [7], Atkeson and Lucas [2], and Levine [19]). In the standard models, there is a centralized mechanism for pooling and redistributing the resource, and the underlying friction is *private information* (e.g., the realization of the endowment or taste shocks). In our model, the resource distribution is decentralized at the meeting spots, and the underlying friction is *pairwise public information* (i.e., people outside a meeting cannot observe the actual

resource redistribution, or even the the actual resource realization), a friction naturally emerging in such the decentralized environment.²¹ Of course, there can be private information in our model (e.g., only the seller knows his resource realization). Our model purposely focuses on issues related to pairwise public information.

Reports in our model share some similarity with communication in repeated games with private monitoring (see, e.g., Ben-Porath and Kahneman [3], Compte [6], and Kandori and Matsushima [12]). The structure of monitoring in our model is close to that of [3], where agents' actions are not public information but they are perfectly observed by a subset of the other agents. A point that is not pursued in this repeated-game literature plays a critical role in our study: If agent i sends a report or message regarding agent j which affects j 's future payoff, then j has an incentive to directly reward i in exchange for a good report.

One should not confuse the inclusion result in Proposition 1 with the inclusion result given by Kocherlakota [15]. In [15], memory is a technology that records the true trading histories; money, as is money in our model, is a technology that generates a special form of reported trading histories. Kocherlakota shows that an allocation implemented by money can also be implemented by memory. This inclusion result, stated as *Money is Memory* in [15], deals with true histories and a special form of reported histories. Our inclusion result deals with different forms of reported histories. Part of this result can be stated as *Money is Reports* (more precisely, an allocation implemented by money can also be implemented by reports). Although the physical environment we study is a special case in [15], our result seemingly can be extended to the general environment there.

Our inclusion result can be related to [15] in another aspect. The inclusion result in [15] requires that memory is not accessed by everyone so that memory is weaker than perfect monitoring. Our result suggests a reconciliation between this requirement on memory and the widespread view in monetary theory that imperfect monitoring of the individual trading histories

²¹Aside from details, Koepl et al [18] and Temzelides and Williamson [24] study optimal risk sharing in matching models comparable to our model with $\rho < 1$. The resource in each meeting spot imposes a constraint on the planner. By their assumptions, the planner infers each agent's private characteristics (e.g., his type, endowment, etc.) after the agent himself has reported those characteristics. Therefore, as in the standard risk sharing literature, the planner's main concern is to induce each agent to truthfully report his own characteristics.

is necessary for money to be essential (see, e.g., Ostroy [22] and Townsend [25]). That is, for the environment in [15], perfect monitoring includes any record-keeping system in which the imperfect monitoring has the form that outsiders to a match do not observe transfers of the good but agents' identities are public information.^{22,23}

Our formulation of noncash payment systems provides a useful way to think about the nature of the cashless economy that some people foresee arriving in the near future. Presumably this cashless economy is a limit of a sequence of economies in which the use of cash declines. One can formulate the economies in this sequence to be economies in which some transactions are perfectly monitored but others are not monitored at all (see, e.g., Aiyagari and Williamson [1] and Cavalcanti and Wallace [5]), or economies in which all transactions are randomly monitored (see, e.g., Kocherlakota and Wallace [17]). Under such a formulation the use of cash declines following an increase in either the portion of perfectly monitored transactions or the probability that transactions are monitored. In the limit, all transactions are perfectly monitored, and, consequently, there is no role for money and hence monetary policy.²⁴ In contrast, under our formulation, even if cash is completely replaced by checks in the limit, there is a role for money, and most likely for monetary policy.

A related point is that our formulation of the checks system provides a way to distinguish the special object of cash from a general notion of money. This differentiation is useful for calibrating a micro-founded model of money. For example, in monetary matching models, money is typically motivated by, as well as interpreted as, cash, but it becomes problematic to quantify money by monetary aggregates other than M0 (see Prescott [23]). Our formulation provides a model in which the notion of money can be mapped to M1.

Our study also provides a useful way to think about two important questions for the theory of payments: Why store-of-value systems and account-based systems are prevalent payment systems in practice, and how they can coexist.²⁵ Our model suggests an explanation involving the abilities of the

²²Inspecting example 3 in [15], one can see the role played by the fact that memory and money reveal different histories of agents' identities in previous matches.

²³Wallace [26, section 2] and [27, section 2.1] suggest another reconciliation: Perfect monitoring with money includes perfect monitoring without money.

²⁴Green [8] makes the same point.

