Poverty, Relative to the Ability to Eradicate It: An Index of Poverty Reduction Failure

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Abstract

In this note we approach the question of relative poverty from a different angle. Fixing the poverty line, we ask: What is the extent of poverty *relative to the resources available in the society to eradicate it*? We argue that the same level of poverty is "worse" if the resources available to address it are greater. We characterize a class of indices that measure Poverty Reduction Failure and provide an empirical illustration of their suitability using data for 94 country observations in 2001.

Key words: Absolute Poverty, Relative Poverty, Poverty Reduction Failure

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1. Introduction

The debate on "relative poverty" has engaged many economists. The way the question has been posed is in terms of whether the poverty line should be absolute or relative, in particular, whether it should rise with mean income. The arguments are typically in terms whether in wealthier societies more resources are needed to acquire the same set of minimum basic outcomes.[1]

In this note we approach the relative poverty question from a different angle. Taking the poverty line as fixed and absolute, we ask instead—*what is the extent of poverty relative to the resources available in the society to eradicate it?* If we view the resources for poverty eradication as coming from those above the poverty line, then an increase in these resources should increase the capacity to reduce shortfalls below the poverty line. If the shortfalls nevertheless remain unchanged, this tells us something about the society in question. We would argue that the same absolute poverty is "worse" if the resources available to address poverty are greater. This takes us closer to bringing inequality explicitly into the assessment of poverty but, as we will show, it is not the same as simply measuring inequality.

We discuss the issue of measuring the extent of Poverty Reduction Failure (PRF) in the next section in an axiomatic framework. Under some reasonable assumptions we characterize a class of measures that are simply interpreted and easy to apply. Section 3 continues this discussion by illustrating the measure for a few simple cases and shows the relationship of our measure to the well known FGT family of poverty indices (Forster Greer and Thorbecke, 1984). The PRF measures and FGT measures are then computed and compared in section 4, in the context of an empirical application using data for a large number of countries. We present our analysis in terms of rank correlation among different poverty measures and our PRF measures for 94 country observations in 2001, and show that the PRF measure has real information content. Section 5 concludes.

2. Measurement of Poverty Reduction Failure

Consider a population of size n, with income distribution $(y_1, y_2, ..., y_n)$. Without loss of generality, let this be increasingly ordered ($y_1 \leq ... \leq y_n$). Let z be the absolute poverty line, which is fixed and

[1] There is a large literature. See, for example, Atkinson and Bourguignon (2000), Foster (1998), Sen (1985).

invariant. Let the number of poor persons, $\#\{1 \le i \le n \mid y_i < z\} = q$. Define the normalized shortfall for each poor individual by

$$d_i = \frac{z - y_i}{z},\ i = 1,...q\ .$$

Let $\mathbf{d} = (d_1,...,d_q) \in (0,1]^q$ be the deprivation vector. Similarly, define the normalized excess income for the non-poor

$$e_i = \frac{y_i - z}{z}, i = q+1,...,n.$$

Let $\mathbf{e} = (e_{q+1},...,e_n) \in R_+^{n-q}$ be the excess income vector.

Define the index of Poverty Reduction Failure (PRF) as $A = A(\mathbf{d},\mathbf{e}) : (0.1]^q \times R_+^{n-q} \to R_+$. Note that by construction the index is invariant to scaling incomes and the poverty line by the same factor. Consider now the following axioms for such an index.

Continuity (C): A is continuous in each argument.

Symmetry (S): Simply permuting individual labels cannot change the index.

Monotonicity (M): For a given $\mathbf{e}$, A is increasing in any element of $\mathbf{d}$ (other elements being fixed). Also, for a given $\mathbf{d} \ne \mathbf{0}$, A is increasing in any element of $\mathbf{e}$ (other elements being fixed).

Normalization (N): For a given $\mathbf{e}$, when $\mathbf{d} \to \mathbf{0}$, we have $A(\mathbf{d},\mathbf{e}) \to 0$ and for a given $\mathbf{d} \ne \mathbf{0}$, $A(\mathbf{d},\mathbf{e}) \to 0$ as $e \to \mathbf{0}$. Failure is minimum when either there is no deprivation or no excess income to redistribute.

Transfer Principle (TR): Fix $\mathbf{e}$. Now if there is a progressive transfer of income from any poor individual to a poorer individual without changing their relative position in the society, then A decreases.