²⁵See Kahn and Roberds [11] in a recent survey of payment economics. In [11], the authors credit Edward J. Green for introducing the questions.

different systems to overcome the friction that outsiders cannot observe the transfer of goods in the meeting. Because the cash system provides no access to the individual payments histories, it must be a store-of-value system. On the other hand, for the checks and reports systems to provide the payment histories, they must be account-based systems. There are some plausible extensions for combining cash with checks and reports. For example, the infrastructure to maintain a ubiquitous checks or reports system may be very costly, in which case it may be efficient for some transactions to be completed with cash.

Appendix

Completion of the Proof of Proposition 1

Proof. Here we provide a proof for part (i). First consider the cash system and let $m_{i,0}$ be agent i 's initial money holding. Suppose that the mechanism T and the equilibrium strategy profile σ implement the allocation $\{\alpha_t\}$. Next consider the checks system and let agent i 's initial money holding be $m_{i,0}$. Now we construct the mechanism T' and the strategy profile σ' as follows. Fix t and $z^t \cup \{(\eta_k^t, m_k^t)\}_{k \in I}$. Fix a date- t meeting between buyer i and seller j . In this meeting, T' is the same as T . At phase I if $t = 0$; at phase I if $t > 0$ and if for each $0 < \tau \leq t$ the start-of- τ distribution of money implied by $z^\tau \cup \{(\eta_k^\tau, m_k^\tau)\}_{k \in I}$ is one of those implied by σ under the cash system given $z^\tau \cup \{\eta_k^\tau\}_{k \in I}$; and at phase II; σ' specifies the same actions as σ specifies under the cash system when the public information is $z^t \cup \{\eta_k^t\}_{k \in I}$ and the pairwise information at the start of the meeting is that i and j hold $m_{i,t}$ and $m_{j,t}$, respectively. At phase I if $t > 0$ and if for some $0 < \tau \leq t$ the start-of- τ distribution of money implied by $z^\tau \cup \{(\eta_k^\tau, m_k^\tau)\}_{k \in I}$ is not one of those implied by σ under the cash system given $z^\tau \cup \{\eta_k^\tau\}_{k \in I}$, then σ' specifies the seller to consume e . ■

Proof of Lemma 1

Proof. First consider the unilateral-reports system. Suppose that the mechanism T and the equilibrium strategy profile σ implement the allocation $\{\alpha_t\}$. So it is without generality to assume that in σ , when autarky arises in a meeting, the buyer makes $r = 0$. For if this is not the case for some meeting, there exists an equivalent equilibrium strategy profile $\bar{\sigma}$ which differs from σ only in that for this meeting r in σ is treated as $g(r)$ in $\bar{\sigma}$, where $g : [0, 1] \rightarrow [0, 1]$ is isomorphic, and in $\bar{\sigma}$, when autarky arises, the buyer makes $r = 0$.

We use the following notation for mapping a report to a money holding. Let f_0 be a strictly increasing function from $[0, 1]$ to $[1, \infty)$. For each $t \geq 1$, let f_t from $[0, 1]$ to \mathbb{R}_+ be defined by $f_t(x) = x10^{-t}$. Let f_t^{-1} be the inverse of f_t .

Now consider the checks system and let agent i 's initial money holding be $f_0(r_{i,0})$. Now we construct the mechanism T' and the strategy profile σ' as follows. Fix t and consider a date- t meeting between buyer i and seller j . In this meeting, for each agent T' specifies the same sets of actions as T . If a sequence of actions leads to autarky under T , then under T' it also leads to autarky. If a sequence of actions leads to a trade under T , then under T' it leads to the same trade but with r being replaced with $l = f_t^{-1}(r)$. Let $l_{k,\tau}$ be the transfer of money from k to $\gamma_{k,\tau}$. When the public information is $\{(\eta_i^t, l_k^t) : k \in I\}$, σ' specifies the same actions as σ specifies under the unilateral-reports system when the public information is $\{(\eta_i^t, r_k^t) : k \in I\}$ with $r_{k,\tau} = f_\tau^{-1}(l_{k,\tau})$. ■

Completion of the proof of Proposition 2

Proof. For the “only-if” part, let $\rho < 1$, and assume by contradiction that some T and σ implement the first best. Consider a meeting at some t between buyer i and seller j , and both agents have some in-equilibrium holdings at the start of t .

First we define two particular money transfers, l and l' , in this meeting. When $e = 1$, let (q^*, l) be the (deterministic) trade at phase II implied by σ following the seller's phase-I consumption $c \leq 1 - q^*$ implied by σ . Notice that transferring l gives i and j the same post-meeting continuation value, denoted w^* . For agent j , w^* must be higher than the post-meeting continuation value if he receives zero units of money, which occurs if he chooses autarky, denoted $w_s(0)$. Because money can be freely disposed, l must be positive. When $e = 0$, let $(0, l')$ be the (deterministic) trade at phase II implied by σ . Notice that transferring l' gives i and j the same post-meeting continuation value w^* , so in particular $l' > 0$, for if $l' = 0$ then the seller's continuation value is $w_s(0) < w^*$. But then for agent i , the post-meeting continuation value if he transfers zero units of money, which occurs if he chooses autarky, denoted $w_b(0)$, must equal w^* . In the rest of the proof, we draw a contradiction by showing that at some meeting, some buyer has an incentive not to give up l' when $e = 0$.

For each meeting between two agents with in-equilibrium money holdings, there must be such l and l' . Now take the union of all such l and l' from all

meetings to form a set, denoted S . This set may or may not be bounded. Either way, we can choose some meeting in some t in which $\max\{l, l'\}$ is sufficiently large in a way described below. To emphasize the association of l and l' with this particular meeting, denote l by $l(t)$ and l' by $l'(t)$. Also, for this date- t meeting, let the buyer be denoted by i , and his pre-meeting money holdings by m . Recall that for i , $m - \min\{l(t), l'(t)\}$ and $m - \max\{l(t), l'(t)\}$ give him the same post-meeting continuation value. In this date- t meeting, when the seller is endowed with zero, suppose that i deviates so that the meeting outcome is $(0, 0)$. We shall show that this deviation is beneficial to i .

In date $t + 1$, if i is a buyer again, then by throwing out some extra money before meeting a seller, he can carry some in-equilibrium holdings into the meeting (so he is not inferred as a defector). So consider a meeting in date $t + 1$ when i is a seller. For this date- $t + 1$ meeting, let $l(t + 1)$ and $l'(t + 1)$ be the equilibrium transfers of money from the buyer when i is endowed with 1 and 0, respectively, *and* when i carries $m - \max\{l(t), l'(t)\}$ into the meeting (so he is not inferred to be a defector). Recall that i , after his assumed date- t deviation, has at least $m - \min\{l(t), l'(t)\}$ before the date- $t + 1$ meeting. If $\max\{l(t + 1), l'(t + 1)\} \leq \max\{l(t), l'(t)\} \equiv L(t)$, then i certainly benefits from the assumed date- t deviation. For in this date- $t + 1$ meeting, he can choose outcome $(0, 0)$, and still can (by likely free disposal of some extra money) carry some in-equilibrium money holdings into any $t + 2$ meeting.

So it remains to show that there are suitable t and i to assure that $L(t)$ is sufficiently large so that the measure of meetings in $t + 1$ with $\max\{l(t + 1), l'(t + 1)\} \leq L(t)$ is sufficiently close to unity. If S is unbounded, then simply choose i and t such that $L(t)$ is large enough so that the measure of buyers with at least $L(t)$ is close to zero. If S is bounded, then it suffices to choose i and t to let $L(t)$ be close to the least upper bound on S . ■

Completion of the proof of Proposition 4

Proof. Here we complete the rest of the proof for $\rho < 1$. First, we define the counterpart of the mapping g in the proof for $\rho = 1$. Still denote this mapping by $g = (g_b, g_s)$, and let i, j, v_b, v_s, π , and z be their counterparts in the proof for $\rho = 1$. To make a distinction, we denote a generic unilateral report by r and a generic report by R . Now let $R = (R^q, R^e, R^s)$ be the report of the meeting between buyer i and seller j and let $x = (R, v_b, v_s, z)$. To define $g_b(x)$ and $g_s(x)$, we consider 8 cases. In cases 1-6, x is such that $R^e = 1$. By defining $r = R^q R^e$, we treat those cases the exactly same way

as their counterparts in the proof for $\rho = 1$. In case 7, $R^e = 0$, $v_b \neq \underline{v}$ and $v_s \neq \underline{v}$. If $R = r^*$, then $g_b(x) = v_b$ and $g_s(x) = v_s$; otherwise, $g_b(x) = v_-$ and $g_s(x) = v_-$. In case 8, $R^e = 0$, $v_b = \underline{v}$ or $v_s = \underline{v}$. Then $g_b(x) = v_b$ and $g_s(x) = v_s$.