Subgroup consistency for shortfalls (SS): $A(\mathbf{d_1},\mathbf{e}) > A(\mathbf{d'_1},\mathbf{e}) \Rightarrow A((\mathbf{d_1},\mathbf{d_2}),\mathbf{e}) \ > A((\mathbf{d'_1},\mathbf{d_2}),\mathbf{e})$ where $\mathbf{d_1},\mathbf{d'_1} \in (0,1]^{q_1}$, $\mathbf{d_2} \in (0,1]^{q_2}$ with $q_1 + q_2 = q$ and $\mathbf{e} \in R_+^{n-q}$.

Subgroup consistency for excess income (SE): $A(\mathbf{d},\mathbf{e_1}) > A(\mathbf{d},\mathbf{e'_1}) \Rightarrow A(\mathbf{d},(\mathbf{e_1},\mathbf{e_2})) > A(\mathbf{d},(\mathbf{e'_1},\mathbf{e_2}))$ where $\mathbf{d} \in (0,1]^q$, $\mathbf{e}_1,\mathbf{e'_1} \in R_+^{n_1-q}$ and $\mathbf{e}_2 \in R_+^{n-n_1}$

Population Principle (PP): If we combine two identical distributions, the measure A remains unchanged. In other words, $A((\mathbf{d},\mathbf{d}),(\mathbf{e},\mathbf{e})) = A(\mathbf{d},\mathbf{e})$ for all $\mathbf{d} \in (0,1]^q$, $\mathbf{e} \in R_+^{n-q}$.

With these axioms we can prove the following theorem.

Theorem 1 The index of PRF satisfies C, S, M, TR, SS, SE and PP if and only if it is ordinally equivalent to $A(\mathbf{d},\mathbf{e}) = A(\frac{1}{n}\Phi,\frac{1}{n}\Psi) = A(\frac{1}{n}\sum_{i=1}^{q}\phi_q(d_i),\frac{1}{n}\sum_{i=q+1}^{n}\psi_{n-q}(e_i))$ where $\phi_q(.)$ is increasing and convex, $\psi_{n-q}(.)$ is increasing. A is increasing in Φ and Ψ.

Proof: The proof is very similar in spirit to that of proposition 1 in Foster and Shorrocks (1991). Hence we omit the details of proof here and give a brief outline. SS implies that $A(\mathbf{d},\mathbf{e})$ must be ordinally equivalent to $A(\sum_{i=1}^{q}\phi_q^i(d_i),\mathbf{e})$. Application of SE establishes the equivalence with $A(\sum_{i=1}^{q}\phi_q^i(d_i),\sum_{i=q+1}^{n}\psi_{n-q}^i(e_i))$. S then implies that $\phi_q^i(.)$ and $\psi_{n-q}^i(.)$ must be independent of *i*. Continuity and increasingness of A, ϕ and ψ follow from C and M. Convexity of $\phi_q(.)$ follows from TR. Finally introducing PP leads to the form as in the statement. It is easy to establish the converse. ■

Example 1: Let $\phi_q(x) = x^\alpha, \alpha > 1$ (satisfying TR) and $\psi_{n-q}(y) = y$. Then we have $A = A(\frac{1}{n}\sum_1^q d_i^\alpha, \frac{1}{n}\sum_{q+1}^n e_i) = A(P_\alpha, \frac{1}{n}\sum_{q+1}^n e_i)$, written in terms of the FGT measure with parameter α , P_α.

Example 2: Let $\phi_q(x) = x^\alpha, \alpha > 1$ (TR) and $\psi_{n-q}(y) = y^\beta, \beta > 1$. This choice is consistent with the idea of penalizing higher excess income at a higher rate. As the imposition of a progressive tax will be able to extract more from the richer non-poor, this seems reasonable. Then we have $A =$ $A(P_\alpha, \frac{1}{n}\sum_{q+1}^{n} e_i^\beta)$.

Example 3: Let $\phi_q(x) = e^x$ and $\psi_{n-q}(y) = y^\beta, \beta > 1$. $A(.,.)$ is similarly defined.

Example 4: Let and $\psi_{n-q}(.) = 1$, then $A(\mathbf{d}, \mathbf{e}) = A(P_\alpha, \frac{n-q}{n})$.

Now consider the following axiom.

Proportionality A proportional increase in the deprivation argument can be offset by a proportional decrease in the excess income argument.

This is formalized as follows. For $x, y > 0$, $A(x, y) = A(x', y')$ if $\frac{x'-x}{x} = -\delta \frac{y'-y}{y}$ or $\frac{dx}{x} = -\delta \frac{dy}{y}$, where δ is a constant.