Next, let φ_t and π_t be defined in the same way as the proof for $\rho = 1$ (while now h_i^t is defined for the reports system). Then we can encode an agent's public history into a statistics in \mathbf{V} by the exactly same way as there. To describe σ , consider a date- t meeting in which a buyer has a public history equivalent to v_b and a seller has a public history equivalent to v_s . When $e = 1$, σ is the same as one in the proof for $\rho = 1$ but with $r \in [0, 1]$ being replace by $R = (r, 1, \text{YES})$. When $e = 0$, σ is such that both agents say yes at phase II. It can be verified that such defined σ is an equilibrium. ■

Completion of the proof of Proposition 6

Proof. For the “if” part, let $\rho = 1$. If $\lambda = 1$ (buyers have all the bargaining power in renegotiation), then we simply apply the proof of Proposition 4 for $\rho = 1$ —hereafter the *previous proof*. So let $\lambda < 1$ and consider the unilateral-reports system. Let v^* , \underline{v} and δ_s be the same in as in the previous proof. Let each agent's initial index be v^* ; also, let T be such that the buyer makes a take-it-or-leave-it offer. (If the seller rejects an offer and autarky is Pareto dominated, then two agents enter the renegotiation process in which the buyer has the bargaining power λ .)

Next, we define a set \mathbf{V} and a mapping g that play the same roles as their counterparts in the previous proof. Let v_+ and v_- satisfy $u_b(q^*) > \beta(v_+ - v^*)$, $\delta_s < \beta(v^* - v_-)$, and

$$\frac{(1 - \lambda)[u_b(q^*) - \beta(v_+ - v^*)]}{\lambda[-\delta_s + \beta(v^* - v_-)]} = \frac{u'_b(q^*)}{u'_s(1 - q^*)}. \quad (5)$$

Let $\mathbf{V} = \{\underline{v}, v_+, v_-, v^*\}$. To define $g \equiv (g_b, g_s)$, let v_b, v_s, r, z , and x be the same as in the previous proof. Let $\bar{z}_+ \equiv 1 - 2(v_+ - v^*)/[\delta + \beta(v_+ - v^*)]$ and $\bar{z}_- \equiv 1 - 2(v^* - v_-)/[u_b(q^*) + \beta(v^* - v_-)]$. By $\beta > \beta_\lambda$, $\bar{z}_+, \bar{z}_- \in (0, 1)$. Now consider six exclusive and exhaustive cases. Each case is the same as the counterpart case in the previous proof. In cases 1, 3, 4, and 6, g is defined exactly as its counterpart. In case 5, g is defined as its counterpart but with \bar{z} replaced by \bar{z}_- . In case 2, if $z \leq \bar{z}_+$ then $g(x) = (v^*, v^*)$ for $r \geq q^*$, and $g(x) = (v_+, v_-)$ for $r < q^*$; if $z > \bar{z}_+$ then $g(x) = (v^*, v_+)$.

Next, let φ_t and π_t be the same as in the previous proof, so that we can encode an agent's public history into some $v \in \mathbf{V}$ in exactly the same way.

Also, to describe the candidate equilibrium σ , let z , v_b and v_s be the same as in the previous proof. There are two exclusive and exhaustive situations. In the first situation, $\pi_t\{v^*\} \neq 1$. Then at phase I the seller consumes 1. At phase II the buyer offers $(0, 0)$, and the seller accepts an offer if and only if the transfer of the good is zero. In the second situation, $\pi_t\{v^*\} = 1$. Then at phase I the seller consumes 0. Let \tilde{r} , $U_b(q, \tilde{r}, v_b, v_s, z)$ and $U_s(q, \tilde{r}, v_b, v_s, z)$ be the same as in the previous proof; and again, without loss of generality, we can restrict the support of \tilde{r} to $\{0, q^*\}$. Given the seller's phase I consumption c , at phase II the buyer offers

$$(q(c), \tilde{r}(c)) = \arg \max [U_b(q, \tilde{r}, v_b, v_s, z) - U_b(0, 0, v_b, v_s, z)]^\lambda \times [U_s(q, \tilde{r}, v_b, v_s, z) - U_s(0, 0, v_b, v_s, z)]^{1-\lambda}$$

subject to $0 \leq q \leq 1 - c$; and the seller accepts an offer (q, \tilde{r}) if and only if $U_s(q, \tilde{r}, v_b, v_s, z) \geq U_s(q(c), \tilde{r}(c), v_b, v_s, z)$. In both situations, the buyer enters 0 in the card whenever autarky occurs.