Now define $u = \log(x)$, $v = \log(y)$ and $A(x, y) = a(u, v)$. Then Proportionality, in terms of these new variables, becomes: $du = -\delta dv \Rightarrow \frac{\partial a}{\partial u} du + \frac{\partial a}{\partial v} dv = 0$. So, substituting for dv and simplifying, we have $\frac{\partial a}{\partial u} du - \frac{1}{\delta}\frac{\partial a}{\partial v} = 0$, hence $a(u, v)$ must be equivalent to (for some continuous function $h(.)$) $h(u + \frac{v}{\delta}) = f(xy^{\frac{1}{\delta}})$ (in terms of the original variables x and y). A similar argument establishes an analogous form for A^P. Thus we have the following result.

Theorem 2: When $A(\mathbf{d}, \mathbf{e})$ as in theorem 1 also satisfies **Proportionality**, then it must be of the form

$$A(\mathbf{d},\mathbf{e}) = A(1/n\sum_{i=1}^{q}\phi_q(d_i), 1/n\sum_{i=q+1}^{n}\psi_{n-q}(e_i)) = f\left(\left\{1/n\sum_{i=1}^{q}\phi_q(d_i)\right\}.\left\{1/n\sum_{i=q+1}^{n}\psi_{n-q}(e_i)\right\}^{\frac{1}{\delta}}\right),$$ where

$f'>0$.

Corollary 1: The index of PRF $A(\mathbf{d},\mathbf{e})$ as in theorem 2 also satisfy N if and only if $\phi_q(0)=0$, $\psi_{n-q}(0)=0$ and $f(0)=0$.

Proof: If $A(\mathbf{d},\mathbf{e})$ satisfies N then $f\left(\left\{\frac{1}{n}\sum_{i=1}^{q}\phi_q(d_i)\right\}.\left\{\frac{1}{n}\sum_{i=q+1}^{n}\psi_{n-q}(e_i)\right\}^{\frac{1}{\delta}}\right)$ also satisfies N. Now take $\mathbf{d}=(d_1,d_1,...,d_1)$. Then we must have $f\left(\frac{q}{n}\phi_q(d_1).\left\{\frac{1}{n}\sum_{i=q+1}^{n}\psi_{n-q}(e_i)\right\}^{\frac{1}{\delta}}\right)\to 0 \text{ as } d_1\to 0$ and $\forall\,\mathbf{e}$.

Hence, because of the continuity of $\phi_q(.)$, we must have $\phi_q(0)=0$ and $f(0)=0$. $\psi_{n-q}(0)=0$ is similarly established. The converse is immediate. ■

Example 5: Continuing example 1 we now have $A = f\left(P_\alpha\left\{1/n\sum e_i\right\}^{\frac{1}{\delta}}\right)$**.**

Example 6: From example 4, it turns out that $A = f\left(P_\alpha\left\{\frac{n-q}{n}\right\}^{\frac{1}{\delta}}\right)$**.**

3. Discussion

Consider a variant of Example 5, where *f(.)* is the identity function. Then the PRF index can be written in terms of the FGT measure with parameter α , P_α, and the overall mean income, μ, as

$$A_{\alpha,\delta} = P_\alpha (P_1 - 1 + \frac{\mu}{z})^{1/\delta}.$$

For example, when α and δ are set at 1, we have $A_{1,1} = \frac{P_1}{z}\left[zP_1 + \mu - z\right]$. Thus A can be calculated using information that is produced in standard analyses of Poverty. Similarly, when α = 2 and δ = 1, we have $A_{2,1} = \frac{P_2}{z}\left[zP_1 + \mu - z\right]$. When δ = 2, then we have $A_{1,2} = P_1\sqrt{(P_1 - 1 + \mu/z)}$ for α = 1 and $A_{2,2} = P_2\sqrt{(P_1 - 1 + \mu/z)}$ for α = 2. Finally, if we consider a concave function as a candidate for *f(.)*, say $f(x) = x^{1/4}$, then for α = 1 and δ = 1, we have $A = \left\{P_1[P_1 - 1 + \mu/z]\right\}^{1/4}$.

This family of indices for the poverty reduction failure provides a useful template to discuss a number of interesting issues. If the poverty of any poor individual increases, so does PRF. If the income of any non-poor individual increases without a decrease in poverty, so does PRF. For a generally poor society, where those above the poverty line are not particularly well off, the PRF is low and the index registers this. Consider two societies where total population size and the income distribution below the poverty line are identical. Any standard measure of poverty will then be the same in the two societies. But if non-poor incomes in one society are much higher, then the PRF in this society will also be higher. A high PRF is also, in one sense, an indictment of wealthy societies that tolerate poverty.

Of course our measure of poverty reduction failure evokes inequality, since it penalizes increase in non-poor incomes without a corresponding increase in poor incomes. But it is not same as inequality. It is easy to show that a mean preserving spread in the income distribution can move the PRF index in any direction, and an increase in any standard inequality measure can coincide with an increase or a decrease on the PRF index. This is also seen in the empirical illustration, to which we now turn.

4. Empirical Application

For our empirical application, we have used data prepared by Chen and Ravallion (2004), for the Worldbank, *POVCAL*, available at http://iresearch.worldbank.org/PovcalNet/jsp/index.jsp**.** The poverty line is $1.08 per day ($32.74 per month) in 1993 PPP prices. This site contains the estimates of poverty measures, Gini coefficients, nominal and real mean per capita consumption at the country. We have used data for the year 2001 (94 country observations). We do our analysis in terms of rank correlation (see table 1) among different poverty measures and our PRF measures. In the table, A refers to our poverty reduction measure, the subscripts picking out different parameter values as stated in the formulae in Section 3 (the first subscript gives the value of the parameter α and the second subscript gives the value of the parameter δ). Further, H is the head count ratio, PG is the poverty gap measure, SPG is the squared poverty gap measure, GINI is the Gini coefficient of inequality and MEAN is the mean of the distribution.

Table 1: Rank Correlation Matrix

	MEAN	..H.	..PG.	..SPG.	..GINI.	.A11.	.A21.	.A12.	.A22.
MEAN	1.000	-0.822	-0.753	-0.705	0.003	-0.324	-0.296	-0.588	-0.558
..H.	-0.822	1.000	0.982	0.950	0.465	0.767	0.720	0.923	0.888
..PG.	-0.753	0.982	1.000	0.989	0.510	0.839	0.820	0.968	0.952
..SPG.	-0.705	0.950	0.989	1.000	0.508	0.859	0.867	0.973	0.977
..GINI.	0.003	0.465	0.510	0.508	1.000	0.777	0.705	0.665	0.616
.A11.	-0.324	0.767	0.839	0.859	0.777	1.000	0.966	0.940	0.930
.A21.	-0.296	0.720	0.820	0.867	0.705	0.966	1.000	0.920	0.948
.A12.	-0.588	0.923	0.968	0.973	0.665	0.940	0.920	1.000	0.986
.A22.	-0.558	0.888	0.952	0.977	0.616	0.930	0.948	0.986	1.000

The first point to note from the table 1 is that the PRF measures A_{ij} , i, j = 1, 2 induce almost the same ranking of countries as each other (the rank correlation coefficient, r_R is greater than 0.92 in all cases). Next, comparing the A_{ij}'s with the traditional poverty measures we find that the rank correlation, r_R, between A_{11}, A_{12} and PG, A_{21}, A_{22} and SPG and finally between A_{ij}'s on the one hand, and H, PG and SPG on the other, are in the range (0.72, 0.98). The relationship is positive in all cases and all measures. Rankings will generally be in the same direction. So, A_{11}, A_{12} contains similar information as PG but *not* identical.[2] For instance, there are significant differences between the rank of A_{11} and the rank of PG for some countries. India-rural has a A_{11} rank of 56 and PG rank of 20. The corresponding ranks for

[2] We checked whether the computed value of r_R is statistically different from 1, the case of perfect concordance, or not using large sample tests and the result is in the affirmative. In fact, with a sample size of 94 countries, a difference of .05 in the correlation coefficient is statistically significant.

Panama are 12 and 50, for Rwanda are 57 and 24, for Swaziland are 16 and 51 and for Uganda they are 41 and 2. Similar conclusions can be drawn for the other combinations.

Comparing A_{ij} 's with the traditional inequality measure, the r_R between Gini and the A_{ij} 's is in the range (0.61, 0.77). Hence, the information contents in these two classes are similar but again there is a significant difference. Finally, the r_R between the Mean and the A_{ij} 's gives $r_R \in (-0.59, -0.29)$. So the ranking is in general in the reverse direction. Richer societies seem to exhibit less Poverty Reduction Failure, although the association is fairly weak.

5. Conclusion

In this note we have axiomatized an index of Poverty Reduction Failure, mathematically related its form to standard poverty measures, and compared the rankings induced by Poverty Reduction Failure to the rankings induced by poverty, for 94 countries in 2001. Poverty Reduction Failure is empirically associated with, but it is not identical to, poverty.

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