For each $c \in [0, 1]$, the construction of g implies the following properties of $(q(c), r(c))$. If (v_b, v_s, z) is such that the values of g_b and g_s do not depend on r (as in case 2 with $z > \bar{z}_+$, case 5 with $z > \bar{z}_-$, and case 6) then $q(c) = 0$. If (v_b, v_s, z) is such that the values of g_b and g_s depend on r (as in case 1, case 2 with $z \leq \bar{z}_+$, case 3, case 4, and case 5 with $z \leq \bar{z}_-$) then $q(c) = \min\{q^*, 1 - c\}$, and

$$\frac{v_+ - v^*}{v^* - v_-} \leq \frac{U_b(q(c), \tilde{r}(c), v_b, v_s, z) - U_b(0, 0, v_b, v_s, z)}{U_s(q(c), \tilde{r}(c), v_b, v_s, z) - U_s(0, 0, v_b, v_s, z)} \leq \frac{\lambda u'_b(q(c))}{(1 - \lambda) u'_s(q(c))}. \quad (6)$$

In (6), the first inequality is strict only if the support of $\tilde{r}(c)$ is $\{q^*\}$, and the second inequality is strict only if $q(c) = 1 - c$. By (5) and (6), $f(c) \equiv U_s(q(c), \tilde{r}(c), v_b, v_s, z)$ is strictly decreasing over $[1 - q^*, 1]$.

To verify that the above σ is an equilibrium, we leave it to the reader to use the same argument in the previous proof. ■

Proof of Proposition 7

Proof. Suppose the contrary. First, let \hat{t} be such that

$$u_s(1) + [u_s(1) + u_b(q^*)] 0.5 \sum_{t=1}^{\hat{t}} \beta^t > u_s(1 - q^*) + [u_s(1 - q^*) + u_b(q^*)] 0.5 \beta / (1 - \beta). \quad (7)$$

Fix agent i , t , and a history of realizations of his endowments and matches. Let $m_{i,t}$ be his in-equilibrium money holdings at the start of t , and suppose

that he is the seller at t . Let $L_{i,t} = \sum_{\tau=1}^{\hat{t}} l_{i,t+\tau}$, where $l_{i,t+\tau}$ is agent i 's in-equilibrium expenditures of money when he is a buyer at $t + \tau$, and let $l_{i,t+\tau} = 0$ when he is a seller at $t + \tau$. (Note that $L_{i,t}$ depends on the realization of meetings from $t + 1$ to $t + \hat{t}$.)

We claim that (i) If $L_{i,t} > 0$ (i.e., agent i is a buyer for at least one meeting from $t + 1$ to $t + \hat{t}$) then $L_{i,t} > m_{i,t}$; and (ii) If $L_{i,t} > 0$ then there must exist some $1 \leq \tau \leq \hat{t}$ such that $l_{i,t+\tau} > m_{i,t}/\hat{t}$. For part (i), suppose $L_{i,t} \leq m_{i,t}$. But then (7) implies that agent i is better off by choosing autarky when in the date- t meeting. Part (ii) follows from part (i) immediately.

The claim applies for arbitrary i and t . Therefore, for any $m > 0$, when t is sufficiently large, it implies that there is a history such that agent i has an in-equilibrium holding $m_{i,t} > m$. For example, one such history is as follows. Agent i is a seller from 0 to t : From 0 to $\hat{t} - 1$ he meets the same buyer; from $\hat{t} - 1$ to $2\hat{t} - 1$ he meets the same buyer who as a seller meets the same buyer from 0 to $\hat{t} - 1$; and so on.

Now let $m_{i,t} > m$, and let agent i be the seller at t . We claim that if m is sufficiently large, then agent i is better off by choosing autarky in a date $t + \tau$ meeting when he is supposed to spend $l_{i,t+\tau} > m_{i,t}/\hat{t}$. By choosing autarky, his current loss is $u_b(q^*)$, but he has at least $m_{i,t}/\hat{t}$ additional units of money at the start of $t + \tau + 1$. Let ρ_m denote the lower bound on the measure of agents with holdings greater than m/\hat{t} . Note that $\rho_m \rightarrow 0$ as $m \rightarrow \infty$. Let ρ_m be sufficiently close to 1, so that with a probability sufficiently close to 1 and with a sufficiently large $\hat{\tau}$, agent i can obtain at least $u_b(q^*)$ as a buyer but also $u_s(1)$ as a seller in meetings from $t + \tau + 1$ to $t + \hat{\tau}$. That is, the lower bound on agent i 's gains from holding $m_{i,t}/\hat{t}$ more units of money at the start of $t + \tau + 1$ can be sufficiently close to $[u_s(1) - u_s(1 - q^*)]/(1 - \beta)$. By the hypothesis of the proposition, agent i is better off by choosing autarky in the date $t + \tau$ meeting. ■

Proof of Proposition 9

Proof. Fix an equilibrium in the modified cash system. Without loss of generality, we can assume that this is a stationary monetary equilibrium. Such an equilibrium can be represented by (w, π) . Here $w(m)$ is the continuation value for an agent who holds m units of money at the start of a date, where $w(m + 1) \geq w(m)$ all m . Also, $\pi(m)$ is the measure of the agents who hold m units of money at the start of each date. Without loss of generality, we can assume that $\pi(Z) > 0$. Let G be the set of money holdings such that if a seller with $m \in G$ meets a buyer with Z , the seller's end-of-meeting holding

is Z —without loss of generality we assume that the trade is deterministic.

Now we construct a candidate equilibrium in the unilateral-reports system from (w, π) . Fix $\gamma \in (0, 1)$. In this candidate equilibrium, at the start of each date, the measure of agents with the continuation value $w(m)$ for $m \in \{0, \dots, Z-1\}$ is $\pi(m)$, the measure of agents with $w(Z)$ is $\gamma\pi(Z)$, and the measure of agents with $w(Z+1)$ is $(1-\gamma)\pi(Z)$. Here $\delta \equiv w(Z+1) - w(Z) > 0$ is determined by

$$2\delta = \sum_{n \in G} \pi(n) \rho[f(\Delta_n + \delta) - f(\Delta_n)], \quad (8)$$

where $f(x) = u_b(1 - u_s^{-1}(u_s(1) - \beta x))$ and $\Delta_n = \beta[w(Z) - w(n)]$.

The basic idea in the construction is to treat an agent with $w(m)$ as an agent with m units of money in the modified cash system, but extend the bound on money holdings to $Z+1$ in a special way. To be more specific, consider a meeting in which the buyer's and seller's start-of-the-date continuation values are $w(m)$ and $w(n)$, respectively. Denote by $v_b(r)$ and $v_s(r)$ the buyer's and seller's start-of-the-next-date continuation values, respectively, when $r \in [0, 1]$ is the report of the meeting. Let $l \leq m$ be such that $r \in [l/m, (l+1)/m)$. There are four exhaustive and exclusive cases.

Case 1: $m \leq Z$ and $n \leq Z$. If $n+l \leq Z$ then $v_b(r) = w(m-l)$ and $v_s(r) = w(n+l)$; otherwise, $v_b(r) = w(m)$ and $v_s(r) = w(n)$.

Case 2: $m \leq Z$ and $n = Z+1$. Then $v_b(r) = w(m)$ and $v_s(r) = w(Z)$.

Case 3: $m = Z+1$ and $n \leq Z$. If $n \notin G$ and $n+l \leq Z$, or if $n \in G$ and $n+l \leq Z+1$, then $v_b(r) = w(\min\{m-l-1, 0\})$ and $v_s(r) = w(n+l)$; otherwise, $v_b(r) = w(Z)$ and $v_s(r) = w(n)$.

Case 4: $m = Z+1$ and $n = Z+1$. Then $v_b(r) = v_s(r) = w(Z)$.

In those cases, m and n and l are treated as the buyer's holdings, the seller's holdings, and the transfer of money in the modified cash system. In cases 1, 2 and 4, the buyer and seller obey the bound Z as in the modified cash system. In case 3, the seller obeys Z if $n \notin G$ but $Z+1$ if $n \in G$; and the buyer gives up one unit of his post-meeting holdings if the holdings are positive. In each case, if the transfer l violates the seller's bound then it is treated as 0.

Indeed, an agent with $w(m)$ and $m \leq Z$ in the unilateral-reports system is just like an agent with m in the modified cash system. Although in the former system this agent as a seller trades with a buyer with $w(Z+1)$, that the buyer has all the bargaining power makes this trade just like no trade to the agent. To complete the argument, we compare an agent with $w(Z+1)$

to an agent with $w(Z)$. The former has the same gains as the latter when both are sellers (because of $\lambda = 1$), or when both are buyers and meet some seller with $w(n)$ and $n \notin G$. The former gains more than the latter when both are buyers and meet some seller with $w(n)$ and $n \in G$, where the extra gain is $f(\Delta_n + \delta) - f(\Delta_n)$. ■

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